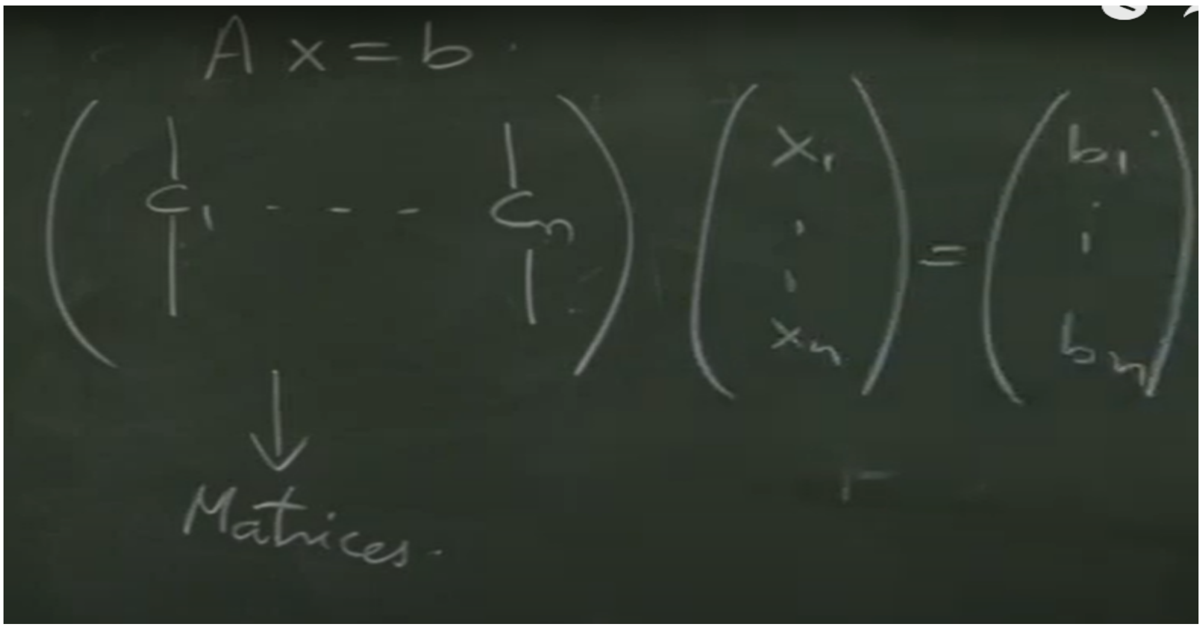
Linear Programming and its Applications to Computer Science
Prof. Rajat Mittal
Department of Computer Science and Engineering
Indian Institute Of Technology, Kanpur

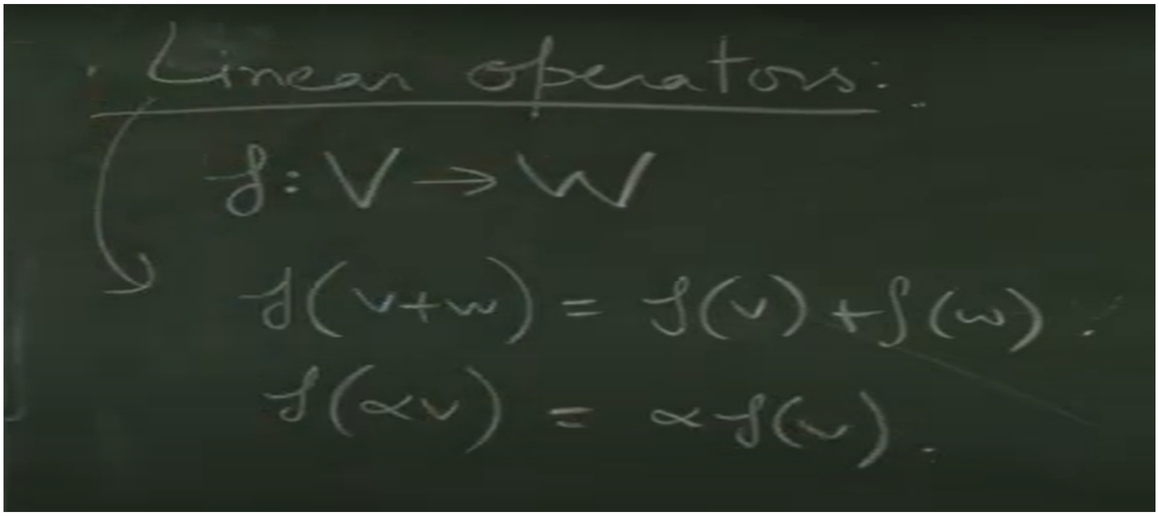
Lecture – 06
Linear Operators

An attempt to solve Ax equal to b right would be let us say this is m cross n. So, there are n columns. A is this b n. Remember you wanted to solve this problem and from whatever we discussed the first thing should be if you want to solve this first we should find a basis of the space spend by C1 to Cn.



This might be a good idea right why if Cn can be written as a linear combination of C1 then it is kind of useless for us. So, we can throw it out we can make our problem simpler we can reduce the dimensions right and if we reduce it to 2 great then we can use the graphical method right. But obviously it seems like a good idea to kind of condense this matrix and that we can study irrespective of this and this.

