Linear Programming and its Applications to Computer Science
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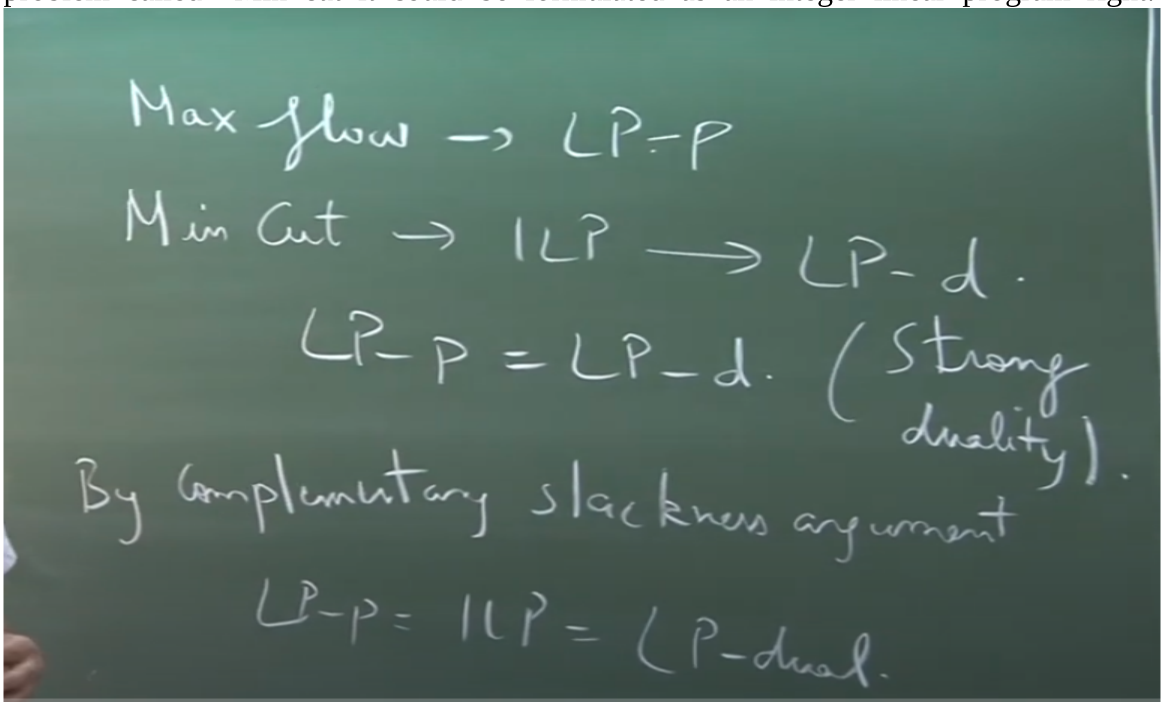
Lecture–42
Primal Dual Approach

Welcome to another lecture on linear programming. Last time we were looking at the Max flow - Min cut duality. What I mean by that is we saw these two problems Max flow as well as Min cut. Realized that Max flow was a Max flow was a what kind of program it was a linear program directly right. And then we saw very related, but a very distinct problem called Min cut it could be formulated as an integer linear program right.



So, this was a linear program where we had to put in con specific constraint that a particular variable is going to be integer then it becomes an integer linear program. Now this integer linear programs or integer programs is a different beast and I would say difficult beast in the sense that once you formulate your programming as a linear program you should start dancing with joy already right.

