Linear Programming and its Applications to Computer Science

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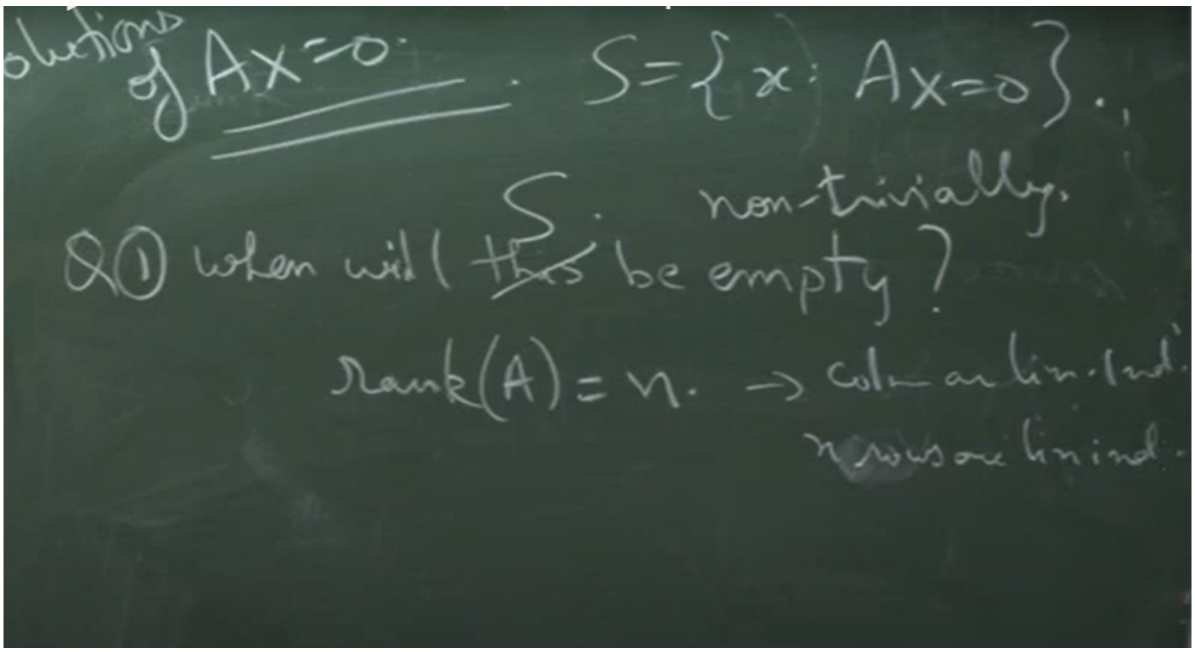
Lecture – 07
Solutions of Linear Equations