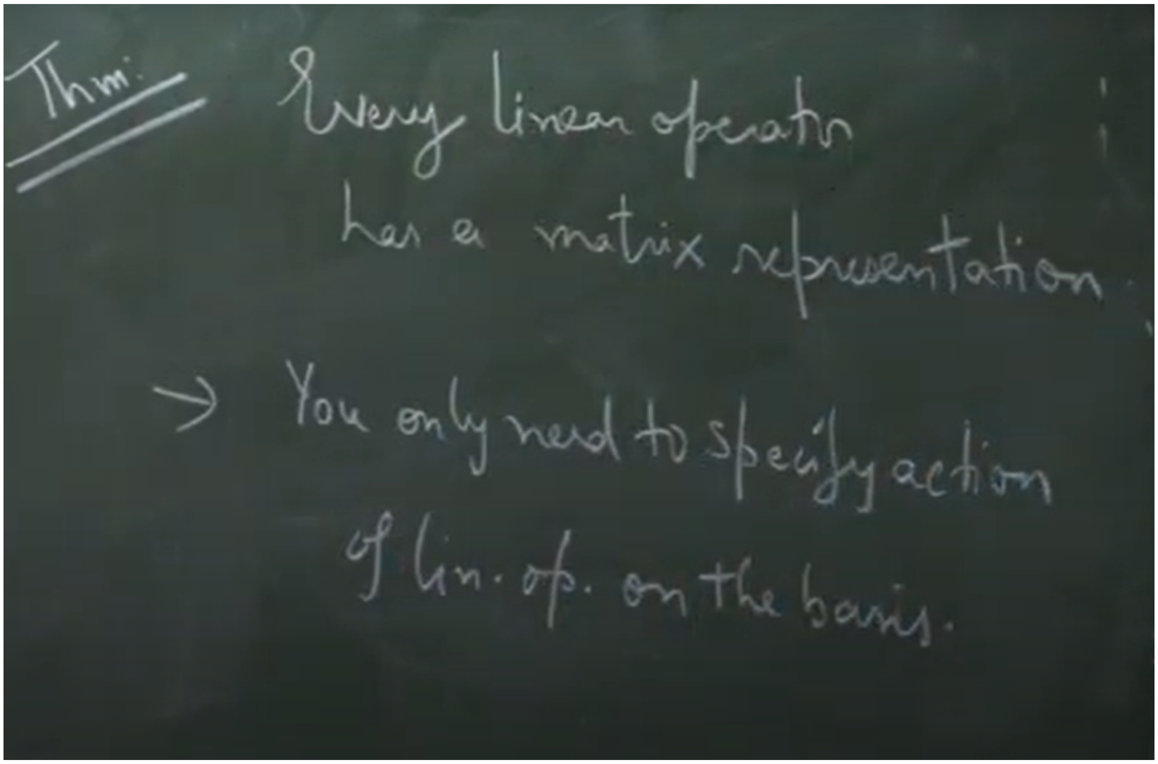
And so let us study matrices. The first thing we want to talk about for matrix is linear operators and can someone tell me what are linear operators? What is an operator? Function on vectors. So, what is a linear operator? Some constraints. Yes.



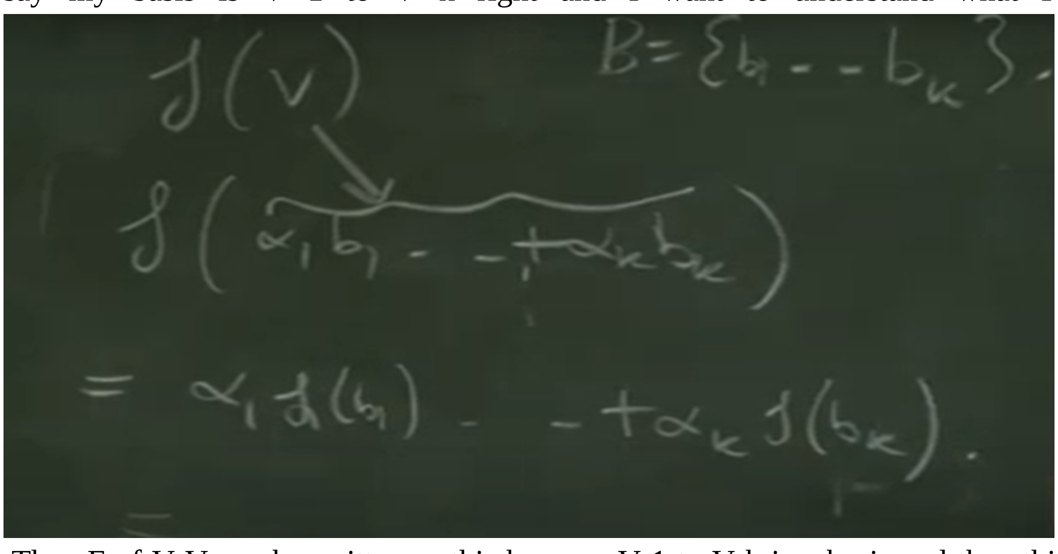
So, operator means I am looking at some function or some operator from a vector space V to vector space W right. Operator means for any vector V here there is a image from vector space W right. And then if I talk about linear operator that means a linear operator acts linearly right. Do we know any example of linear operators? Good. So, how are the linear operators? What are they? Which space are they going from and what space are they going to? So, let us say matrix M is some m cross m matrix right.