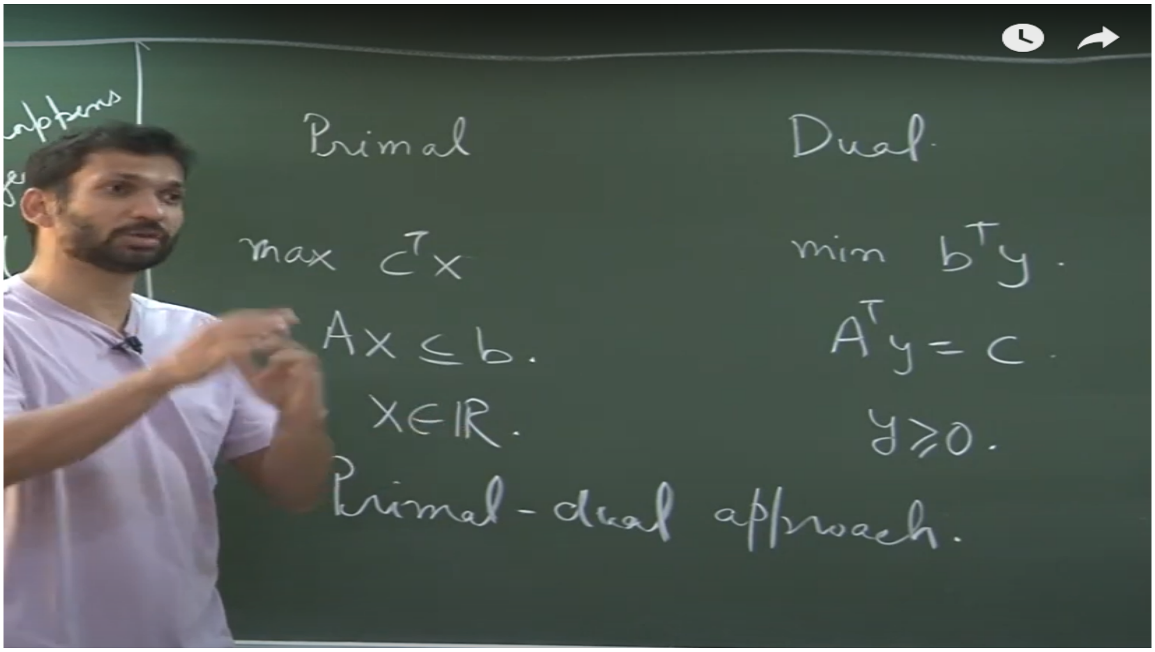
It is a linear program you know a lot about it you write it as ILP you are like what is the big deal about it right. The reason last time I emphasized there are problems which are very, very difficult and can be formulated as an integer linear program. So, now what to do with this? This does not help mostly the strategy is you take this integer linear program and convert it into a linear program. What do I mean by convert my problem does not remain exactly the same it is a different problem it is a relaxation of my original linear program.

So, I am not saying. So, this is equality this is equality this I do not know and the entire game to be played is what can I say about the relation between this ILP and this LP right yes. For the case of Max flow and Min cut problem this turns out to be great. What I am doing by great by using some argument which relied on complementary slackness it was not simply complementary slackness because of this nature of this problem and properties of complementary slackness we can actually prove that these two. So, this follows by linear programming, but actually all three are equal and if you remember what we did was we took a solution here using that solution we are able to generate a cut of the same value.

So, the optimal solution here produces a solution here which had the same value correct this is what we were able to achieve. So, now this proves that these three are equal in effect for Max flow problem this is equality right. And what will be dealing with in future is when we relax this integer programs what can we say about it and what happens in general is it equality is it not equality is it a good approximation this is depends on the situation this depends on the context and we will see multiple such contexts. So, this relaxation rounding technique is has been a success in last twenty years in many many cases there are many many papers which do this technique. And I hope by the end of two three weeks you would know what I mean what is this relaxation rounding.

For today we take a detour, for today what we are interested in is even if you have seen this complementary slackness condition you might say that oh this should somehow help me in finding a solution it is giving me, telling me what conditions to put to find my optimal solution. So, can I use these to find my solution easily what we are going to do today is learn this generic approach to solve linear programming. First important thing is this is an approach this is not an algorithm this is not like simplex algorithm I will outline the approach and then in some cases this will bear fruit in some cases it would that is why I am calling it an approach not an algorithm like simplex or ellipsoid or interior point. So, in some cases this helps us in some cases it does not one of the cases where it helps us is Max flow - Min cut. So, plan for today is understand this approach this is very nice intuitive approach once we understand the approach then actually ask can we apply to one case.