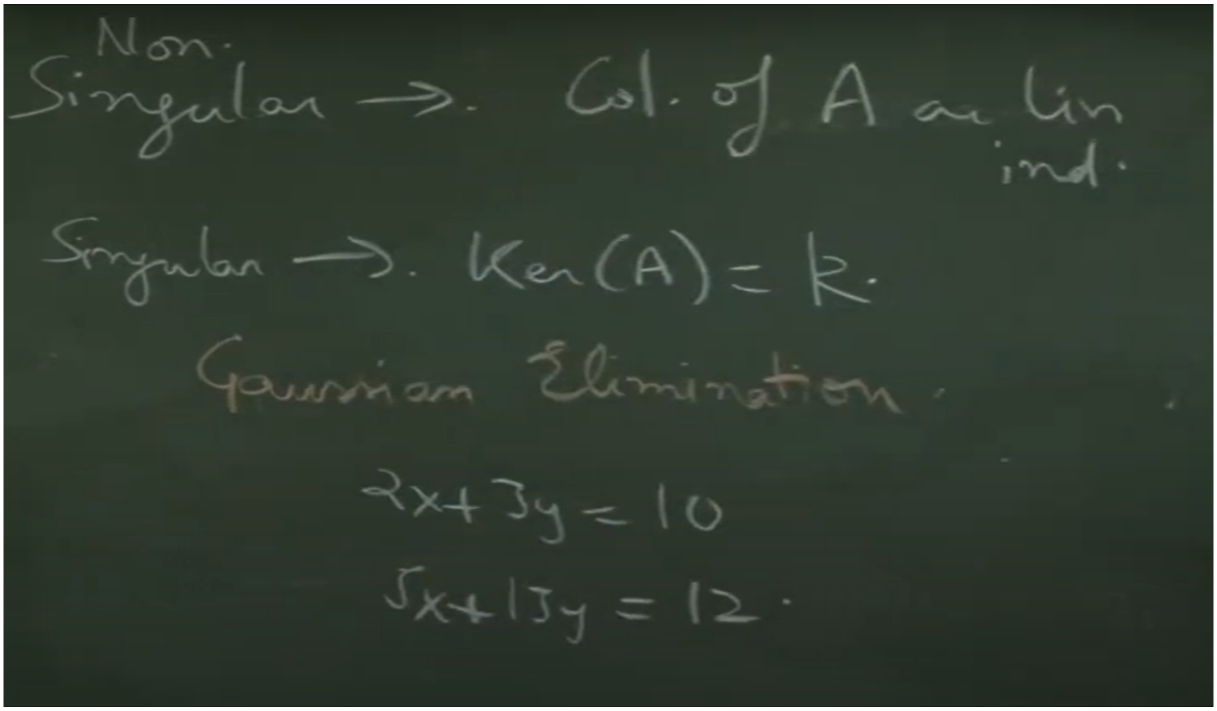
Remember our old problem, we wanted to find out solutions of Ax equal to b and b equal to 0. In some sense the simplest case, this is talking about the kernel of A. So, the first question is what does this look like? And actually this question is very easy to answer. Where will the solution set? When I say this, I mean let us say S is my set of feasible points in this constraint system. When will S be empty? Non-trivially empty, good. So, it is never empty because 0 is always there, but when is it non-trivially empty? When rank of A is n.



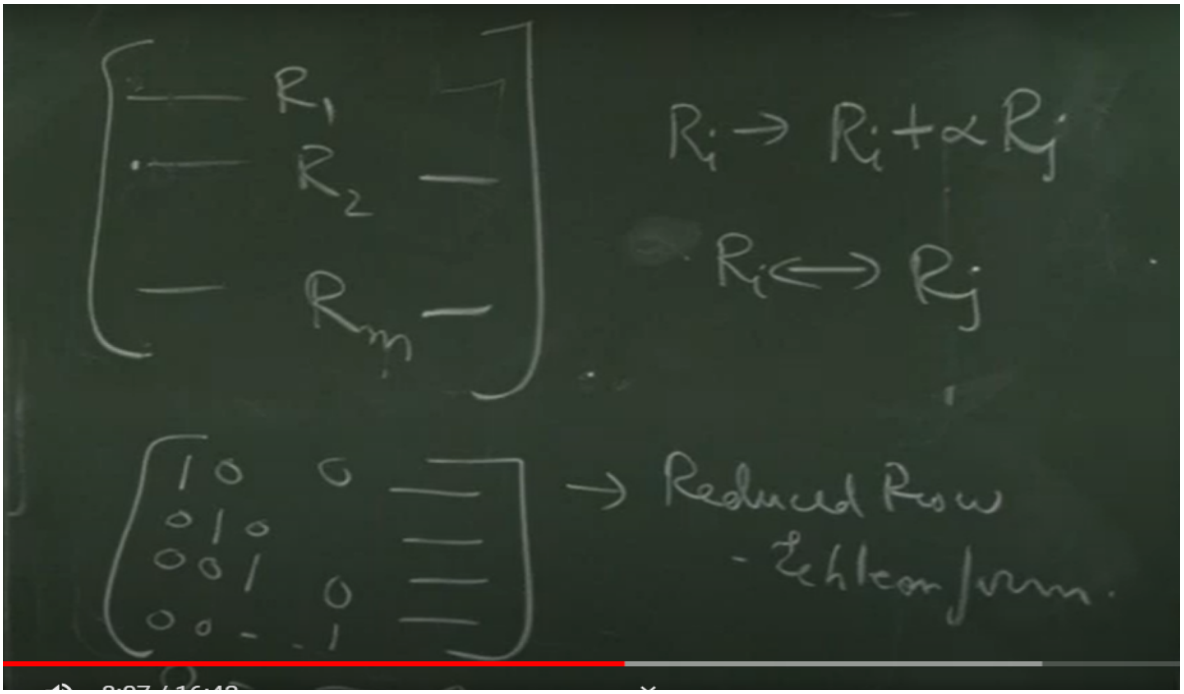
Now, because of whatever we have discussed, there are multiple ways to say it. We can say that columns are linearly independent or n rows are linearly independent ok because S being empty means kernel is phi with 0. Right so, dimension of that subspace is 0. That means the size of the images or dimension of the images is n.

