Linear Programming and its Applications to Computer Science
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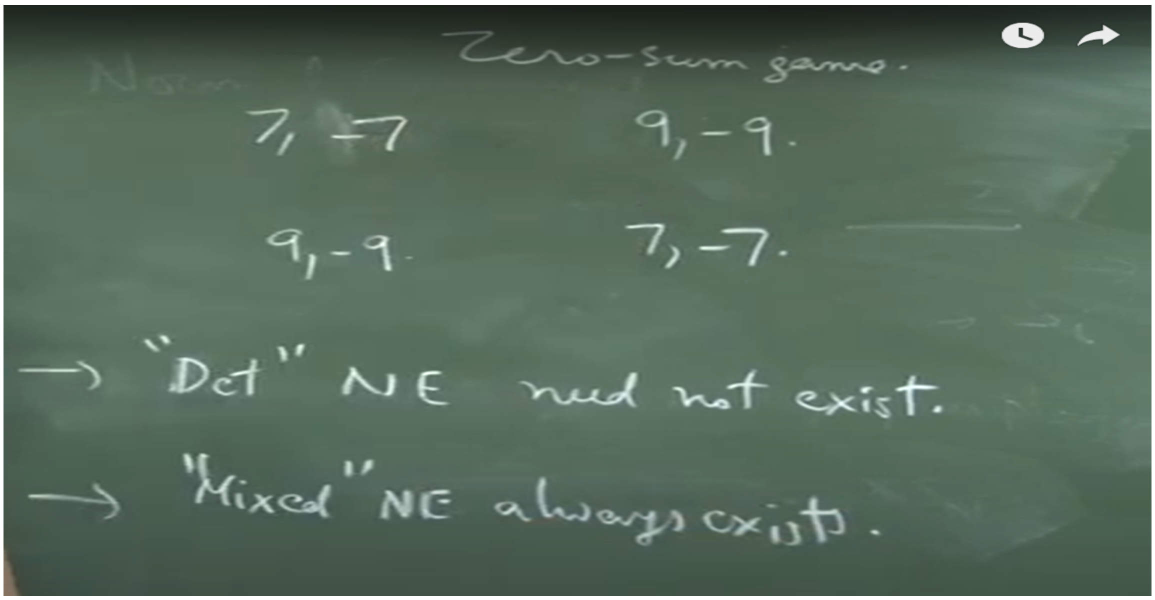
Lecture – 32
Nash Equilibrium

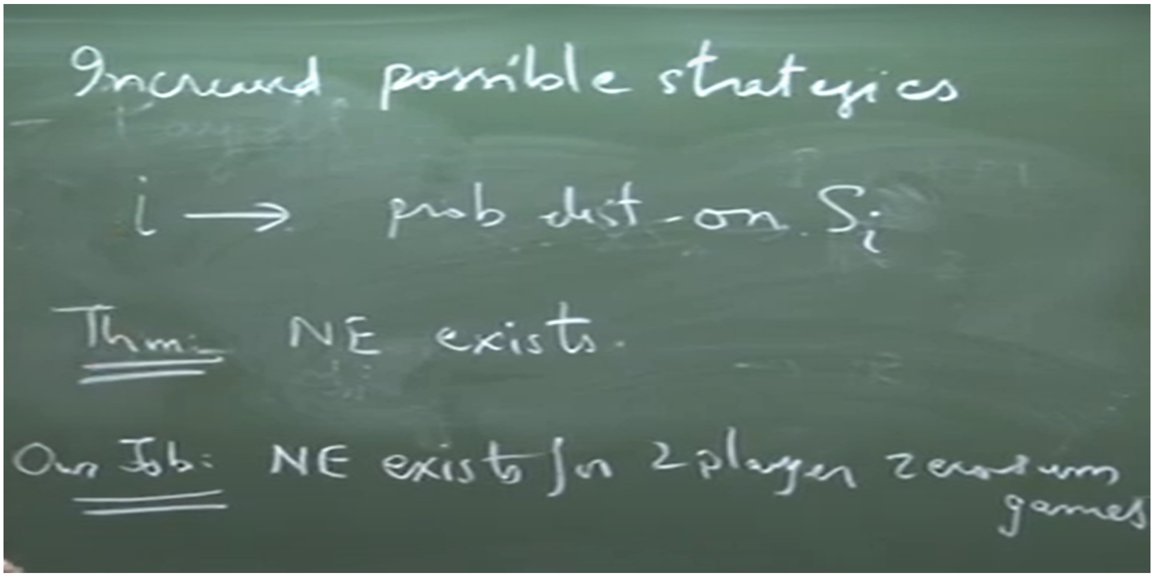
Nash could show was that always exists. What do I mean by mixed any? By mixed any I mean that players can play randomized strategy.