And we will see that when we apply to this you obtain what we call Ford-Fulkerson algorithm you might have heard of Ford-Fulkerson or Edmonds-Karp or whatever there are multiple names variations. But that is what and this is like augmenting paths if you have heard of these terms you would have learn this in algorithm course. Cool the idea is clear what we are going to do today let us start. And since I want to torture you first question is to tell me what is the duel? So, sorry go ahead senior people who are very smart can also figure out what is the duel or if you want to tell me directly please go ahead I would be happy.



So, Minimize , good. So, now, you are getting a hang of these things right. So, I have to make things more complicated I will. So, this kind of duals now you can very nicely take you do not have to convert into standard form and all that you realize why is the variable here I have to put a constraint because of the variable x here. It is this constraint and this comes up because of less than equal to right good.

And now the primal dual approach starts with an optimal solution not an optimal solution obviously otherwise nothing to do some solution of primal right. And this we will assume we will say assume b is greater than 0 for this discussion. So, that you already have a solution of primal. So, now, once we have a solution of primal what we know is that. So, again idea is can we improve x right, but now using complementary slackness condition what we know is that a dual solution will only be positive or non zero if this condition is loose right.

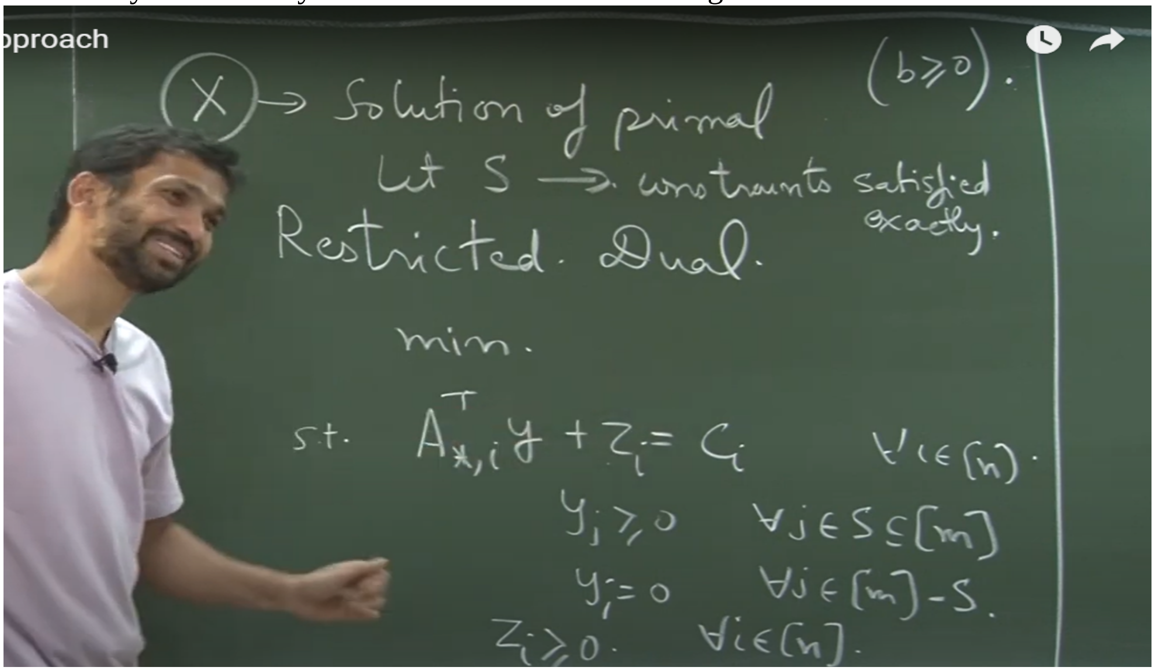
So, x is not optimal, but if x was optimal it would have satisfied complementary slackness right. If it had satisfied complementary slackness then the corresponding dual there is no corresponding dual as such, but let us assume again this is on heuristic intuitive picture. The y would only have non zero coordinates where this constraint is not tight correct. So, then we are not going to find a dual solution what we are going to solve is some will be some y which is almost a dual solution. It is not going to be a dual solution, ok.