You said it is a linear operator right. I agree with you. But if it is a linear operator it is going from V to W. What is V? What is W? Is also no no no V is R n and W is R m. It takes a vector remember m a equal to b this has to be n cross 1 and what we get is m cross 1 correct.

So, m is basically a linear operator you can check you know that agreed. And again one of the fundamental theorem is when you when you hear of linear operators when you will hear of matrices very nice. It is not just that matrices are examples of linear operators, but every linear operator has a matrix representation right. Now, someone says they have a linear operator and you want to know that linear operator. How will they give you the linear operator? By set of equations how, what equations will be there? No no no the remember what is linear operator? It is a function from V to W right.



The dumb thing would be to to take all possible elements of V and give its give its output, but that is going to be a long list. Exactly. So, this is another fundamental thing that this is true because we have a linear operator right. So, once I give you the action on the basis and I want to know what is the action on V? I will first write it as yes. So, let us say my basis is V 1 to V k right and I want to understand what F V is.

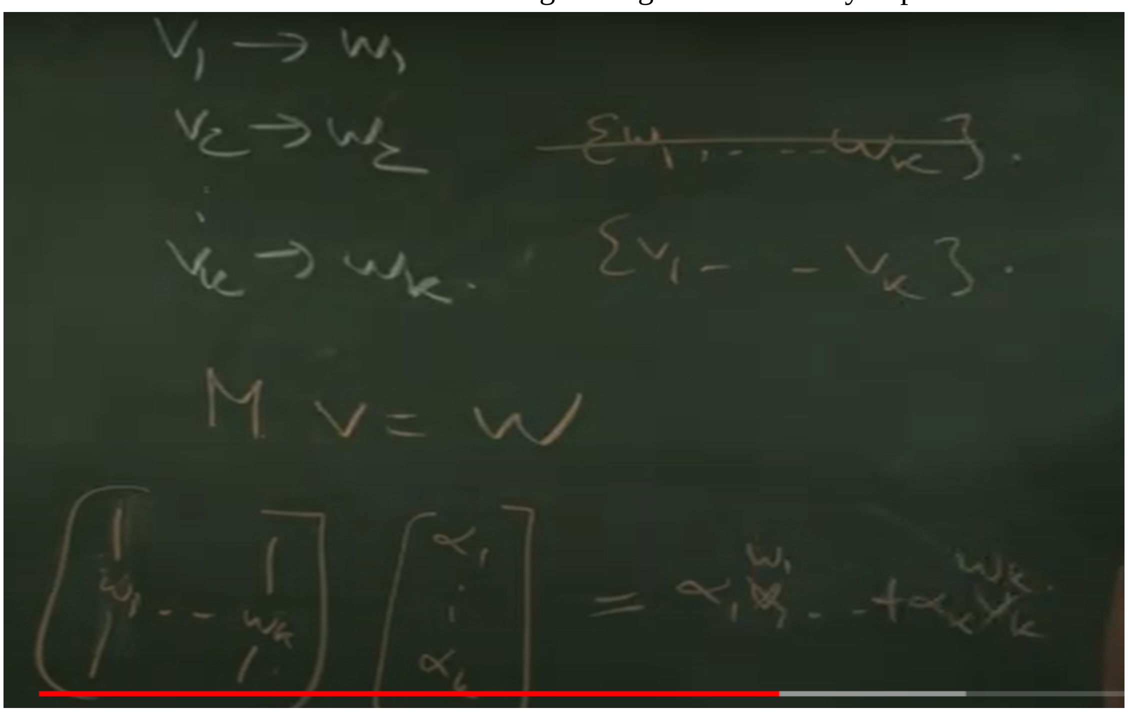


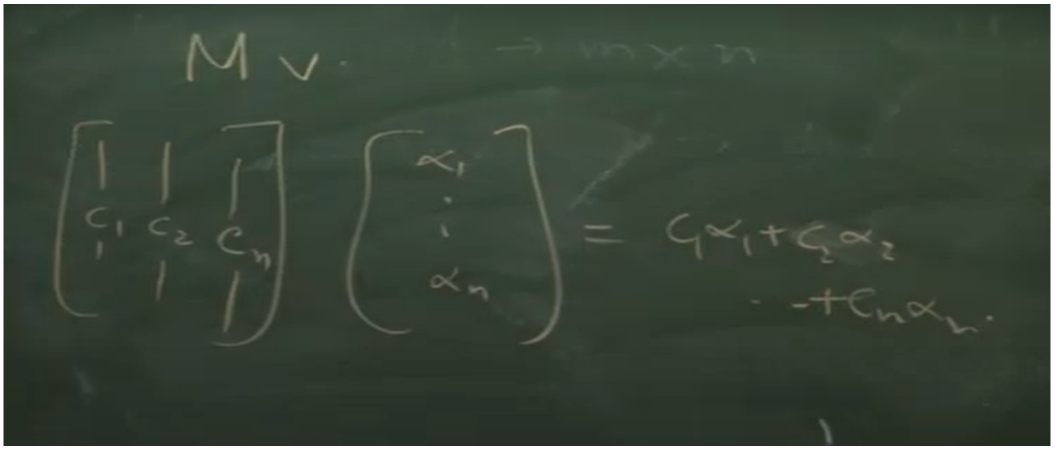
Then F of V V can be written as this because V 1 to V k is a basis and then this is and someone has given you these values on the plate right. So, you can compute this thing. That means as Soham said if you know the action of the basis then you know the action on the entire vector space correct. Now, I have a way to give you the linear operator and every linear operator has a matrix. What is the matrix corresponding to this? Very good.

