Linear Programming and its Applications to Computer Science
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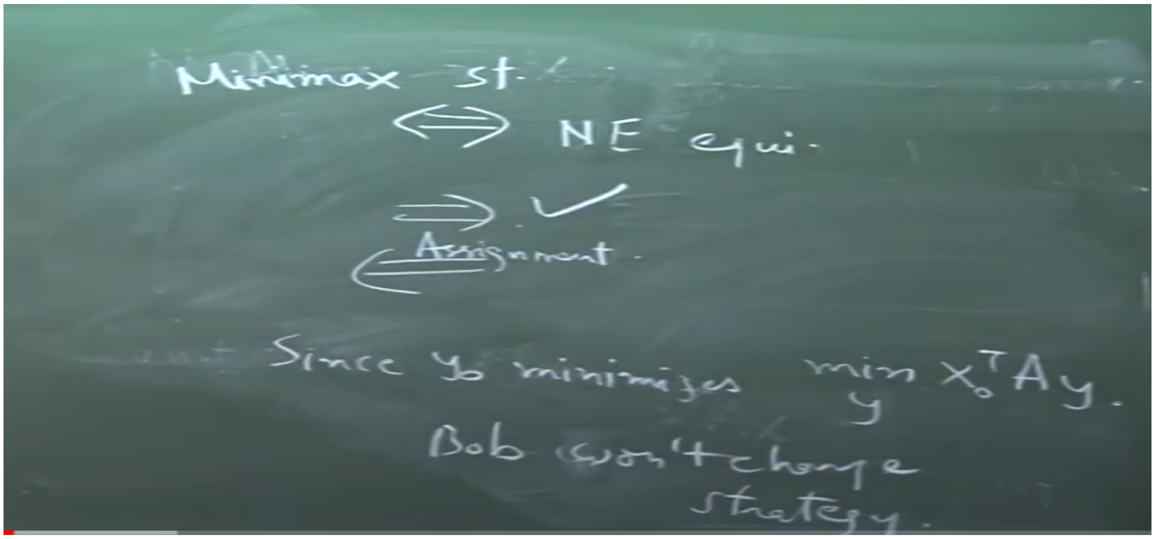
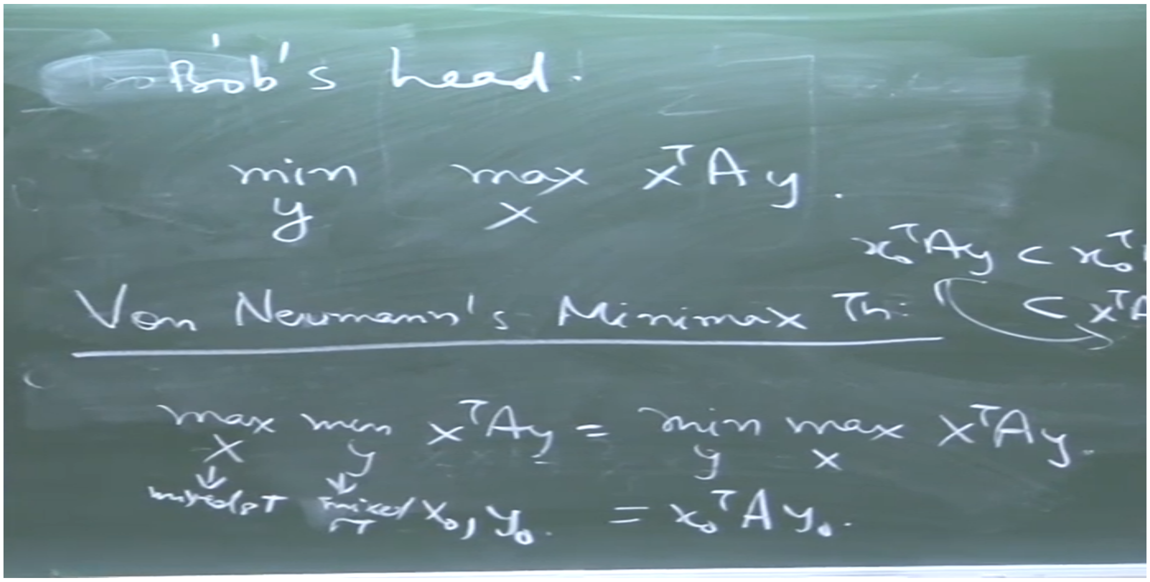
Lecture – 33
Minimax and Nash Equilibrium

We have two ways to come up with this strategy. One is this Nash equilibrium, other is Von Neumann strategies let us say. No surprise they are supposed to be Nash equilibrium is a Minimax solution. A Minimax solution is a Nash equilibrium. So this direction we will do. Sounds good not difficult you already see kind of equivalence and even this direction we have kind of already done.

