**Linear Programming and its Applications to Computer Science**
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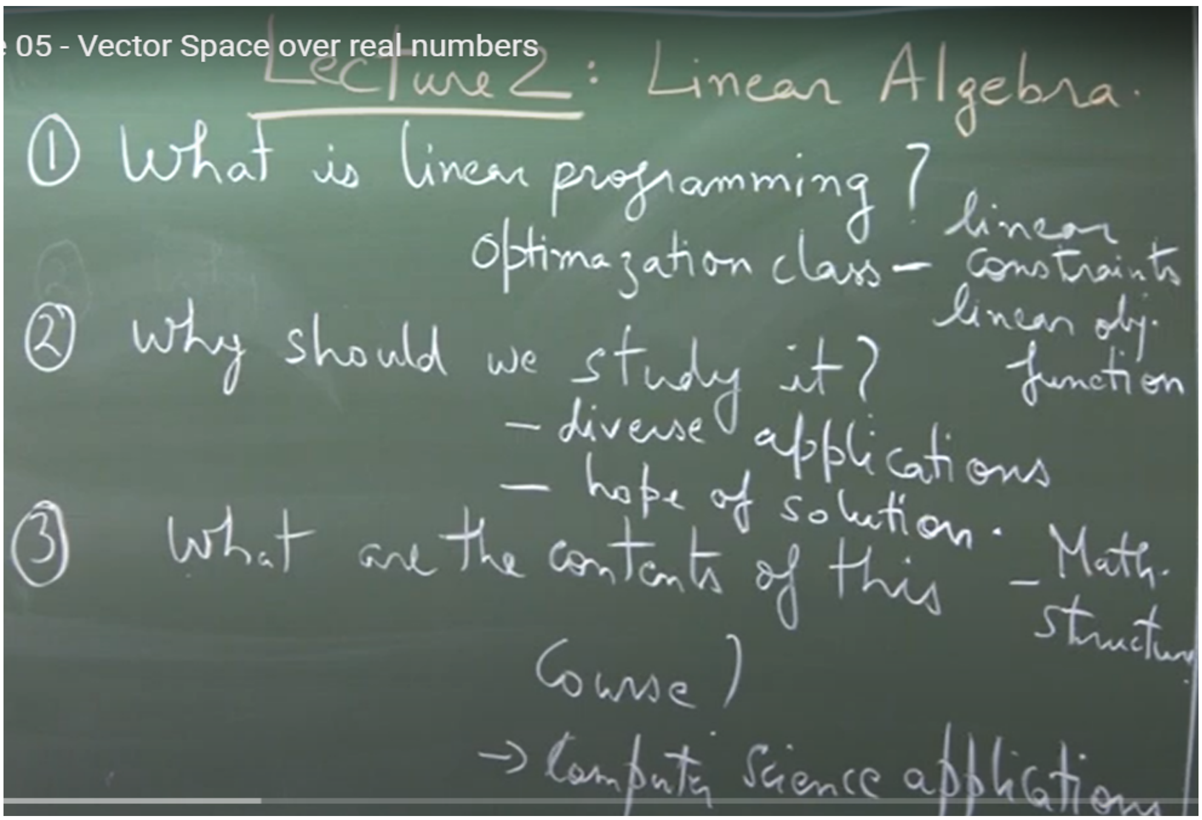
**Lecture – 05**
**Vector Space over real numbers**

Welcome to the second lecture on the course on linear programming. We saw in the first lecture these questions. We wanted to answer these questions for our first lecture. So let us just revise them. First question we answered was what is linear programming? So would someone like to tell me what is linear programming? How would you like to describe it in one sentence, two sentence probably a paragraph? Very nice. It is an optimization problem or optimization class where what was the restriction? Linear constraints as well as linear objective function.