So, the question to ask is so every linear operator has a matrix representation and this is something which you forget once we fix the basis. And then since you answered so nicely I give you the freedom to choose any basis of your choice. So, I think Dev knows the answer, but what about others? Fix your favorite basis what will be the matrix for this linear operator? You can how can it be identity that is not correct. I am not saying V1 is equal to W1, V2 is equal to W2. So, you have to find a basis such that you can easily write it in the matrix representation.

So, which basis will you choose and how will you write as a matrix? Again we are doing these exercises so that you become familiar with linear algebra right that is the aim. So, think about it. Let me simplify the problem for you. You know V 1 to sorry I want to write the thing in this representation. So, then when I write this as k coefficients these are the coefficients of V 1 to V k and I should get the corresponding W what is M? Right yes what are the entries of M? So, here I will give alpha 1 to alpha k and when I said that my matrix is written in this basis this is what I mean sorry.

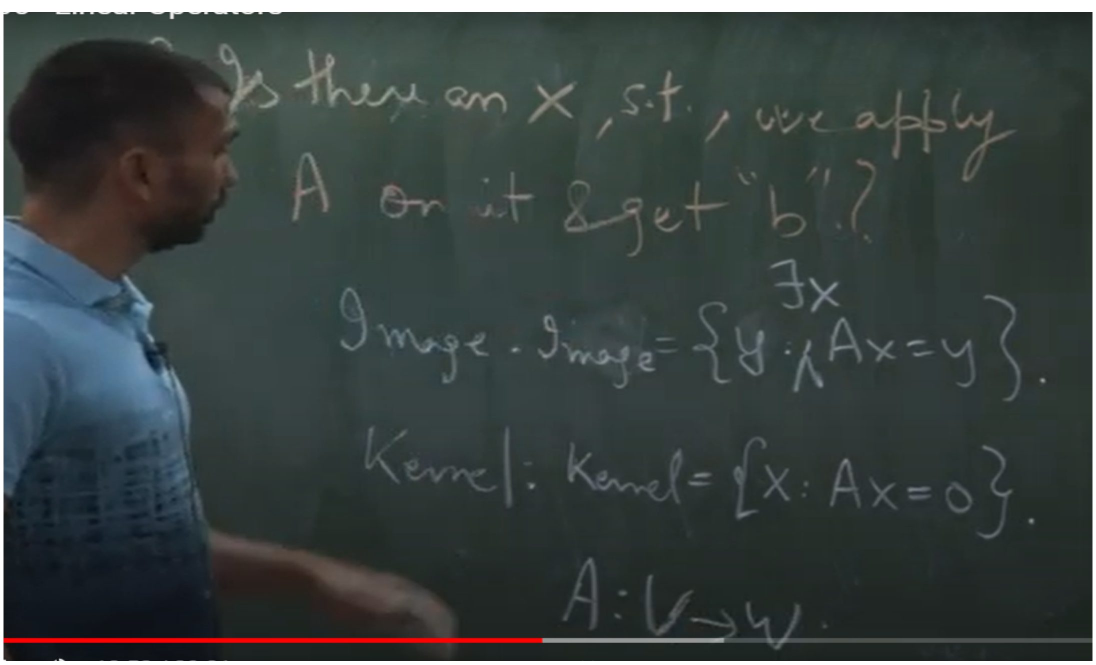
So, I choose my favorite basis here on which I am applying and here also I apply my favorite basis like Did I do something wrong? No. . Sorry alpha is our scalars.