Dimension of the image being n means rank of A is so great. There are two cases, I should say non-singular. This is when columns of A are linearly independent. The other case is singular where kernel of A has some dimension, let us say k. But how do we find these solutions? So, we have to describe a vector space whose dimension is k.

So, we want to find a basis which has k elements. How do I describe this vector space? The answer is Gaussian elimination. See, it is a good idea to believe your instructor. When I said Gaussian elimination is the answer to everything and that is why we like Gauss so much. These are fundamental things.



So, do people know what Gaussian elimination is? Yes, no? So, probably once I describe it then you know. So, how do you solve this bunch of linear equations? Multiply so that one of the coefficients goes away. If you have seen linear algebra, you have these. How many rows are there? We can replace a row by R 1 plus alpha j R j or something. So, any R i can be replaced by Rj, remember this operation and you can, does this remind you? People have seen this, right? Linear algebra course, we do this.



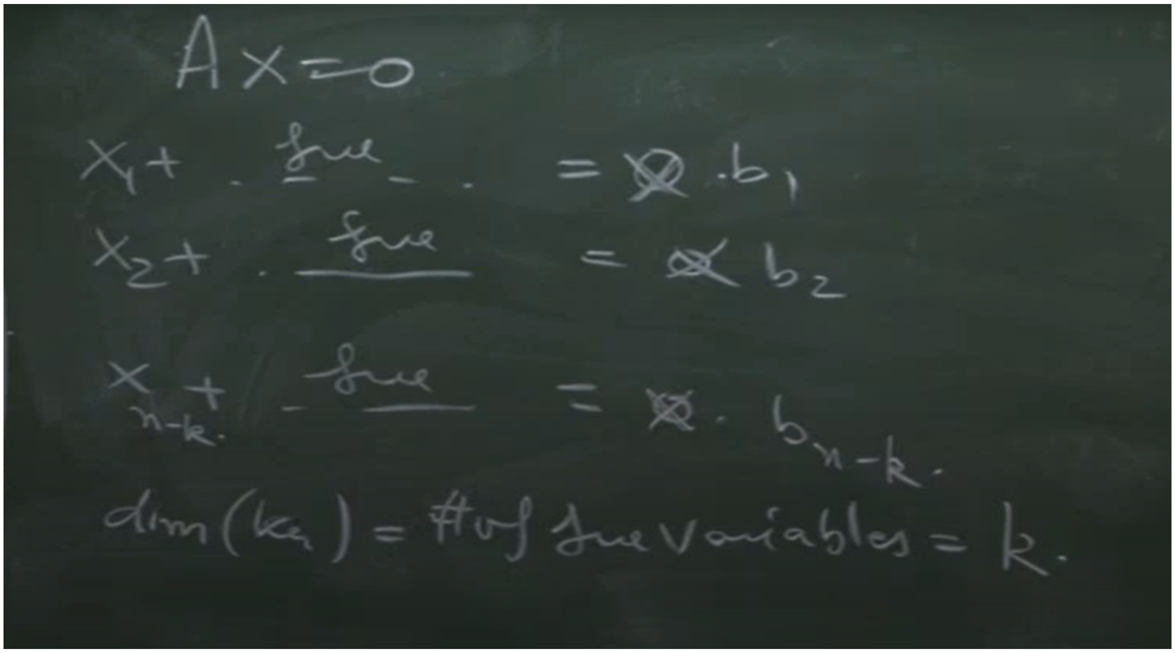
This is Gaussian elimination. This is the simple strategy which we will apply here, kind of try to remove one variable at a time. Right, so, once you have this operation allowed, why are these operations allowed? Because they do not change linear independence, right. This is why this is allowed. And without changing the image, when we do these changes, what we can do is, make the matrix look like a diagonal matrix.