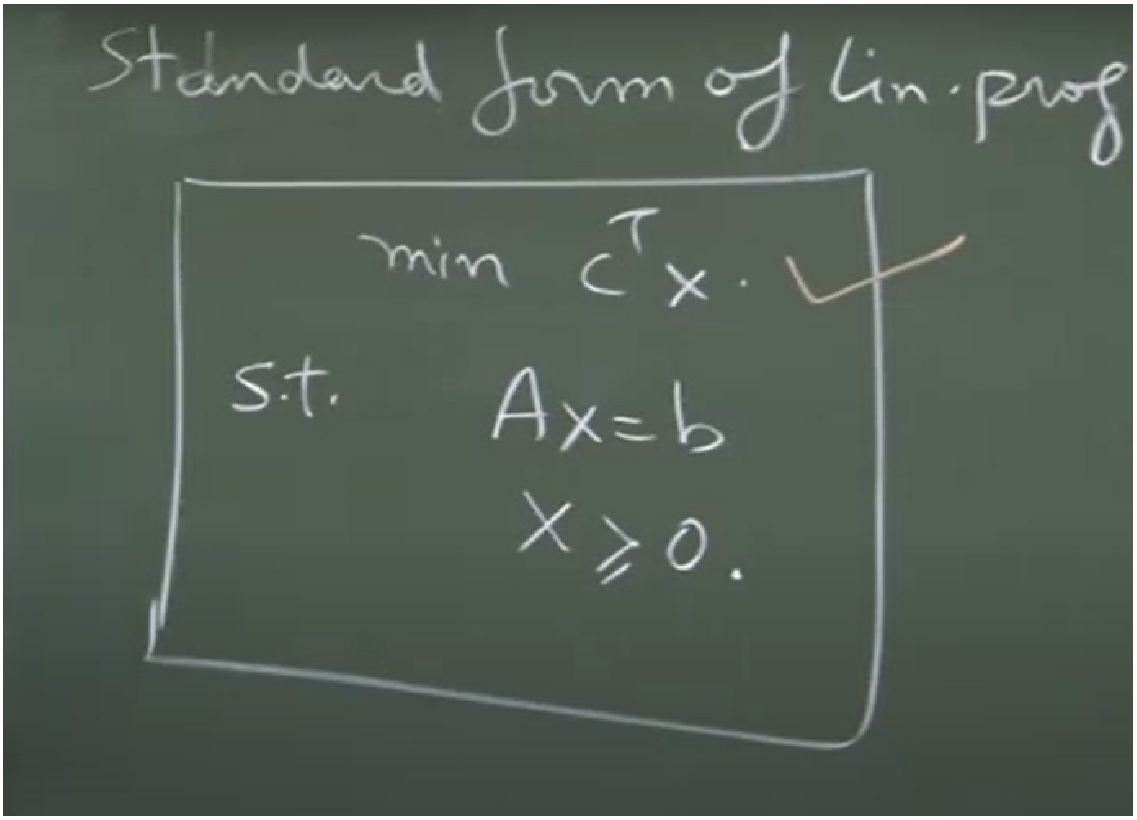



An optimization problem where we have linear constraints and linear objective functions is basically the class of linear programming problems. Second question is why are we worried about studying it? Why should we devote an entire course on this linear programming? We kind of saw which subclasses were important. We asked when would I say this subclass is important, right and linear programming fit all the reasons. What were those reasons? Very good.

It has diverse applications. What else? Easier? Yeah, it has hope of solution and now actually we know that we can solve it and this abstract idea of having a years mathematical structure, right. This is the reason we are studying linear programming. What are the contents of this course? You go to course website, you can find them out. But mainly the focus is going to be computer science applications.

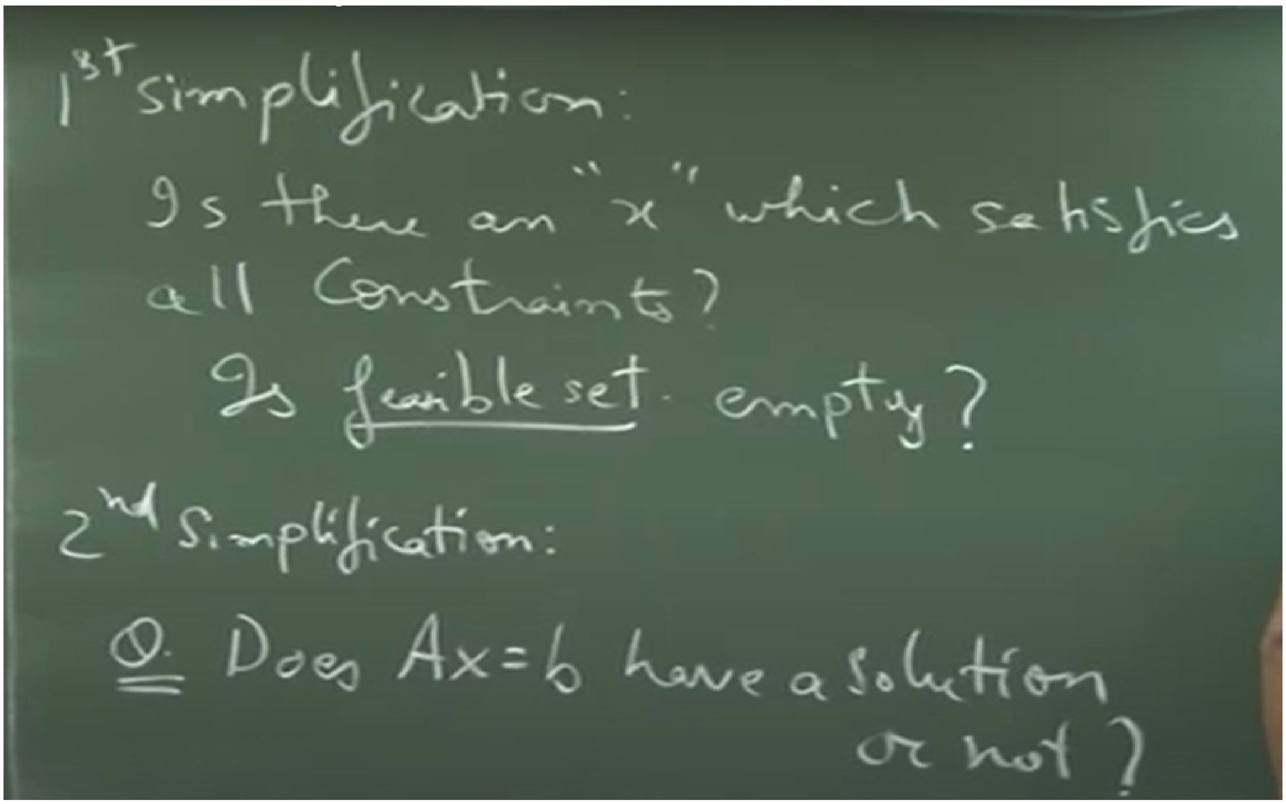
Once we get to the background mathematically what is needed to study, how do we come up with solutions? Once we do that then most of the part would be how to use this knowledge to various applications in computer science that will form the bull work of this course. We have the answer to these three questions. Let us move forward. If you remember we discussed the standard form of linear program and can you tell me what was the form?

So, answer right. So, this is the standard form and clearly I wanted objective function to be linear. That is great. I wanted my constraints to be linear with the positive constraints. So, all my constraints are linear.




But the question is does it cover all kind of linear constraints? We do not know yet, but we will see that this covers all kind of linear constraints. Whatever linear constraints you can think of we can formulate them in this language, but that is for later. For now believe me that any linear program can be written like this. And once this is true what should we do? And kind of the first half of the course the story would be we would be like these explorers who are trying to figure out the world of linear programs.

Suppose you are the first person who comes up with oh this class seems interesting. Then what will be the next thought? You want to solve it, but you want to solve it as a researcher what will be your game plan? What do you want to do? When you have a big problem in mind it is hard to solve. What do we do? Change it in the sense that we simplify it, we break it down into smaller steps. Right and when I look at it I will say oh forget about the objective function.




Let me just even find out whether there is an x which satisfies this condition. So, my first simplification would be is there an x which satisfies all constraints. And from the lecture one we actually know we can write it as the question which says is now what should come here? What is S? It is called feasible set. And then even this has this complication.