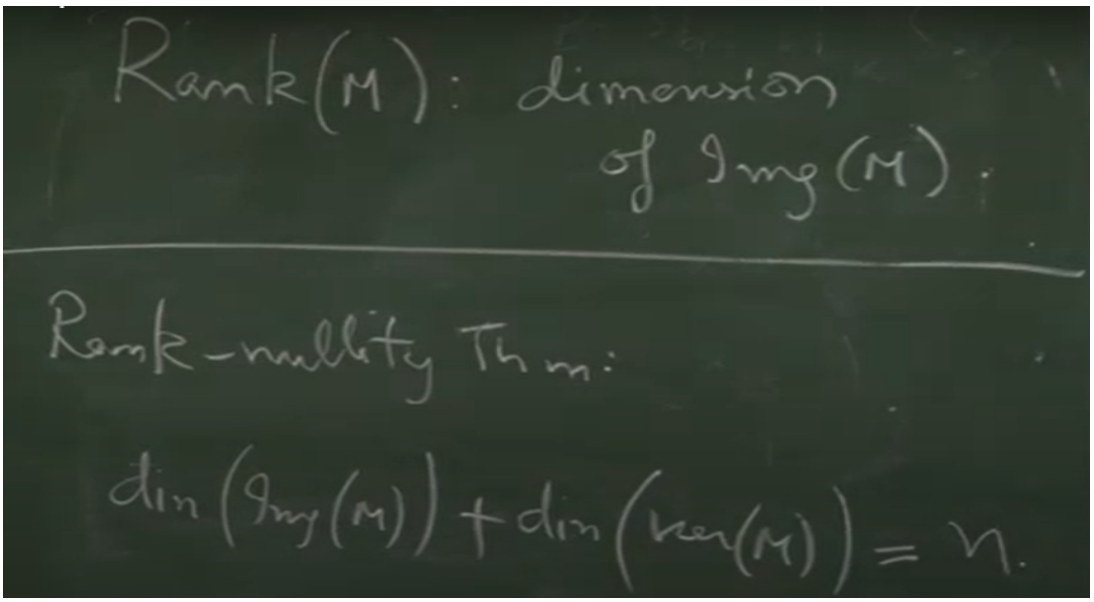
. Yes W 1 as. No W 1 you cannot write in terms of V 1. This is your matrix representation and the idea is this times this is exactly this. This is the way to interpret your matrix multiplication. So, when you write M V and these are the columns. This is this is how we multiply matrix with the vector.

So, then this is the matrix which is the which captures the linear operator. Good we wanted to talk about Ax equal to B. A was a matrix or a linear operator and we want to find out can is there a vector X on which I can apply A and get B. This is the question we are asking. So and Then, if you want to study this obvious things to talk about are the image and the kernel of the matrix. What is the image? Image is the set of all Y .



So, then we are asking is B part of the image of A and kernel is the did I do something wrong? So, it is a set of all Y's such that there exist an X .Ax is equal to Y make sense is this ok. is this mathematical notation with you right. And this is the image and the kernel it is the set of all X such that Ax equal to zero ok. and then if A is a linear operator from V to W then image is in which vector space W right it is a subspace of W it is a subset of W and not just that it is a subset of W actually if we have two elements of the image we take the linear combination it still is the element of the same subspace. So, image is a subspace of W kernel is a subspace of does anyone have this question in mind why image and kernel are vector spaces? This is ok. sounds good.

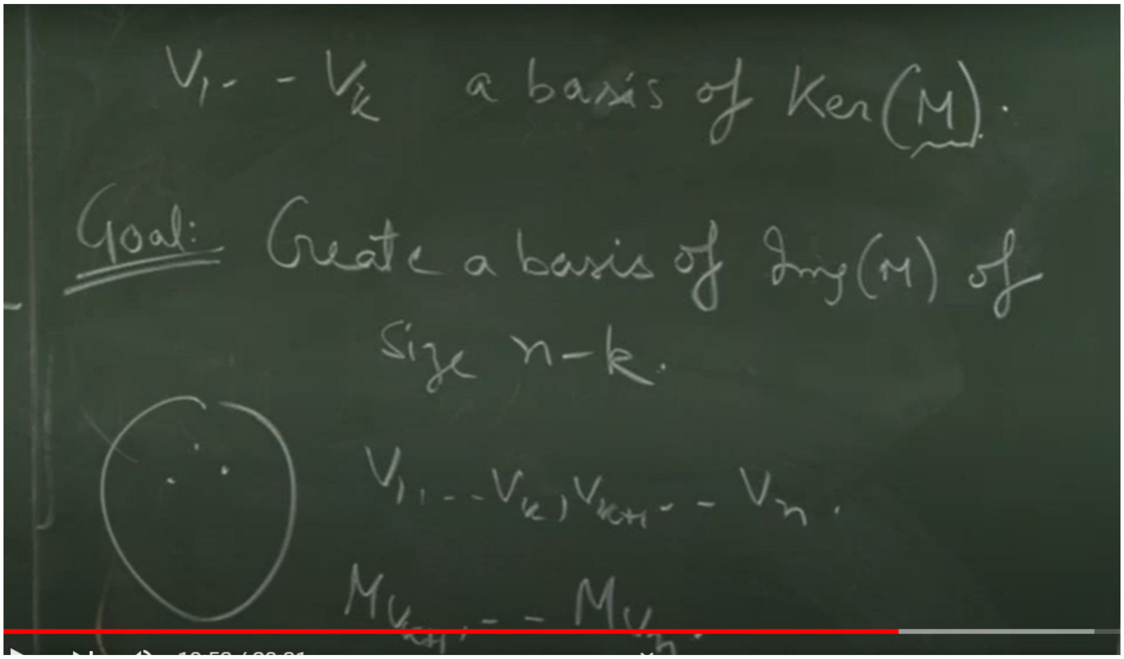
Then next concept is rank of a matrix which is basically the dimension of image, ok. and one reason to reduce kernel and you will see it behaves very nicely it is kind of the orthogonal part of the image I do not want to say orthogonal because they are coming from the different vector space, but they are complimented to each other and one of the way to put it is the rank nullity theorem right which says dimension of image of M plus let   say M is an M cross and matrix.