So, we are going to write the restricted dual. So, the idea is and this restricted dual will not be used in the algorithm at all. I am going to and even the formal proof I do not have to talk about restricted dual it is just for the idea. So, what is going to happen is I realize that if there was a y which was optimal then they would have satisfied complementary slackness then my y would have been of some form that will give us restricted dual and whenever we have a linear program what do we do with it, take it is dual again. So, now we will take the restricted primal which is obviously not going to be this the restricted primal will allow us to improve our solution x, ok.

So, let us see let me write out restricted dual you will see the idea. So, let S be the set of indices such that. So, let S be the set where do I want it to be tight or do I. So, constraint in primal constraints satisfied exactly right. So, these are the ones where A x is actually equal to b.

So, I will assume m constraints and n variables. So, then my restricted dual. So, this y again is not going to be exact is not going to be feasible solution because we are trying to artificially create let it satisfy complementary slackness. So, this is the slack in that and this is for all constraints, but the actual thing which is interesting is that my y j is only equal to zero for all j in S remember S is a subset of m for everything else y j is equal to zero exactly. So, when this is tight the corresponding variable could be lose remember that their multiplication is equal to zero.

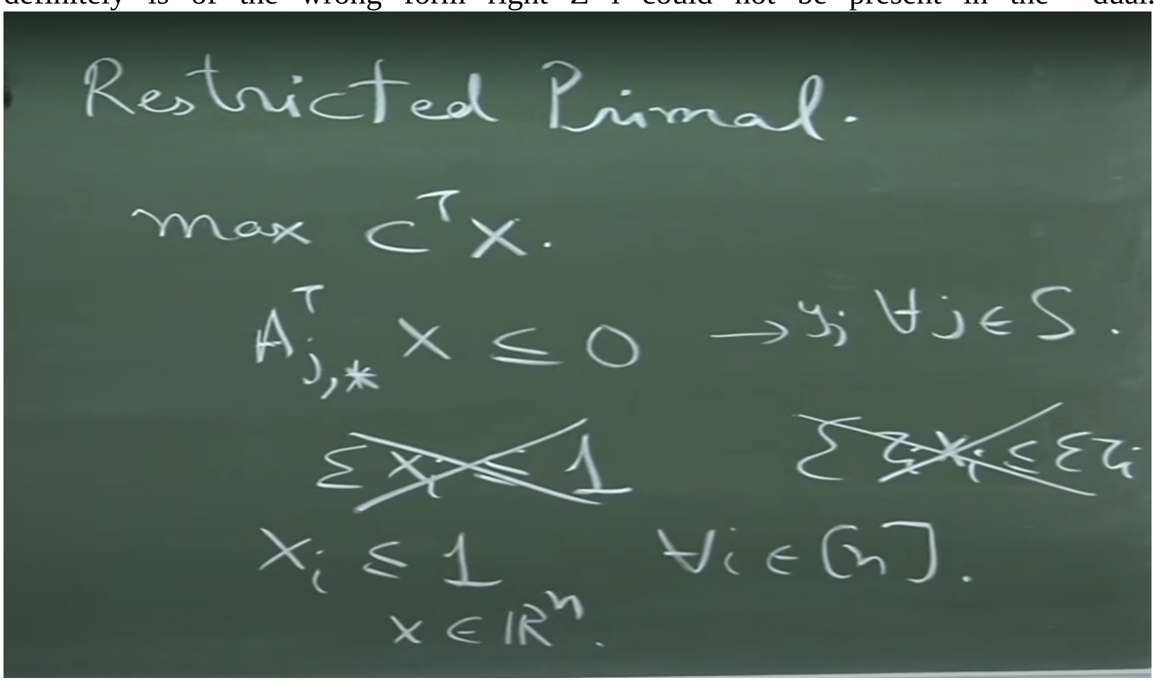
Intuitively what happens is this constraint time the variable is equal to 0. So, if this is tight you have already got a zero then other thing could be anything if this is not 0 then you know that the corresponding variable has to be zero. So, this is kind of our solution and let us to make our objective function easy we will say that our error is just on one side. So, this is just trying to artificially force complementary slackness conditions, but then yes. Why are we assuming that I did not.