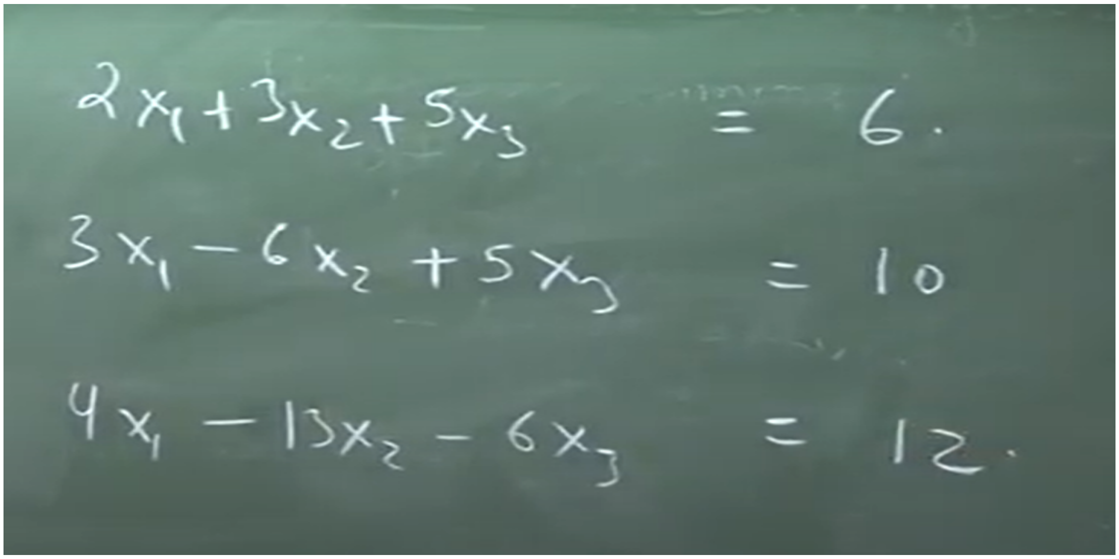
So, what would be the second simplification? Does x is equal to 0 has a solution or not? It has. We skip it we go to Ax equal to b has a solution or not. So, the first question which we are going to tackle is. And this is a question which is pretty appealing because we already know this has been studied.

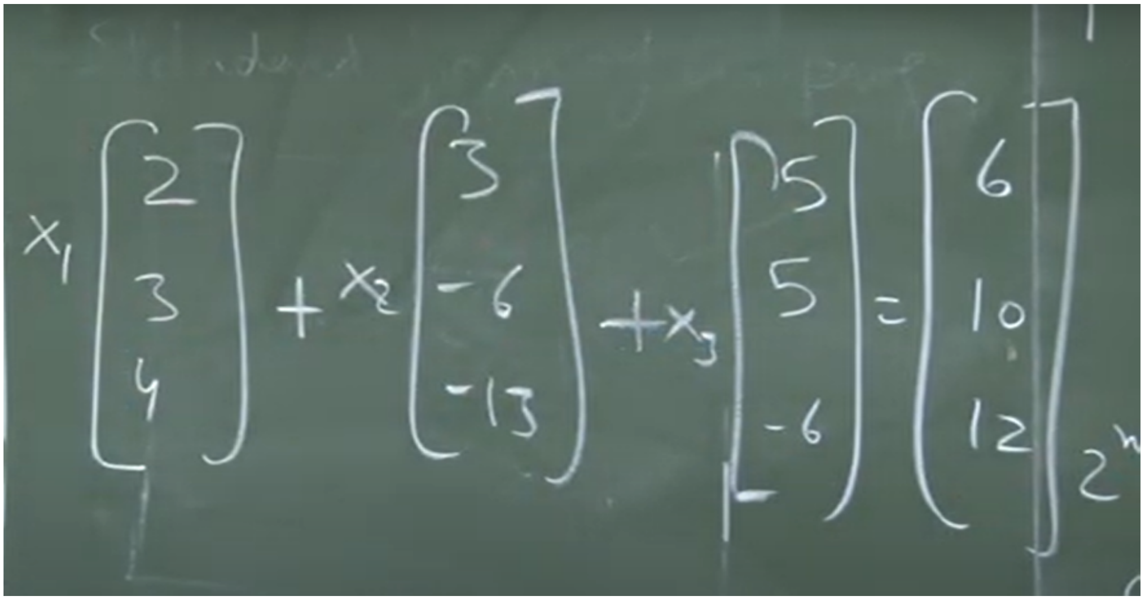
As a researcher once you find a sub problem which is already been studied we are happy. Because we know that Ax equal to b is this fundamental problem for linear algebra not just linear equations. But I would say linear algebra at least 50 percent of it. Right. This is I think the beginning of linear algebra.

We know that Ax equal to b is a fundamental problem for linear algebra. That is great there is lot of work on that. So, we should learn what has been done for this equation and that is going to be the topic for today. We want to understand how people have looked at this equation how they have solved it. I think this is a refresher for you.

If this feels easy that is great that is a good sign. If it does not then you should work more on it should make sure that this seems ok. Let us start does Ax equal to b have solution all the time. No sometimes it has sometimes it doesn't. And the way we start looking at it is by viewing it in a slightly different way.

If I give you an easy set of equations probably in two dimension easy set of like two or three equations in two dimensions you can obviously solve it. But you realize that as soon as the number of dimensions increases number of variables increases the life becomes difficult. But at least computationally we have a very nice way of handling this. And the reason being somehow when we look at this equation right we do not view it as set of equations.