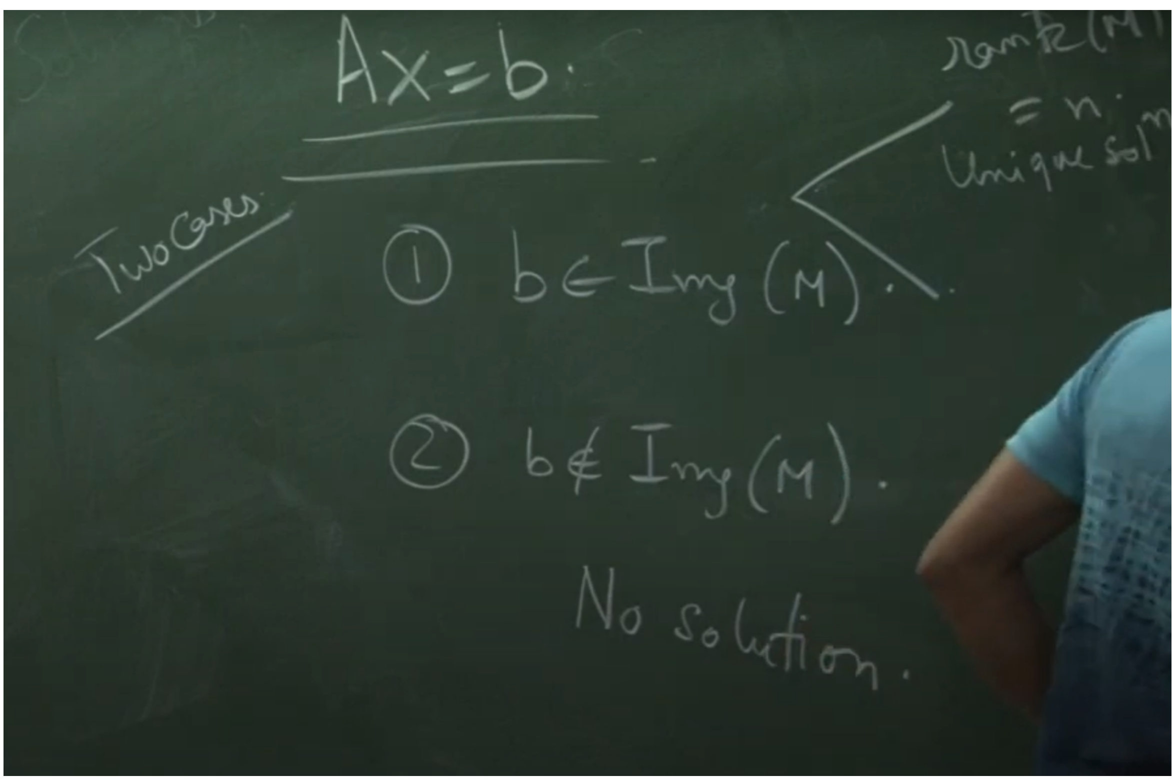
Whereas, A X is equal to B, easy to solve, when A is diagonal. So, all these attempts are trying to make this matrix look like a diagonal matrix. This is what you are doing even here. So, what do you do? Suppose, there has to be a row whose first entry is non-zero. If it is not, then I can forget about the first one.