Yeah I mean to answer. So, because again this is all heuristic it turns out that when we take the restricted primal it helps us in this case. Now, since Z i is greater than equal to 0 there is a nice quantity which I should optimize here what do you think is that quantity. If I had not put Z i greater than equal to 0 then I will have not nothing to write here right. So, restricted dual in some sense is saying let us find the best possible complementary or the dual solution which has S less an error which is close to feasible.

This close to feasible is captured by the Minimization and again I am not saying that this is the LP to solve what I am saying is intuitively it makes sense in some cases it helps. So, there is no formal proof here it is not like this is the only way to do it there might be different ways this is one such way right. And now will you tell me the dual? Now probably you might need a page and paper pen and paper. So, this is not really a constraint right you can remove this constraint and you can say that what you will remember is you have only cardinality of S many variables right. So, that means there is only just one constraint sorry one set of constraints there are N these constraints and I have cardinality of S many variables cardinality of S plus 1 Z wait what am I doing S plus cardinality of S plus M many variables N many variables right N of these at cardinality of S many yi's.

So, that means the variable here is x . This is the constraint corresponding to this is the constraint corresponding to yj. So, how many of these constraints will be there? Modest right. So, this is for all j element of S correct what is the constraint for Z i that is easy actually remember what we used to do to take the dual. One suggestion was how many people agree. Oh you are saying another option is Z i X i less than. Summation Z i this definitely is of the wrong form right Z i could not be present in the dual.