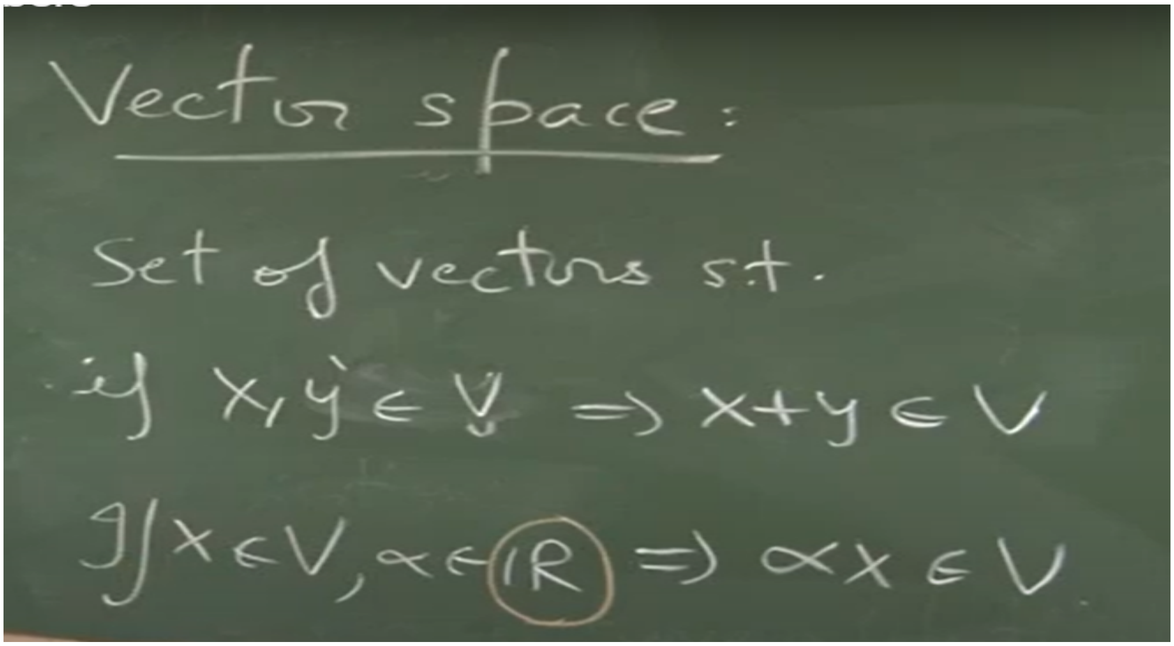



We view it as matrix. What else that is not the fundamental insight. Not as a matrix that is not it. Yeah that is the reason why we call it a matrix. But I would say that the way to look at this equation is this. Let me know if I miss a constant and what is going to be on the right hand side.

Nobody agree nobody would disagree that these two are the same things. But the question about linear equations we have converted it into the question of linear combination of vectors. And we are asking are there coefficients are there numbers which when multiplied by this vector this vector this vector are they allowed to give me this vector. And this is what we will look at. And once you start looking at it you get many obvious constructs of linear algebra.

First of them would be a vector space. What is a vector space? That is a bad bad definition. It is not set of vectors. Set of all. Good set of vectors which satisfy some condition span of x or y.

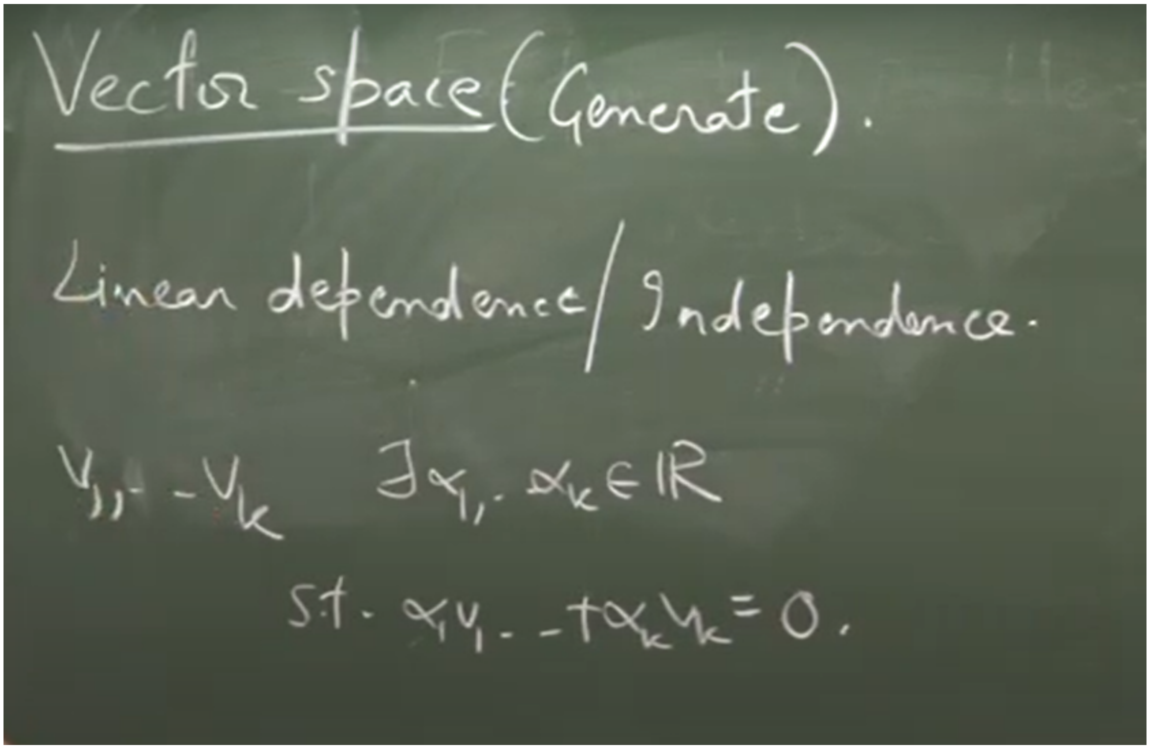
The idea is that we want to denote the set which is obtained by all possible linear combinations of this. Right, we can ask what vectors these three vectors can generate. That feasible set we want it to be a vector space but the better way to view it is set of vectors which have some constraints. You can make it very difficult but for us a vector space is a set of vectors such that x plus y if and y. It is a set of vectors where any such linear combination is contained inside the vector.