This is the proof right because Y naught is the one minimizing this. If my strategy is fixed at X naught there is no incentive to change Y for more right. The entire proof was this exactly it is saying that Y naught minimizes this quantity right. So let me just write it since Y naught minimizes notice this is much simpler now right.

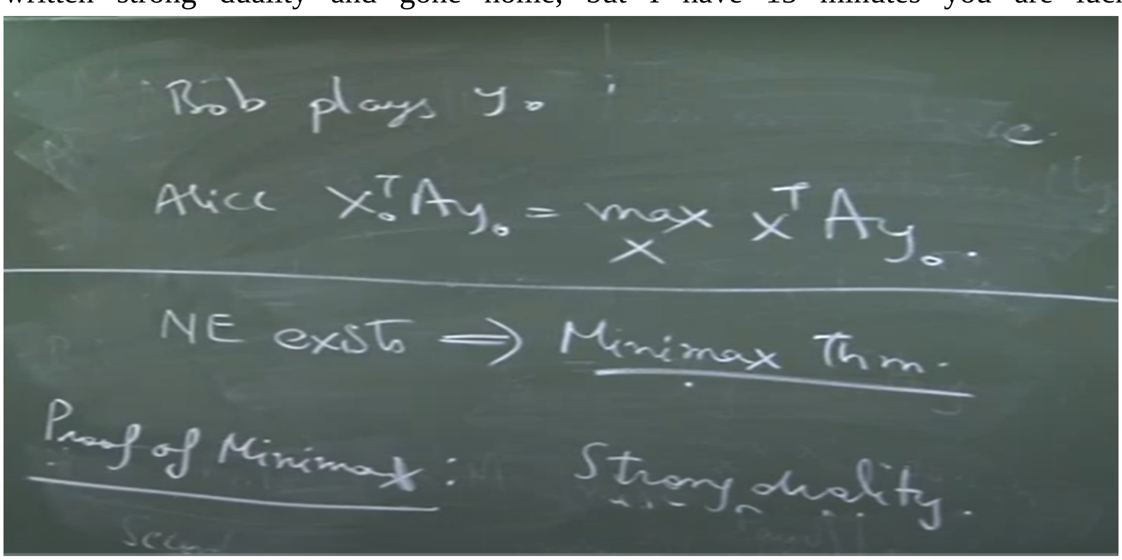


This is X naught is fixed now the only variable is Y right. So if LSS strategy is fixed I am trying to find out what minimizes this and I figured out it has to be Y naught. This is Bob change strategy. Bob is already at the best possible strategy for X naught right and the same thing you can do for LS. You fix Y naught Bob's strategy is fixed there is no point for LS to change X because she is already getting the maximum payoff.



She is getting X naught transpose A Y naught which is. So, if Bob plays Y naught LS gets X naught transpose A Y naught which is right and again same reason X naught maximizes this thing when I fix Y naught. So that means, why would LS change X naught to something else she is already at the maximum payoff. No, but I will only change if I get a better payoff equal payoff there is no point to change. Good so any exists is equivalent to now showing the minimax theorem.

Do you see this now? If von Neumann's minimax theorem is true this minmax is equal to max min that means, there exist X naught and Y naught. So, if the payoff is X naught transpose A Y naught and we know X naught Y naught is a Nash equilibrium. So, if I show von Neumann's minimax theorem then I have shown existence of Nash equilibrium. So, this is our task now to prove minimax theorem and if the time was 3.15 I would have written strong duality and gone home, but I have 15 minutes you are lucky.