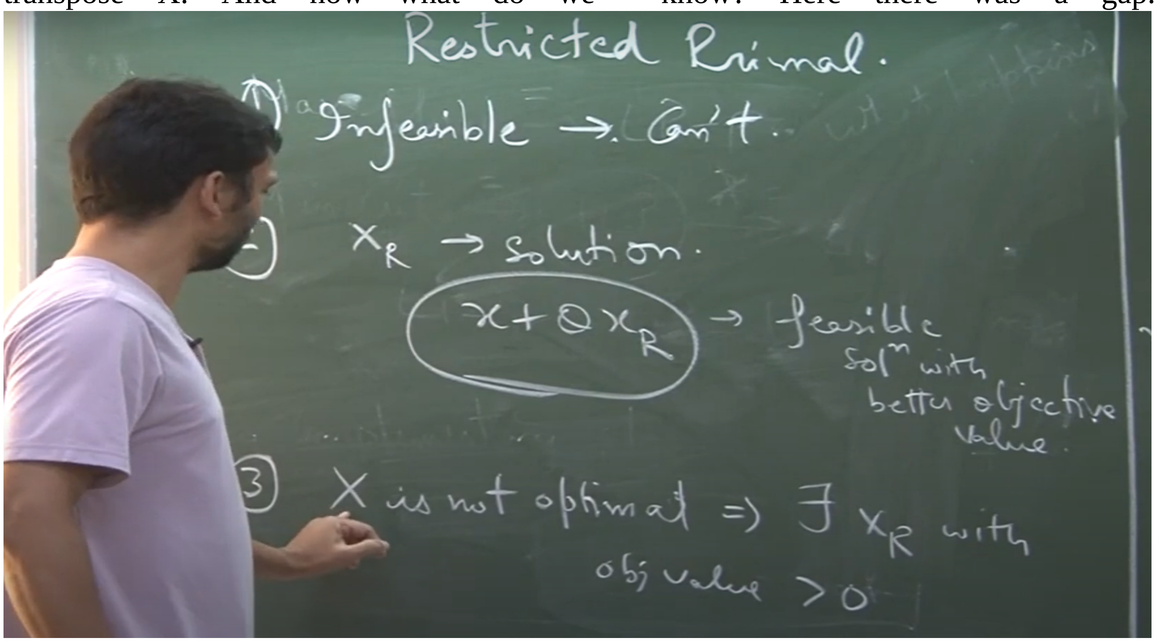


So, you are forgetting how we how we take the dual. So, this is wrong that is why a pen and paper sometimes help this is also wrong if you are not convinced which variable does this correspond to yes right there is no variable which this could correspond to right. So, what will happen is you will have constraint corresponding to Z i. So, the constraint is X i less than equal to 1 for all i and X i greater than equal to 0 you know because equality. So, sounds good this is my restricted dual, ok.

And now let us say we solve this problem. It could either be infeasible or if it is feasible it will have some solution right. So, one option is now once we have the restricted primal one option is it is infeasible why cannot it be infeasible. X equal to 0 satisfies it no it depends on the sign in this a j star no why cannot a j be negative a j star be negative. greater than equal to 0 right. So, correct yes it has a feasible solution great.

So, got it good now the two points which we will observe is suppose it has a solution let us say X R is a solution. Then what you can show is that X plus theta X R is going to be a feasible solution with better objective value sorry. X was your solution of the primal this is what you are started with right this is what you wanted to improve. And what this shows is that this the solution of this if you have a solution of this this will allow you to improve. Actually if I want to be completely correct remember this right what do we know about this dual what is it is can you give a lower bound on its value.

Yes object there is only one value 0 right it cannot be if it is 0 what does it mean? What does it mean by perfect? Exactly no not primal X is optimal right that is what you want to say correct. So, that is awesome if the value here is 0 it is designed in such a way then this is optimal. That means if this is 0 then X is optimal if this is greater than 0 then and then I have given this as an small exercise, but there is nothing much to check we have a C transpose X which is positive. So, if I do this this will obviously be it will increase C transpose X. And now what do we know? Here there was a gap.



So, now I can use theta to multiply it and have some and this something which I think in mid sem also there was a question. So, you kind of increase theta so much in the simplex method we also do the same thing we just increase theta so much so that no constraint is violated. So, this tricks are old, but again I am giving you the intuition that this will be an improvement. How much of improvement all that I am not specifying again because this is a heuristic. In some cases this will be a great algorithm and sometimes it is not, ok.

And another thing we you can show is that if X is not optimal implies there exist an X R with objective value strictly greater than 0. And that is not hard what you can show is that if take the optimal solution you look at the difference between the optimal solution and X and just scale in that direction that is going to be your X R. So, I think I have gotten one direction wrong what will happen is I want to make sure that so probably this would be greater than equal to 0 or something it should be the opposing direction. So, this there has to be an opposing direction. So, I do not remember which way it is probably it is greater than, but what will happen is that this theta will stop like as you increase theta this constraint will be violated if your theta is very large right.

And that is why you have specifically looked at only those j's where there was a gap right. So, I think I think this has to be the correct one I will correct my notes and you can probably verify. The point Pratish was raising was if I write the solution like this and if my restricted primal was like that even if I take theta is equal to infinity it will still remain feasible. Because you have the only this constraint and as you add more and more X this is becoming more and more negative. So, if it was satisfying AX less than equal to B before it will keep satisfying AX less than equal to B before.