In some sense it is some complete set where inside it whatever whenever I do this kind of an operation still falls inside the set itself. Right and if you are an algebra fan then you might notice I have used real number here. For our course we will only be worried about vector spaces over real numbers. But in general you can define vector spaces by putting any field whatsoever here. It could be F2, it could be F3, it could be polynomial modulus some irreducible polynomial whatever but this is potentially any field.

We do not have to worry about it if you are only worried about this course. In general vector space is very strong but here we fix the field to be R. So whenever I say what are the coefficients, what are the scalars they are always good. And now once we have the vector space we can ask how can we generate it? We saw that if we take all possible elements of the vector space V they will obviously generate the vector space V. But is there a better way? And it turns out since we are dealing linear with this linear combinations this has a very nice answer.

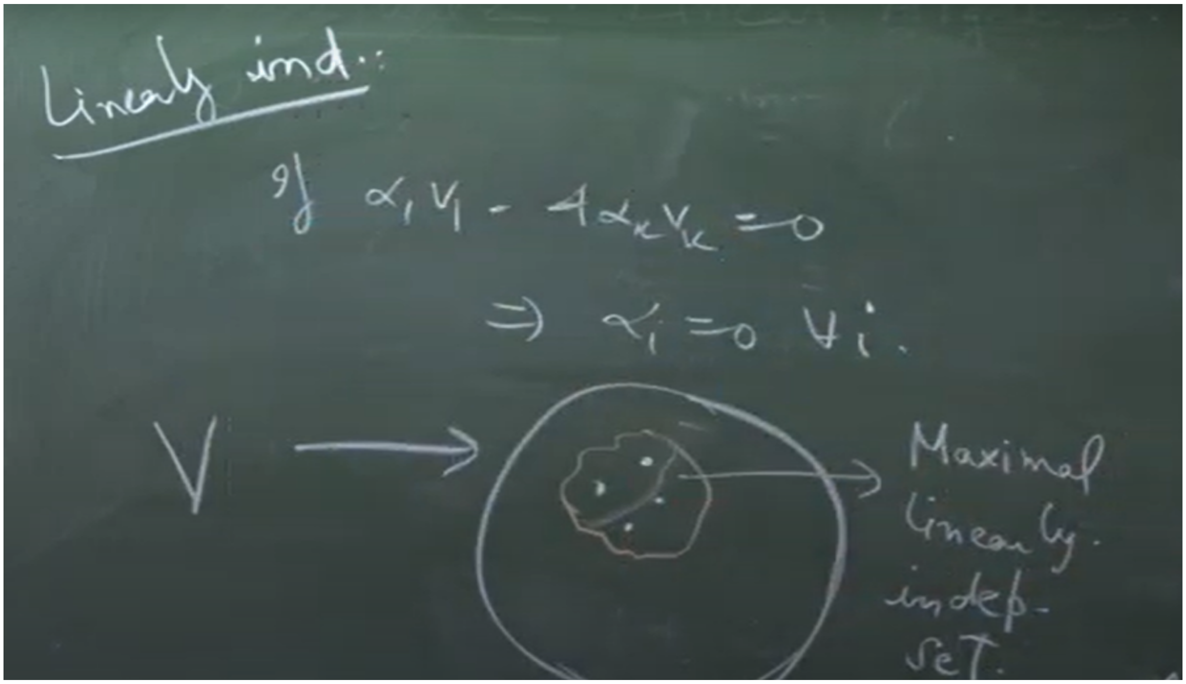
And this comes from the notion of and independence. Again I am going quickly through it because I am giving you a refresher. If any of this is not clear feel free to interrupt me, feel free to ask. So when are the vectors V1 to Vk called linearly dependent? Other way to say it is at least one vector I can write it in terms of other vectors. Other way to say it is if the coefficient of V1 is nonzero then I can forget about V1 when I am trying to create the generator of the vector space.




And if the vectors are not linearly dependent they are linearly independent and a nice way to put that is. Here I should say there exist an I. So this is the set of linearly independent vectors. And now if you think about it we had the vector space. This is vector space V if I take all elements of it this obviously will generate all the elements in the vector space.

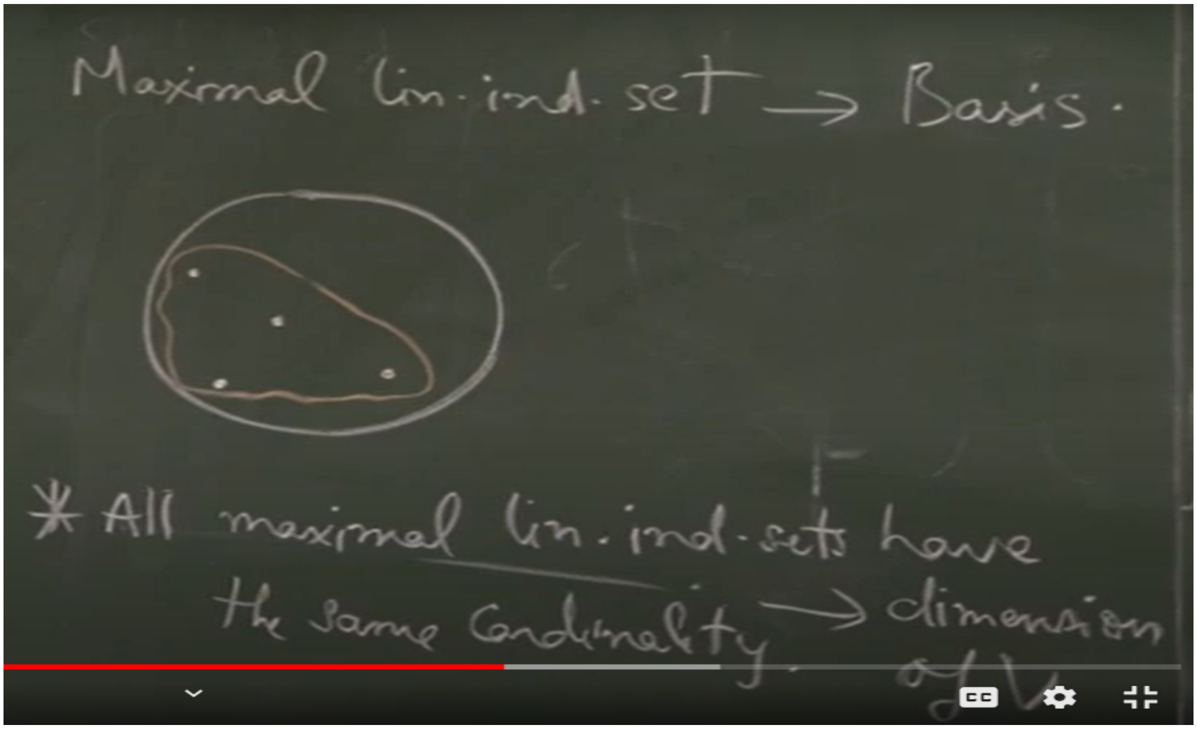
Is the idea of generation clear? Generation means you apply some coefficients add them up and get another vector. And now but with very high probability they will be linearly dependent. But now let us say I start with one vector. Can one vector be linearly dependent? If it is not zero cannot be linearly dependent. Right, So then this is first set and the idea was again you take another set.