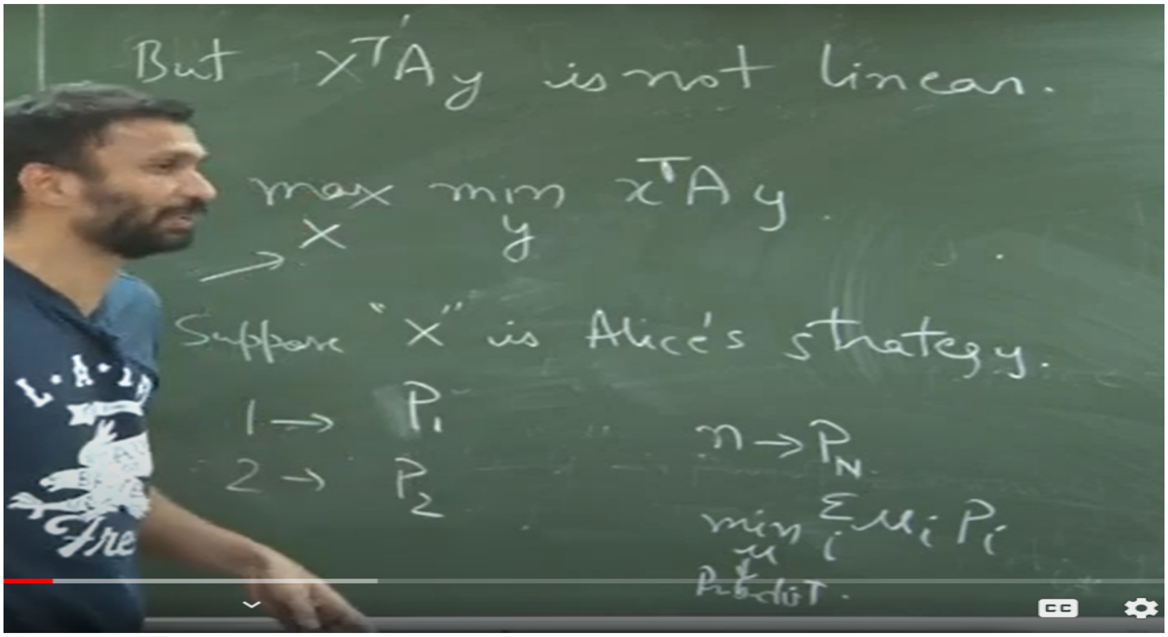
So, I will say more yes this is going to be a strong duality that means, this is going to be a primal problem this is going to be the dual of it. This is the idea if someone tells you the idea what is going to be your objection. Why if someone gives you this much idea why should you not be happy first thing. If you are happy then we can go why should you be not happy if someone just tells you oh this is strong duality this is an LP the dual of it is this an LP what is the first thing which comes out sticks out. This is not linear what are the variables here X and Y right this is a i j X i Y j this is not linear.

So, the idea is great, but is not linear clear. So, first thing is to convert into a linear program. So, the proof of von Neumann's minimax theorem the proof is basically how to view this as a linear program this is an exercise in converting this problem into a linear program. Let us convert we have another option this is what Alice wants to do right. Now let us say Alice's strategy is fixed.

So, suppose X is Alice's strategy right and suppose Bob knows about X right how will you calculate Y it is much more simple than that. What about linear program how will you if you if you knew X. So, you are not still you are thinking of as a linear program or whatever maximize this my minimize that I am thinking of like if you give it to your younger brother or sister right. What will be they will say oh X transpose what if I play strategy 1 what I what if I play strategy 2 right what is the right, but this is the start right I am saying this is the thought process what if I play 1 I will come to the mixed strategies later, but 1 2 right. So, 1 will have some payoff P 1 or P 1 2 will have some payoff P 2 how many studies are there for Bob n P n right 1 obvious thing was to O which has the.

Minimal smallest payoff. Minimal smallest payoff or, but now we are allowed for mixed strategies. So, I can say yes yes yes probability distribution I forget about linear programming for few minutes forget about linear programming. Summation P i. Let me tell you another problem right we are all motivated by crimes suppose you are a thief no suppose you are a thief unfortunately you are a lazy thief.