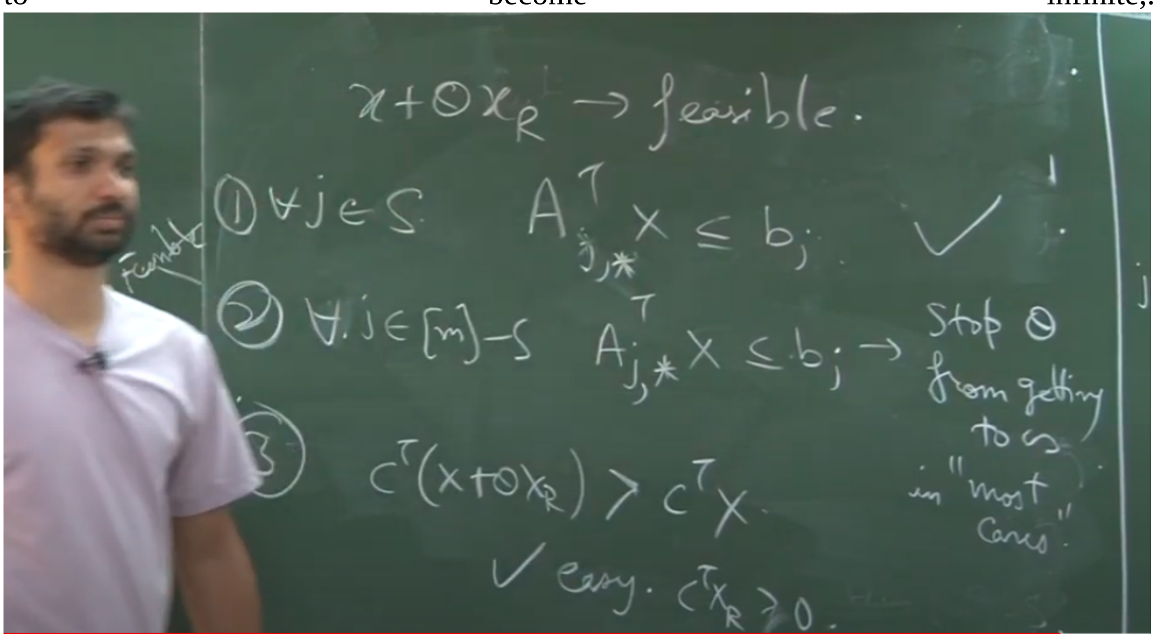
And then on the top of that if theta is infinity then this objective value will become infinity. So, clearly it is thank you. So, good that is why you should know how to take the dual even if your instructor does not right. Yeah, so, if you think about it because this is going from Minimization to Maximization this has to be greater than equal to 0. So, that means see again remember the big picture the exact reason why you had chosen these constraints were.

So, that there is a gap between those constraints and 0. So, you have a gap you have some freedom to move around. And even though this is greater than equal to 0 since you have freedom to move there is some small theta I can use. So, that this is still feasible make sense. So, the entire big picture remains exactly the same you are using this and if you have the correct sign here it will stop you from making theta infinity.

And this will make sure that you always have a direction to improve, correct. Now, at this point I have not given you any guarantee about how much this improvement will be it might happen that first time my improvement is half then 1 by 4 then 1 by 8 and so on and so forth. And you know I might not even get closer to the optimal right. So, all those guarantees are not there and I am not going to give you those guarantees because this is a approach this is this is an approach this is not a algorithm. In simplex algorithms you are moving from vertex to vertex right, but here as you move about I would assume that whatever was tight before will remain tight and you will get more and more tight constraints or something I do not know, but yeah actually yes yeah.

Now, I am thinking probably. So, I think the definition of S. I think we should write the dual exactly first tell me what is the dual is it greater than equal to 0 or less than equal to 0. No, no, no, no see the remember these are only some of the constraints it no, no, no see because remember these are not the entire constraints right there is only S many constraints what about the other constraints. So, something here was saying A X was Minus 2 right, now when you do X plus theta X R what happen to those constraints well that constraints still remain satisfied we have no, no you are confusing between constraints and variables the constraints will remain I am constructing a new feasible solution for the primal right X R the variables are less constraints are the same. We have mod S many variables X corresponds to no we have n many variables in the restricted primal we have n many variables right because there are n many constraints here and again you are confusing between the number of variables we have restricted are in the dual not in the primal right. So, once again tell me I think it should be less than equal to zero.