So, V is R to the n. How do we prove it? What will be your attempt to prove it? This everyone knows this right rank nullity theorem yes no. Is there anyone who has was seeing it for the first time? There is no shame in saying yes for this question right.

All of us have some flaws in our background it is ok, but I need to know if this is new to you. Ok. If this is not new to you then you should be able to prove it. Let me start with you know let so now, this is let us say this is a basis of ok. and then I want to create a basis of image of M.

What size should it be? I think I have helped you enough. Now, can you tell me how many people know the answer to this? One ok. others . So, if I am given a basis of V 1 to V k how will I create a complete basis of the vector space? You remember this diagram right this nice diagram. Take one if it is not you take another one if you still get did not get the maximum independent set another one right and we were given a guarantee that every time we will stop after n after we have picked n vectors right.

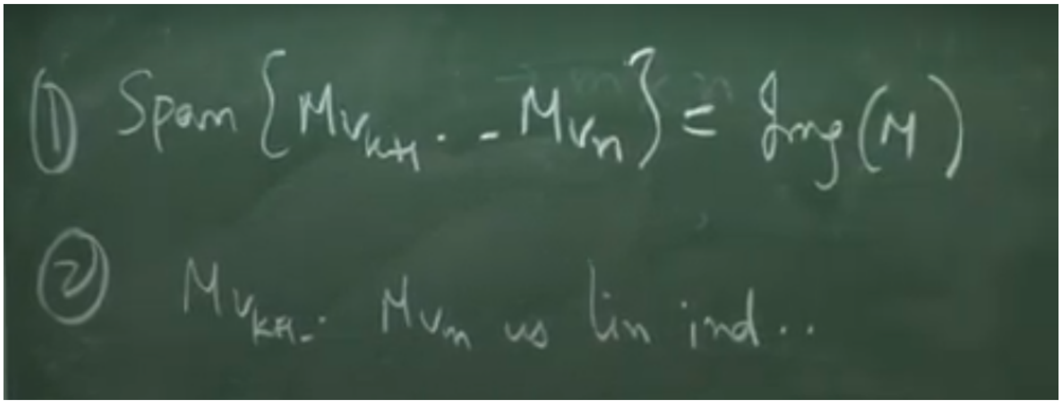


So, why do not we start with that? So, we have V 1 to V k this is linearly independent because it is a basis.

Let us find something which is not in the span of V 1 to V k. Now, can you tell me what is the basis of image of M? You are giving a wrong answer, but I know you know the right answer. What is a? I do not know what a is, but ok. remember it cannot be V k plus 1 to V n. Why cannot be V k plus 1 to V n? It is a wrong space right. I have to go to the w and what is the way to go to w? Apply.

So, my claim is that this is the basis of the image. What do I need to prove for this? You tell me what do you need to prove for this and then I would not ask you to prove it. Wait, wait let anyone else answer. What do I need to prove? The back row is out, this row is out only either you guys or you. What do I need to prove to show that this is a basis of image of M? How do I prove that something is a basis? Linear? Good.

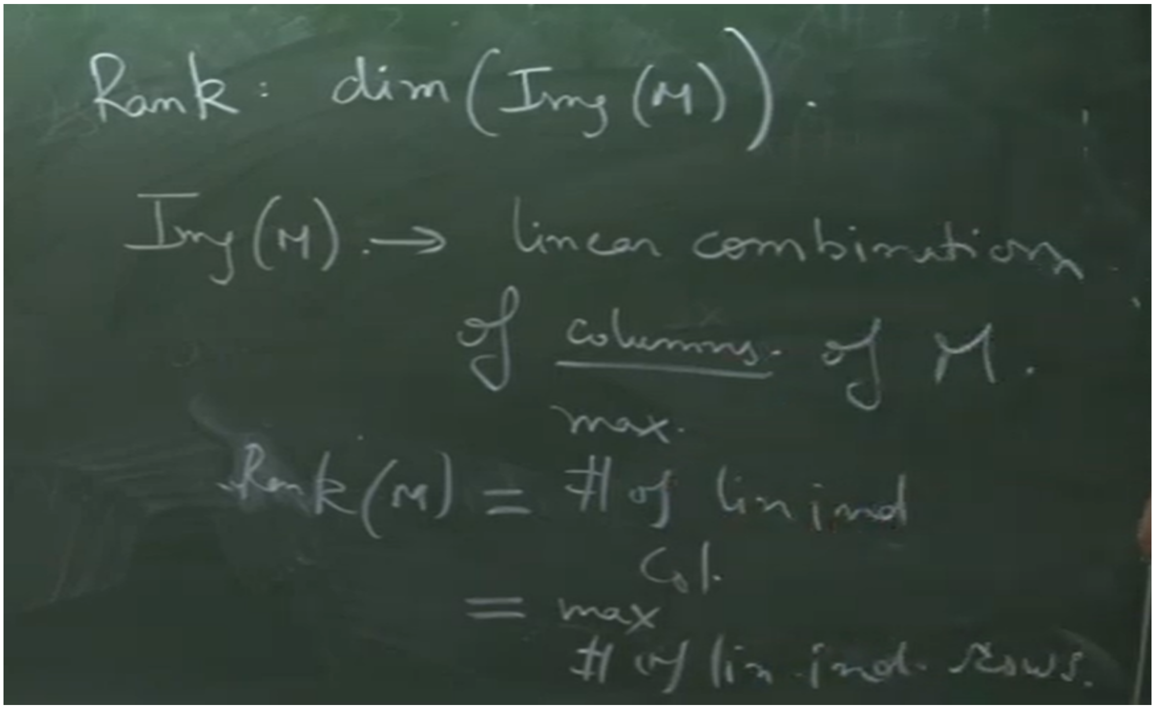
So, I have to show anything here can be generated using these vectors. What else? Wait, wait what else do I need to show for it to be a basis? Yes and both these parts are easy. If you cannot do it once again look. at some linear algebra text. This is basically saying that some linear combination of V k plus 1 to V n is in a kernel of M. It is a counter example which is a which is contradiction right.