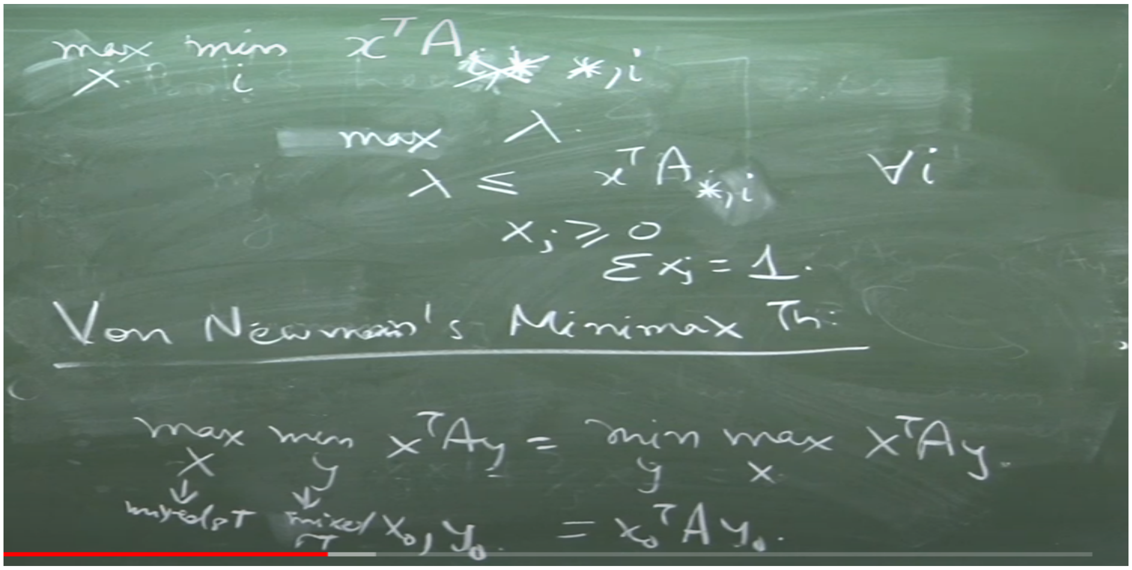
So, you only have a 1 kg bag you go to a house the house has silver gold bronze right because again as a thief if you go to a house you can take 1 kg of n combination of metals you have silver gold silver bronze what will you do you will put all the gold in your bag why would you put silver why do you take silver or bronze right assuming gold is more expensive than silver more expensive than bronze sometimes silver is more expensive I thought you knew that no? Swarnapadak, Rajat padak, Kasya padak.



So, what will you do you will take 1 kg of gold obviously right. So, now, here you can take P 1 with some P 2 with some probability P n with some probability want to maximize minimize take the minimum thing here and just put all your weight there you see it yes. Yes, what is the other way to say it? .

Yes, yes, the other way to say it this is minimum or a maximum for a linear program lies at a vertex. 1 comma 0 0 0 0 0 1 comma 0 0 0 what is this your feasible region it is this simplex of 1 0 0 0 0 0 1 0 0 0 0 you are optimizing a linear over it the maximum will lie at a vertex. So, I hope now you are convinced that this is true and this is the essential part once you see this then it is easy. So, now, let us convert it into a linear program I should not erase this now. So, rest of the proof in this much space.