So, again the small thing was what will stop theta from going to infinity. Something has to and there is what I am telling you remember there are some constraints here which are exactly satisfied and for some constraints there is a gap in the restricted primal with X we have no guarantee over those constraints which had a gap. Here for the constraints for the constraints in S we have a guarantee that this thing is less than equal to 0 for the constraints in M Minus S we have no guarantee about X what I am going I am trying to say is if j is in M Minus S we have no control over this quantity agreed because we have only constraints here, but what we know for these j's is we have a gap for if for all of them if this thing was negative then probably yes the actual value of the primal was infinity, but my point is what will happen in most of the cases is that one of these things will actually be positive with X R one of these things will be positive. And now when you take theta as you increase theta initially for that particular j you had value Minus 2 then it will become Minus 1 and then 0 will stop it because this has to satisfy all M many constraints here. Once again the overall picture is the same which is that this is going to be our new solution the only confusion was there should be something which stops theta to become infinite,.



So, now I am saying this is a feasible solution. So, what do I need to check? This is satisfied. So, let us do it since now the role of state dual is over sorry X i that was just like artifact of dual right. So, when you want to prove that is why probably I gave it as an exercise I want to show that this is a feasible solution of this L P correct. So, then what I am saying is I have three cases j is element of S, a a j star transpose X should be less than equal to b j. And I want that C transpose X correct if I want X plus theta X R to be a better solution it should be feasible which is conditions 1 and 2.

And then it should give me a better objective value which is condition 3 this is fine right if I want to show X plus theta X R is a better solution here I want to satisfy these three things. Now this is feasible this is objective value. And now what do I know about X R. I know about X R that this is less than equal to 0 for just in S. So, in S what did we know about X this condition was equal to 0, X R will only make it negative this is fine everything is linear that is great right. So, with X this was actually equal to b j with adding X R this negative.

So, we are fine in this case. This is the constraint which will stop from making theta infinite because here we know that there is a gap this is not exactly equal to b j. But as you start increasing theta this might get violated right. So, these constraints so you know whatever is the gap divided by theta in some sense the Minimum over that that would be the theta you will pick. So, this in most cases when it does not stop it your primal probably had not probably your primal had the objective value to be infinity. So, it is and this is very easy that is just because C transpose X R is also positive.

This C equal to 0 it will not happen that means you are already at the optimal we will stop. So, this is actually you are getting some increase right. So, probably this makes things more clearer because we made a mistake then corrected it ourselves. So, sorry this positive again if this is 0 then I am at the optimal right. We remember the restricted dual was only optimal when the value was 0 right.

So, then we have nothing to do ok good. So, this was the overview now let us apply it ok. So, if you want to encapsulate our approach we had an original primal we had a feasible solution for it using complementary slackness we construct a restricted dual. Remember restricted dual is a function of the feasible solution you cannot say give me the restricted dual for Max flow that does not make sense. Give me the restricted dual for Max flow when my objective my feasible solution is X then only I decide this. So, this takes a feasible solution looks at complementary slackness and creates a restricted dual correct.

And now how do we go from this to this? Yes this is the standard duality which none of us understand and always get it wrong. So, that means lot of questions on taking duals right this gives us restricted primal. And why are we happy about restricted primal because there is this relation that the solution of restricted primal allows us to not update improve by improve the original solution. And I am writing this one to one in the sense that if this is not optimal there will be a feasible solution here if this is optimal the feasible solution will have value 0. This is the over view. Sorry, oh complexity and if you think about it what have you done you have converted solving LP into solving another LP.

So, this solution is only. So, it is a good question this is only helpful if somehow we have a better complex like optimized thing to solve restricted primal. Oh, you mean here it might not like it can be like this 1 plus half plus 1 by 4 if your strategy of solving restricted primal is very stupid then yes. So, it is the game is about whether this restricted primal can you get intelligent solution for that and that we will see in next flow. Then we have to show that that converges I am not that good in analysis.

So, I will leave it to you. Yeah, so, again this will only be helpful if somehow the restricted primal is easier to manage as compared to original primal and this will happen in the Max flow case.