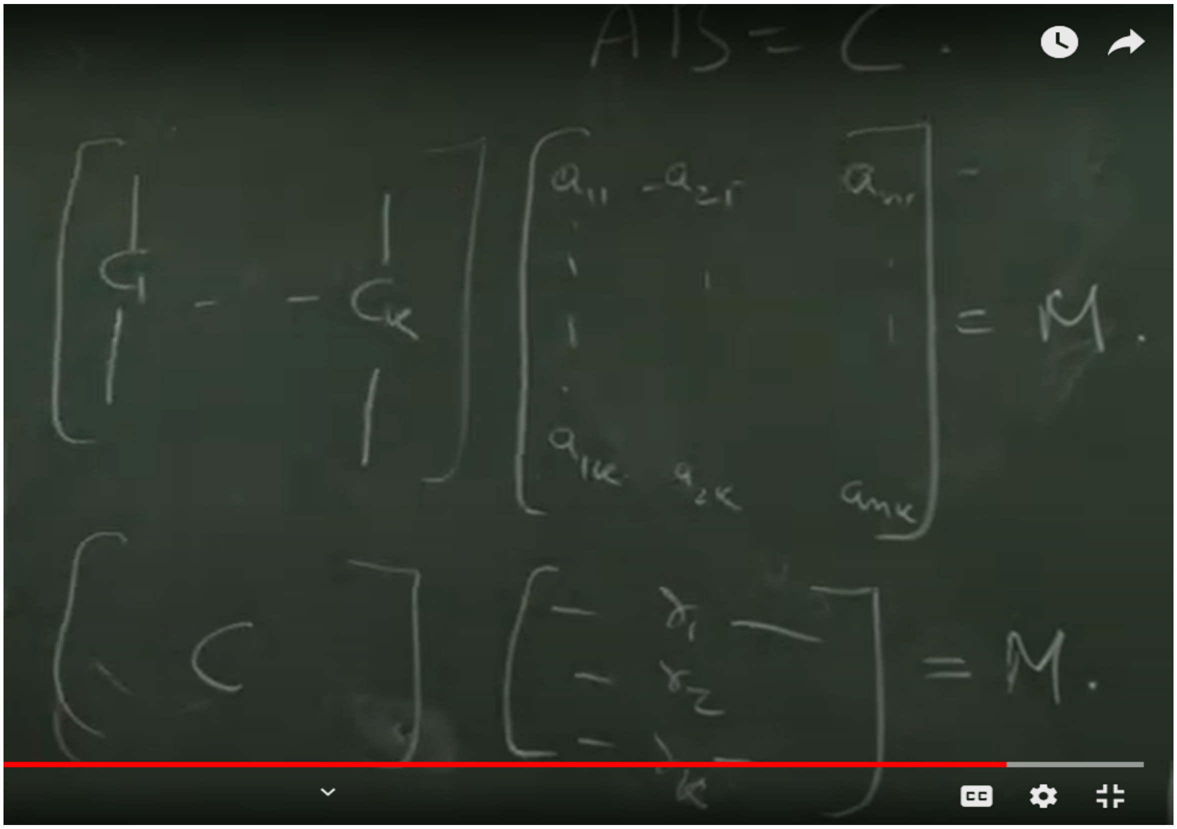
And this will use the fact that you can write any vector as a linear combination of V 1, V 2 up to V n. So, both these parts are ok. But I leave it as an exercise all of you should make sure that you can do this.

Great. Let us move on. Let us see more properties of image of M or the rank. So, the rank was the dimension of image of M right. But I also know that image of M is linear combinations of what should come here? Columns of M right. We just saw what was M times V. It was basically taking linear combinations of the column.