See if it is linearly dependent or not. If it is linearly independent you increase your size. Probably you can get every vector in the vector space through these two. Is there a question? So if you can get everything that is great. Suppose not then there is some element which you cannot get. By definition it is going to be linearly independent because I cannot write this element as a combination of these two.




So then these three and then at some point I will end up at a maximal (very good) maximal linearly independent set. ok And this maximal linearly independent set is called Basis. but clearly this is not unique. Right I can draw the same picture. These might not look the same but they are same according to my drawing. And then you can start with this vector instead of vector here.

Then you will find some vector here, then some vector here and then you realize that everything is a linear combination and this is the basis. This is another maximal linearly independent set. And what is a very nice theorem which makes our life easy in linear algebra? Yes, so that means all these maximal linearly independent sets are going to have same size. This is called the dimension of the vector space. You just said it.

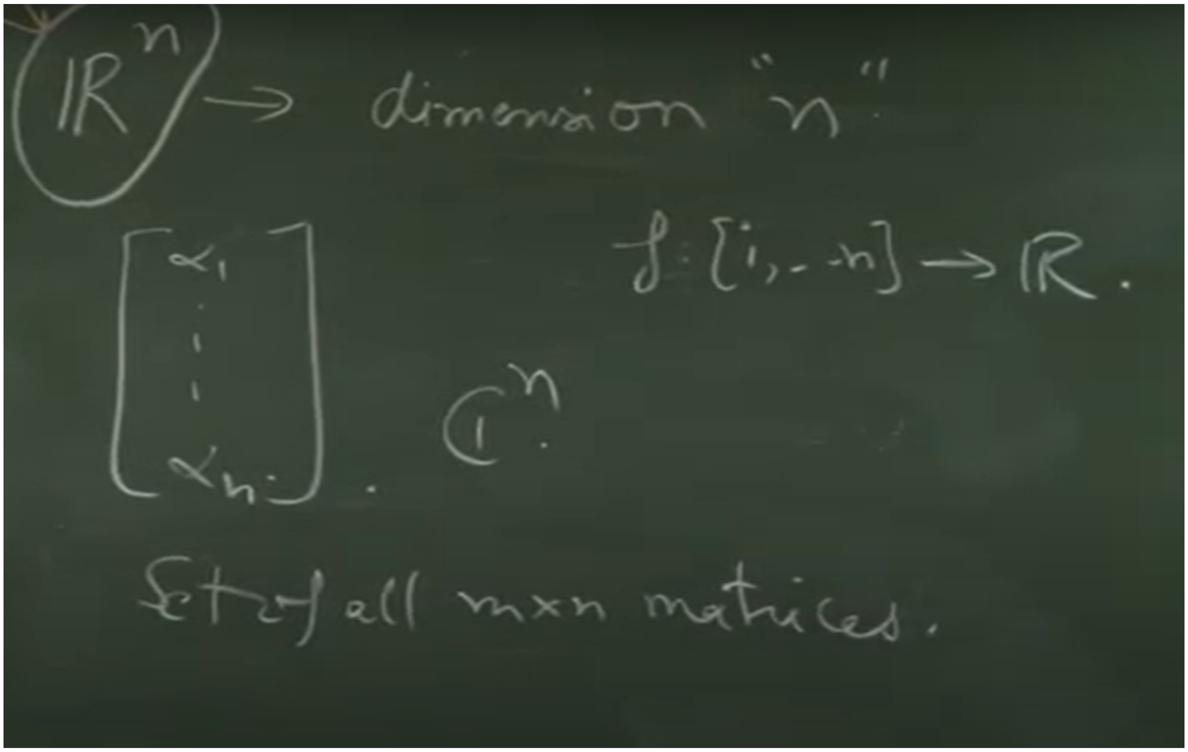



Correct. So this should surprise you. This is not at all evident from the process we were doing. Whatever information we had, it is not clear. Probably just because of the first we were very lucky. The first vector was so nice that this time only three will suffice.

No, it does not happen. Every time we need four. Everyone believes this statement. Right, for every vector space we have a single dimension. How many people can prove it? Interesting. Take a look at it but for linear algebra whenever you are stuck the best answer is Gaussian elimination.