They can say with half the probability I am going to complain with one with other half probability I am not going to complain. With one by third probability I am going to complain with two third probability I am not going to complain. This is a mixed strategy.

And then we are going to analyze the payoffs and see that as there exist an equilibrium where no player has incentive to switch their strategy. So now we are increasing the possible moves. We had these deterministic strategies we can even player probability distribution over these strategies. And then we will understand what the payoff is and all that we will do it.

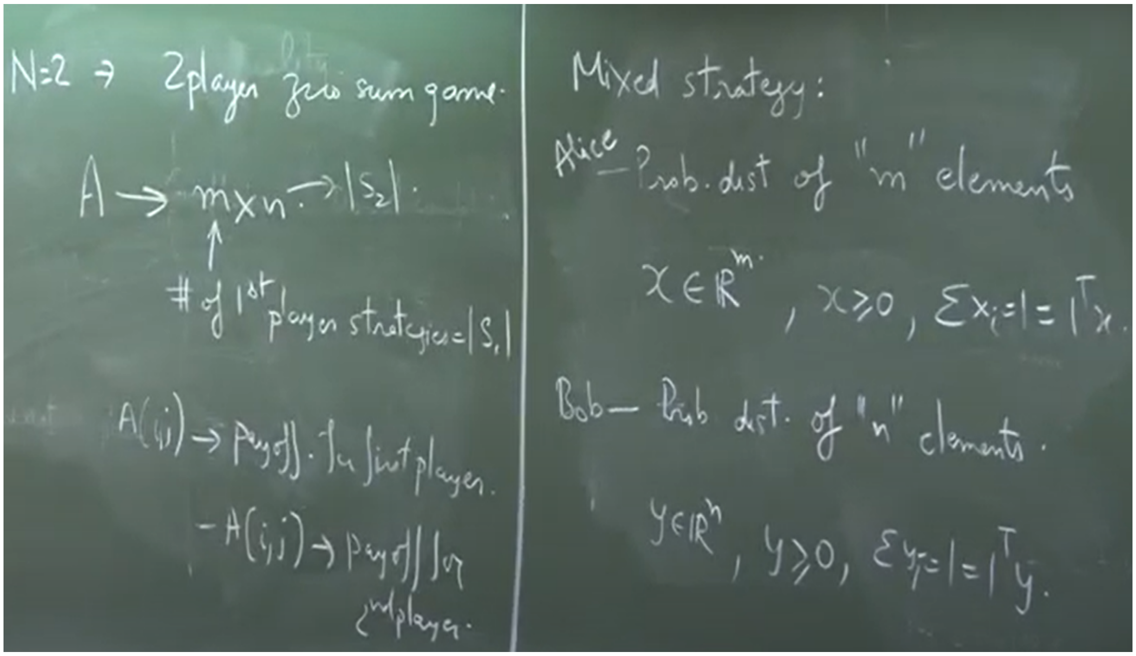
But at the first what we have done is increased possible strategies. For example, ith player can play any probability distribution on Si. Once we do this then we are going to prove the theorem that NE exists not just for 0 some games for games finite games in general that is a big theorem. We are not going to worry about it. What we are going to show is and obviously this is why Nash header beautiful mind.



That is the reason why he got no good place. I think it is so at least for this it is just the idea of it the proof now specially with strong duality is actually not very difficult. And John Nash was famous because his PhD thesis was like 17 pages long which was like the best thesis in years. Generally the PhD thesis starts from like 60 70 pages and goes up to 200 300 400 pages. So, clearly the proof was not difficult coming with proof the idea was very nice and that we will see.

So remember n equal to 2 we are talking about 2 player 0 some game. That means, basically our game will be described by a matrix A. m is the number of first player strategy or size of S 1 size of S 2. And A i j is the payoff is the payoff let us say for first player. What about the payoff for the second player? In this case my game is described by just a simple m cross n matrix.