So, this is columns of M right. That means, my rank of M is equal to number of or maximum number of linearly independent coefficients. And if you believe in the sanctity of this world then this should also be maximum number of no. There is no good reason to believe that this is equal, but we know that this is actually equal right. So, we know that column rank is equal to row rank. And if you are seeing this thing for the first time this should surprise you.

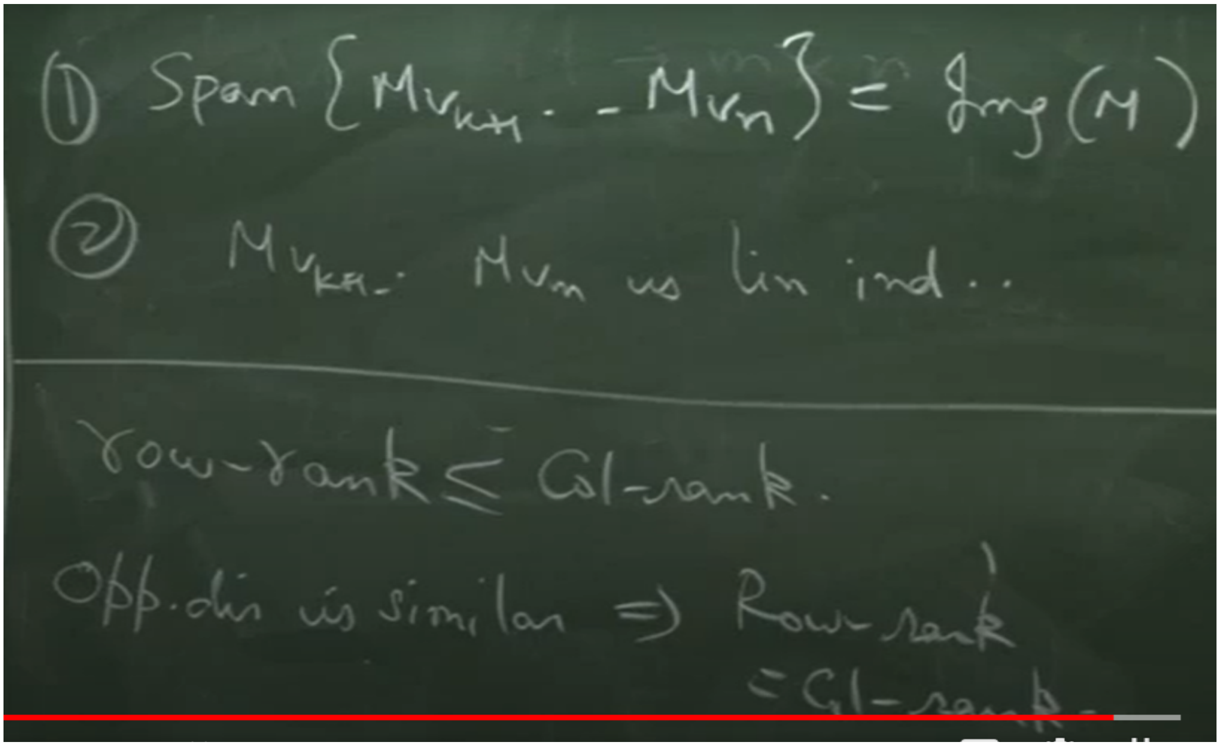


There is no good reason to believe it right. You have a matrix you can view it as bunch of columns. How many linearly independent columns are there? Or you can see it as a collection of rows. How many linearly independent rows are there? It turns out that this is going to be the same number. How many people have seen this before? How many people have not seen this column rank equal to row rank? So, now people who have seen it, what is the any answers, any two line answer, any two words answer? Yes, ok.



So, one of the natural proof which comes which proves this is Gaussian elimination, but I am going to give you a two image proof. You can say it is two image or one image that is up to you, but I think that will highlight why this is true. So suppose the column rank is k, what does that mean? Everything here can be represented in terms of k columns. Let us say some C 1 to C k. There is some linear combination here which will give me C 1.

There is some linear combination here which will give me C 2. So, I can have some numbers here A 11 up to A 3. This is the definition of in some sense is saying that column rank is k. Because there are k vectors which I have picked from here and there are some linear combinations which make it m. right and now yes now you just view it differently.



You can write this as row 1, write this as row 2 as r k. What is the first row of m? It is the linear combination of r 1 to r k with these coefficients. What is the second row of m? It is the linear combination of r 1 to r k. This is just understanding what your matrix multiplication is right So, when you write A B equal to C, you can view it as you take the columns of A, multiply it with the coefficients in the first column of B, you get the first column of C.

Or the very way to put it is you take the first row of A, multiply it with the rows of B, you get the first row of C. So, then I just think of them as k rows and I see that every row in m can written down in terms of k vectors. That proves that if you can show that every element of m, you can write it in terms of every row can be represented in terms of k vectors that shows that what does this show about row rank? Row rank is less than equal to column. Because this need not be linearly independent, but then you can run the same argument in the opposite direction. We had all these we have all these nice things and this allows us, remember our old problem we wanted to find out solutions of A X equal to B and B equal to 0. In some sense the simplest case is talking about the kernel of A. .