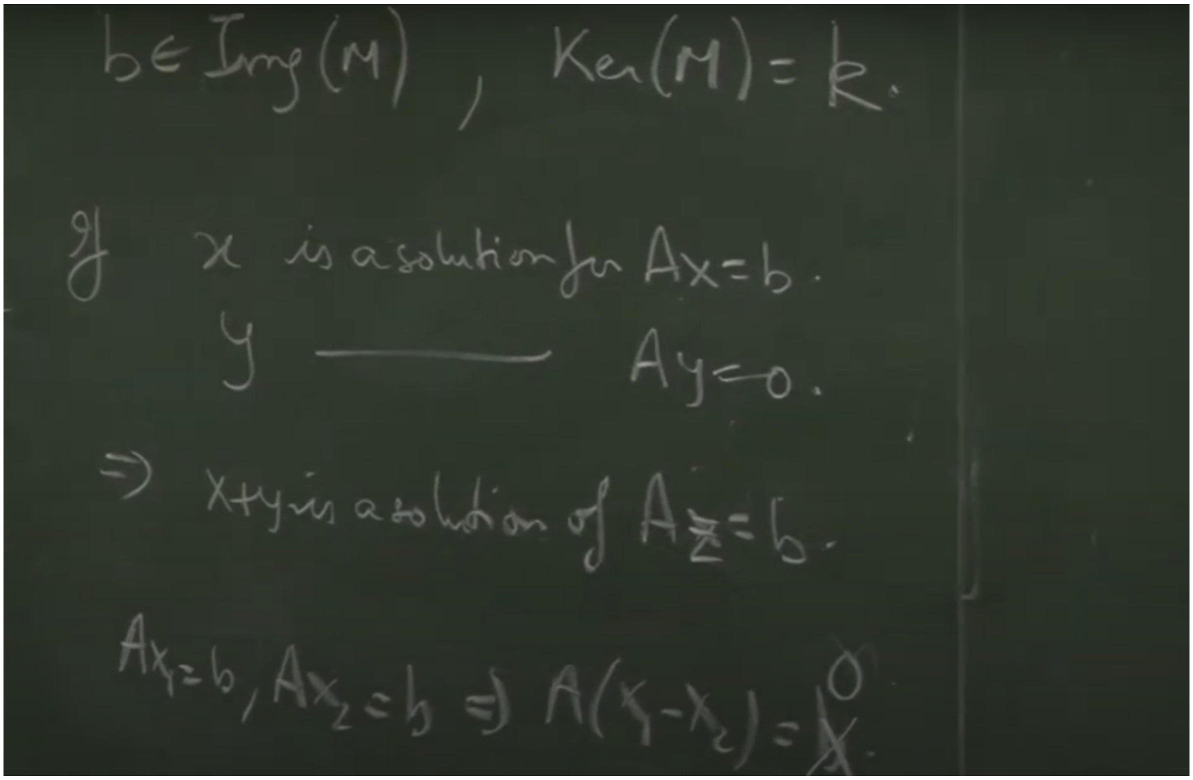
So, get this at the first place, use that to make all the entries below zero. Now, look at the remaining rows. There has to be a place where the second entry is non-zero. If it is only in R 1, and then we can move the column here or something. But otherwise, take this and then make everything zero here.

So, that there is only one thing for that. Does this remind you of Gaussian elimination? Something which all of you have done. So, when you do all this and even if I allow you to rename your columns, right renaming column means, switching the columns, then your matrix gets into this nice form. This is 1 0 0 0, everything below here is 0, there can be some entries here which I do not know, right. So, this is called the reduced row echelon form.

Seems like a distant memory when you studied this and pass the course, right. Welcome back, these things were useful that is why you studied them, right. So, you can you can do this, but once you do this, your life is set. Now, you can solve linear equations. Why? So, let us worry about AX equal to 0.