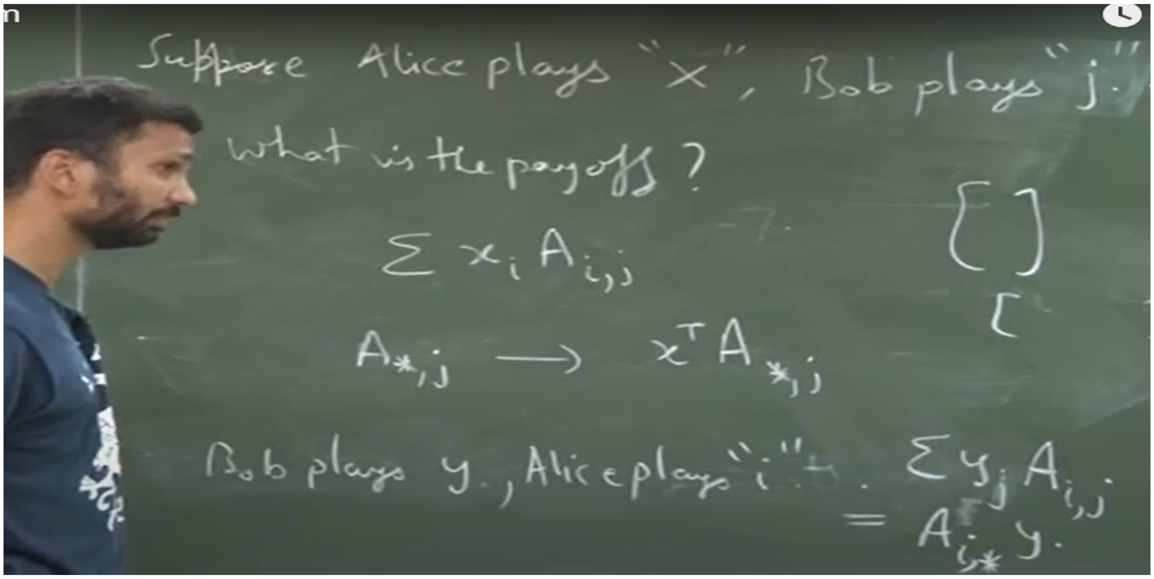
I am interested in that matrix trying to find out whether there is a mixed Nash equilibrium. I will stop calling it mixed that is what I mean by Nash equilibrium. Nash equilibrium means when we are allowed to play mixed strategies. So, we are just condensing all the information in A with the notation or with the understanding that this is the payoff of the first player. If we want the payoff second player we just negate the entries.



So, now let us talk about mixed strategy. What do we know? A mixed strategy is a probability distribution. So, let us say mixed strategy for first player do you want to name the first players Alice is involved. So, for Alice is the first mixed strategy is it is a probability distribution on how many elements? This is the number of strategies possible remember probability distribution on S 1. That means I can describe it by a vector like this such that right this is my description of a mixed strategy.

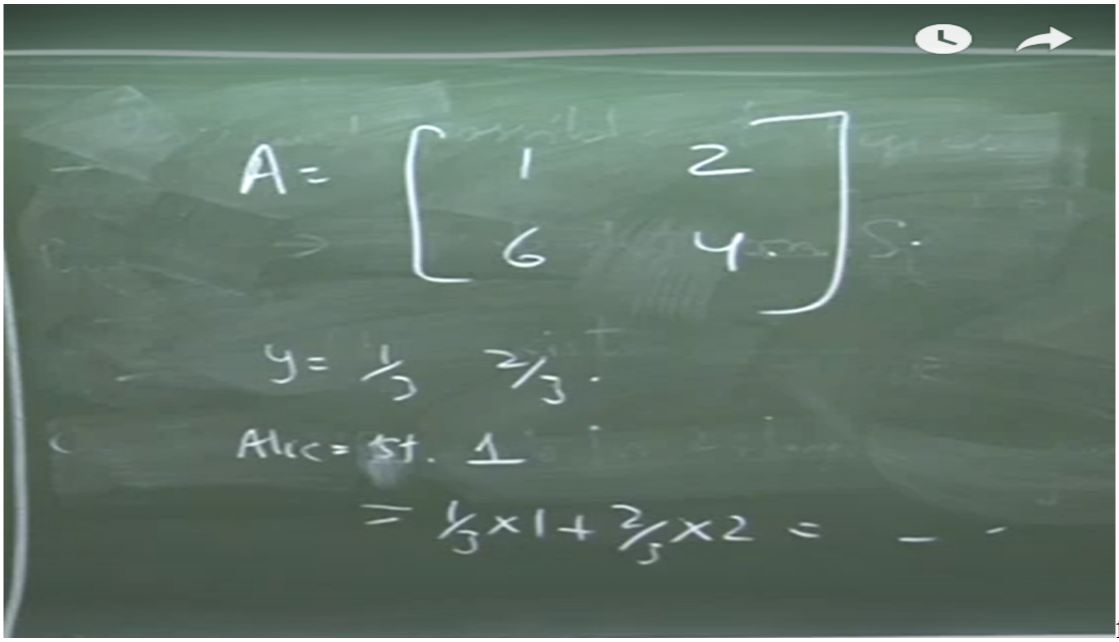
So, I can similarly write mixed strategy of Bob which is going to be a probability distribution over N elements. That means it is a vector such that each entry is positive. So we have described what a mixed strategy is. The first question is suppose Alice is plays X, Bob plays G a deterministic strategy. Question is clear and again payoff now will always be in terms of Alices payoff right.