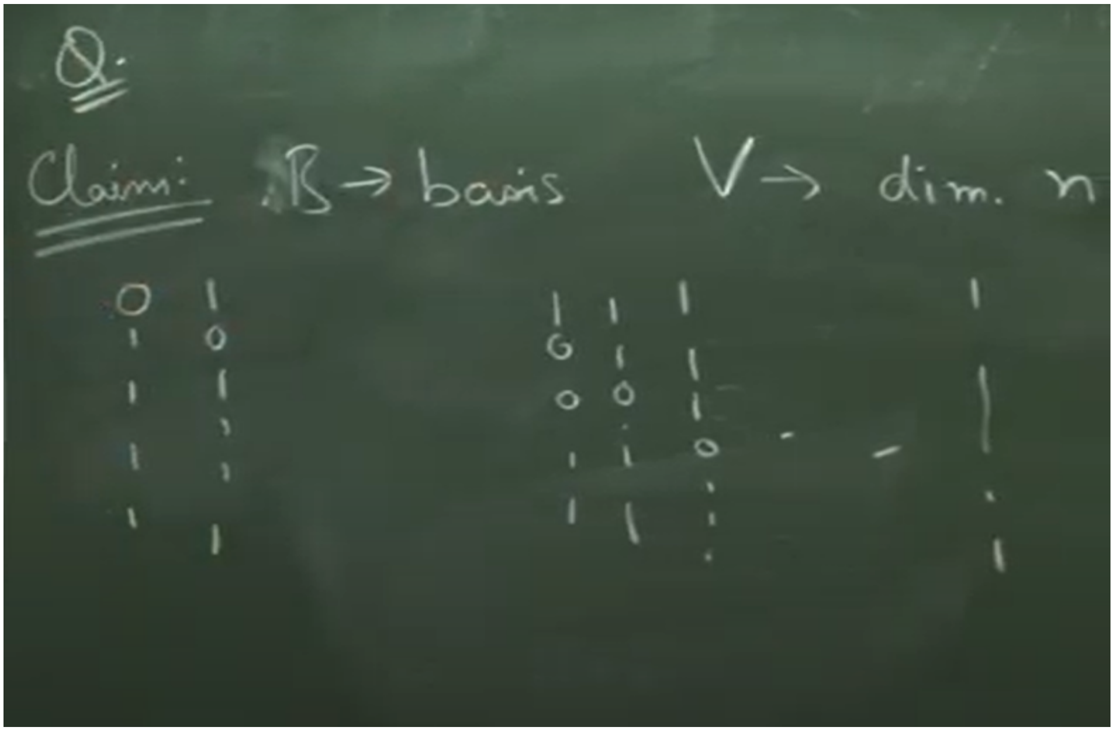
Right? In life the best answer is zero. In linear algebra the best answer is Gaussian elimination. Ok It turns out most of the things in the beginning you can prove by applying Gaussian elimination. But yeah, it is not hard to prove but you have to get a start somewhere. You can look at it but you should definitely know that is one of the fundamental theorems.

This is very important. The kind of vector spaces we look at R to the n, this is the real vector space with dimension n. And if you are familiar with it from your twelfth class this is basically all the vectors of this kind which have n real numbers. And this is going to be the most interesting vector space for us.. There are various ways to describe it. You can also say this is the set of all functions from 1 to n to R.




So alpha 1 is the value at 1, alpha 2 is the value at 2, alpha n is the value at n. Also we have different kind of vector spaces C to the n set of all. So there are lot of vector spaces but for us this is the important vector space. You should understand this. This notation might be new but I am sure you have seen this vector lot of times.

The name for it is going to be R to the n and why R to the n? Because our coefficients, our numbers are coming from real and the dimension is n. Let us see if a linear algebra is ok with you. This is the claim. Let us say I have a basis B for a vector space which has dimension n. Clear any basis which you know of for general vector spaces for the vector space like this? Why 3D in general for even n dimension? 1 0 0 0 1 0 0 right standard basis.

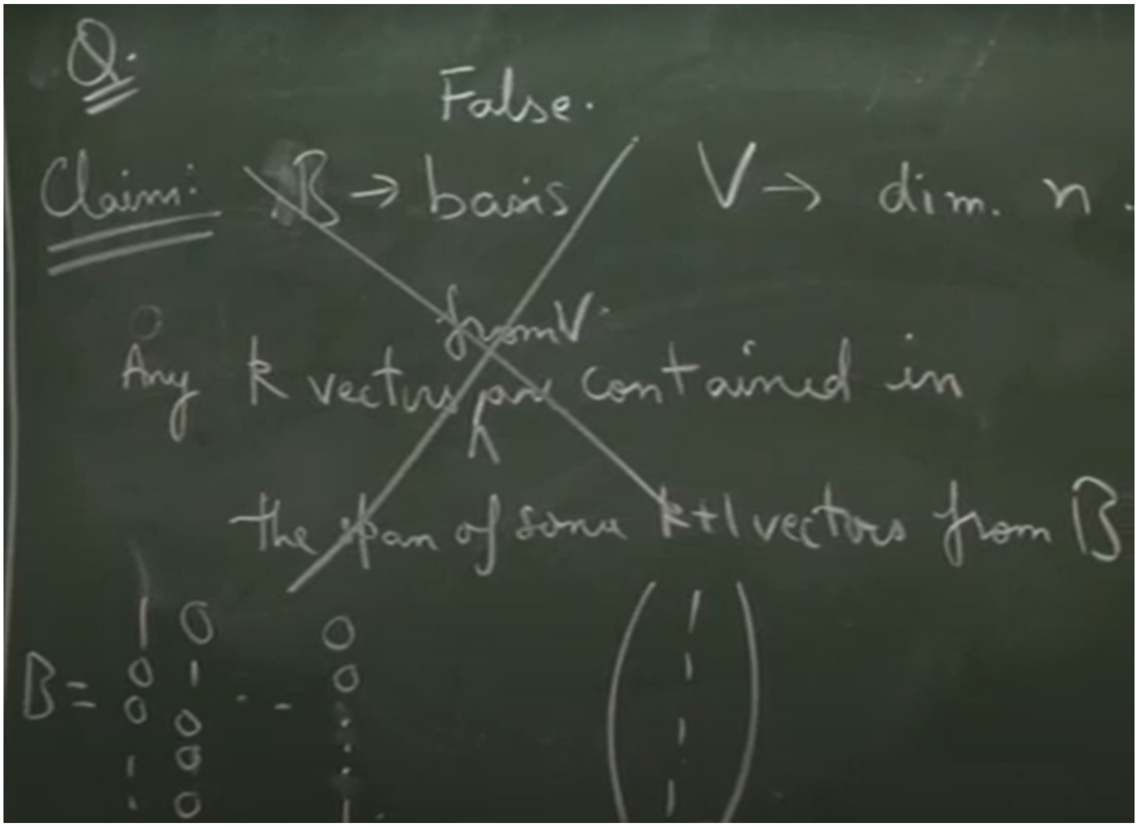



Anything else? n dimension I do not want 3 dimension. Can't you create another basis? Sorry, no don't cheat. You cannot use the same vectors with some scalar multiplication. You can do 0 1 1 1 1 0 1 1 1ok but this is probably harder to see as a basis but I would say that a better vector space or this is a basis and actually any nice triangular form will give you a basis. Right ok great