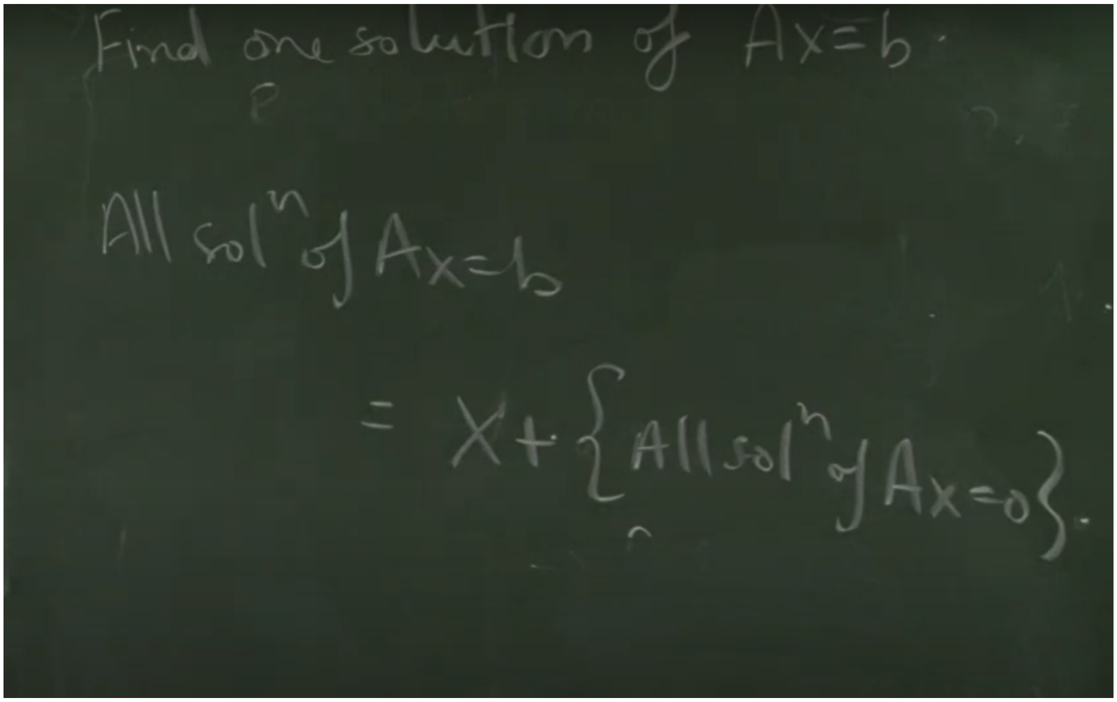






It is the simplest case. We divide our variables into two sets. These are called the leading variables or the awesome variables and these are called the free variables. Why are they free? We do not care about them. You can do whatever with them and still I will have a solution, right.

So, now why is that true? Because if you look at Ax equal to 0, what is the first equation? It is saying X 1 plus something in free variables. What is the second equation? It is saying X 2 plus something in free variables, right. So, then you can set. So, actually yes, whatever value you set for the free variables, this will give me some value of X 1, X 2 up to X n minus k, so that I get AX equal to 0. So, what is the number of free variables? It is the dimension of my matix.



This sounds ok. So, all this was to make sure that we can solve AX equal to 0. Do we have any questions on this up to AX equal to 0, right? And then, yeah, actually it does not take a much bigger leap to solve AX equal to 0, right. Because now, but then if you kind of think deeply about it, there again going to be two cases or probably three, so two cases one is right. Simple, this partitions everything. So, if B is not an element of image of M, then no solution. Ok.

That basically means that, sorry, yeah, there will be a some kind of a contradiction, right. So, now, but when B is element of image of M, there are two ways. One is when rank of M is n, right. Then what do I know? The columns of A are linearly independent. That means, there is a unique solution.

So, the case we are left with is and kernel of M has some dimension k, right. Then what are the all possible solutions? And here we can take help from AX equal to 0. How? If X is a solution for AX equal to B and Y is a solution for AY equal to 0, then X plus Y, A let us say. Do I need to prove this? Fine, right. If AX equal to B and AY equal to 0, A of X plus Y is also equal to B, right.

And on the other side, if AX1 equal to B and AX2 equal to B implies, sorry, yeah, thank you. That means, now we have a very, very nice description of the solutions. Find one solution of AX equal to B and then, right. Given this, now it is very easy to describe all solutions of AX equal to B. You pick a one solution, you pick one solution and then all solutions of AX equal to 0 are going to be there. Ok.

And how do I, how do you pick one solution? Again, the free variables, they will allow me to write something about the X1 to Xn minus k, ok. So what we were to summarize, what we were able to do today was, we were able to look at AX equal to B as linear combination of columns. Find out these properties and that allowed us to have a nice description of solutions of AX equal to 0 and solutions of AX equal to B. We know these are subspaces, we know their dimensions, we know the relation between these dimensions, right. Now with this, we will start exploring linear programming in the next lecture. Thank you. .