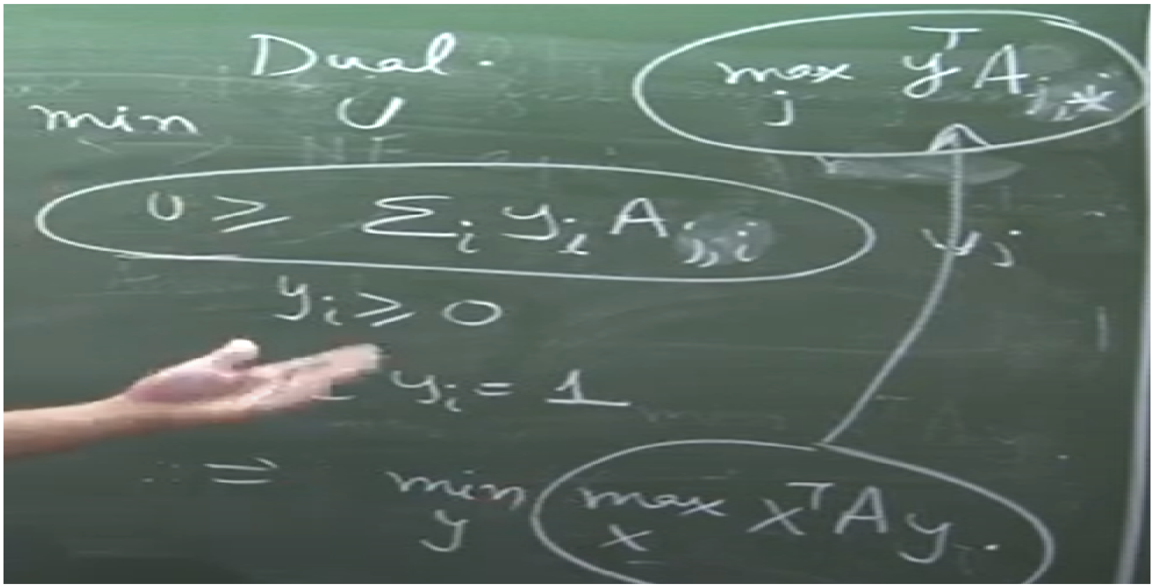
So, then whatever is the payoff rate the payoff would be a star i or a i star this is the column right it is the column of no row right how do I write it as a linear program now. So, let me just see this always is less than equal to greater than equal to sorry maximize the payoff such that for all i. I think this is star i in this case Bob's strategy is fixed sorry Alice has a Bob's yeah Bob's strategy is fixed right and that is that payoff is given by a column i is Bob's strategy right and second place that means the payoffs are given in the column it is a star i right. So, do you see that these programs are equal since I want to maximize lambda, lambda has no other constraint. So, it is going to be the minimum over all these quantities clear once more why are these two programs equal lambda if you want to maximize lambda and you have these constraints how will you put lambda you will take the minimum over all these quantities and put the maximum lambda is less than 2 lambda is less than 3 lambda is less than 4 maximum lambda what is lambda 2 right.



So, these two things are equal right because again this implies pick lambda to be min over i star i. Great now what to do I should know what to do yes whenever you see a linear program take the dual and once again now if you can take directly take the dual it will be helpful instead of converting into standard form and then taking the dual. So, u is the multiplier here y's are the multiplier here right what is the number of y's number of columns in a match right u. Now, if you sum up for lambda you will get summation y i equal to 1 the constraint corresponding to lambda remember lambda is a variable here right. So, for lambda variable when you look at the constraint it will say see there is no lambda here for you.

So, u is out the picture it says summation y i equal to 1 right you can match up everything this is less than equal to that means y has to be greater than equal to 0 this is equal that means u is unconstrained we have seen this multiple times right less than equal to we can check the sign by the meta inequality right equal means the corresponding constraint is this again is of the same form this is we have done a transpose y x a x right. So, this changes to this right do you want to take the dual yourself or you trust me if you have to take the dual by yourself and trust me then you will obviously trust me if the choice was just trust me or not then probably you do not trust me if I was taking the dual, but again whenever you have to take the dual remember write the variables create the constraints corresponding to each variable here again the constraint here does not give you the constraint there the summation y i is coming because the lambda the coefficient this is not just 1 equation these are n equations each of the equation has coefficient 1 that is why you are getting summation y i right stop matching constraint to constraint constraint here corresponds to a variable there and with that you can match up everything. And now you should be excited because what is this?

Right. So, i is varying right summation over i this is the j th right this is the this is the row right I have to bobs strategy is fixed L s is this is L s is responses to bobs j th thing right this is no I am probably saying the opposite way, but will work out right.



Again this I can replace by this you are right this is j star. So, this is j thank you right. So, this is equal to this because the thief is smart he or she is going to put all the gold in the bucket right there is no point having a mixed strategy when y is fixed. If you fix y the best response here is the deterministic strategy which maximizes this if there are multiple determinate strategies which maximize this any linear combination is also sounds good.

So, yes. So, for a 2 player 0 sum game this maximum is equal to minimax the optimal solutions give us the Nash equilibrium. So, this also tells us how to compute Nash equilibrium because this is a linear program we can apply our solutions to this.