Now we know lot of basis. Let me ask a more mathematical question. We are given a basis for a vector space of dimension n and someone claims that any k vectors are contained in the span of some k plus 1 vectors from B. So, B is a fixed basis, V is a vector space. The claim is if you pick any k vectors from V, you can always find some k plus 1.

There are lot of k plus 1 choices and choose k plus 1. So, you can find some k plus 1 vector such that all the k vectors can be represented in terms of these k plus 1 vector. Right Is the term span clear? Span means all possible linear combinations of these k plus 1 vectors. We take the vectors and write alpha 1, V 1, alpha 2, V 2. Is this claim correct or not correct? V is some fixed basis. You know how this problem came about? In my first course in IIT Kanpur, someone proved it using this claim.

Do not stare at it, work. That is why you should always have a pen and paper. I do not know you find out. If you find a counter example and you can show me the counter example in your. Write the counter example concretely and then I can look at it.

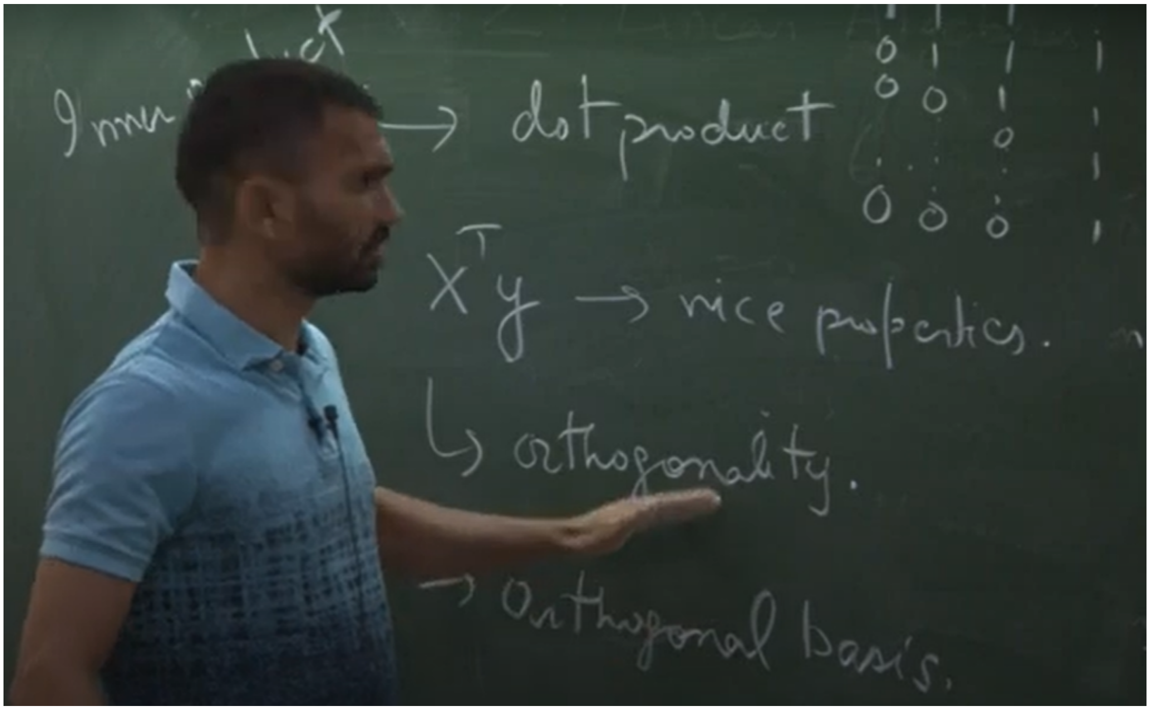



Time to vote. How many people in favor of this claim? Raise your hand. You agree? You agree right? By intuition right? You agree. How many people do not agree? To answer your question, I failed the student who wrote this. I did not fail him but he did not get the marks for this.

Let us say we have our favorite standard basis. This is the vector. This is not even there in any n minus 1 collections. So my k is equal to 1 here. I take this vector that is not even there in any minus 1 things of the basis. I am giving you this example because people get confused about this because you know we can go from any basis to any basis. Generally, the dimensions are the same but dimensions do not mean that you know this basis say you can always select some vectors which capture the entire thing.

What is? So this is my standard basis. This is b. Oh my dash dash and my 1 seems the same. That is my yeah. One thing you can be sure of is that my drawing skills are pathetic but yeah. So this is the standard basis and if you take this vector, it cannot be written down as a linear combination of even n minus 1 of these.

So this is false. Okay? Good. Another interesting thing which is sometimes useful is the inner product space and I am just saying it because r to the n is also an inner product space. What does it mean by inner product space? What is an inner product? Yeah. Do not worry about the mathematical definition but this is going to be the dot product for this. Everyone knows what a dot product is? Yeah. So basically it is dot product for us and it is basically this operation which has nice properties and if you are interested in finding out the nice properties, look at Wikipedia or any place which is the definition of inner product.




But what this inner product does is allows us to talk about orthogonality. It allows to measure angles between different vectors and then we can talk about orthogonal basis. So the basis which I wrote, this was a basis but this was not n. This was clear to everyone. So I will not go into detail because this is not something we require a lot but we know that r to the n has lot of structure.

It is not just a simple vector space but it has an inner product defined on it. Sounds good? Okay. Let us move on to more interesting stuff.