So, what is now we can talk about payoff in the game? My understanding Alices payoff is positive whatever I am talking about if I want to take what is J transpose? Bob plays J means J is a deterministic strategy right. It is one of the strategies out of these N. So, I am simplifying summation right. So, this is saying that I am picking the jth column of A right jth column is this is a nice notation to denote jth column of A right star means I can put anything here j entry means fixed. So, the payoff on the other hand if Bob plays Y and L is plays I.



So, Alices payoffs when Alice is playing I it is written in ith row of A right. Whatever happens when Alice plays I am just restricting to look at the ith row of A and then each of these are played with this probability. So, this is summation this is questions about this that depends on the row I am considering it as a column vector you write it as this way or this way I can right. See this is going to be N dimension that the transpose depends on whether I am writing it like this or I am writing like writing it like this right. So, it is a mixed strategy.

Let us 6 4 and now if let us say Y is saying 1 by 3 2 by 3 and Alice is plays 1 strategy 1. Yes, this is this I is a deterministic strategy right I that is why j I am using different letters just to say that oh this is a deterministic strategy then we will do the hard part also, but I want to start with this right. So, if this is the case what is the payoff tell me if Bob is playing this Alice is playing this what is the payoff 1 by 3 times 1 2 by 3 times. Whatever it is right this is what we want to do you want to take this row wait with respect to the probabilities questions if this is not clear ask now because things are going to get more complicated. So, this might be if you have seen it before this might be easy if you are getting stuck here clarify I am here not asking the payoff of Bob I am only asking I am now always asking payoff of Alice is define it as payoff of game payoff of game is what Alice is gets what Bob has to give.



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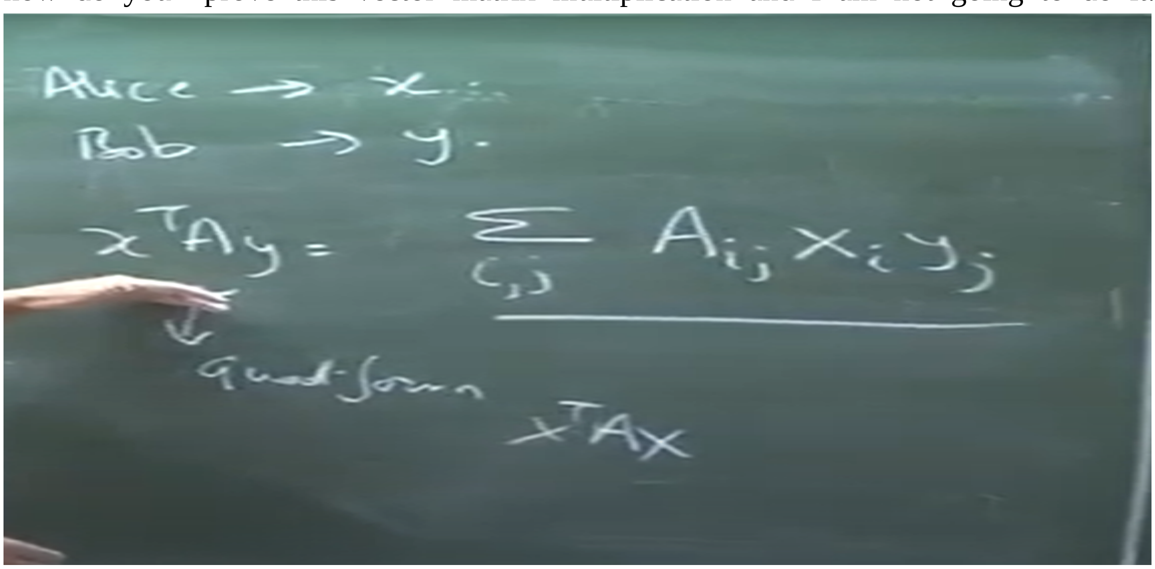
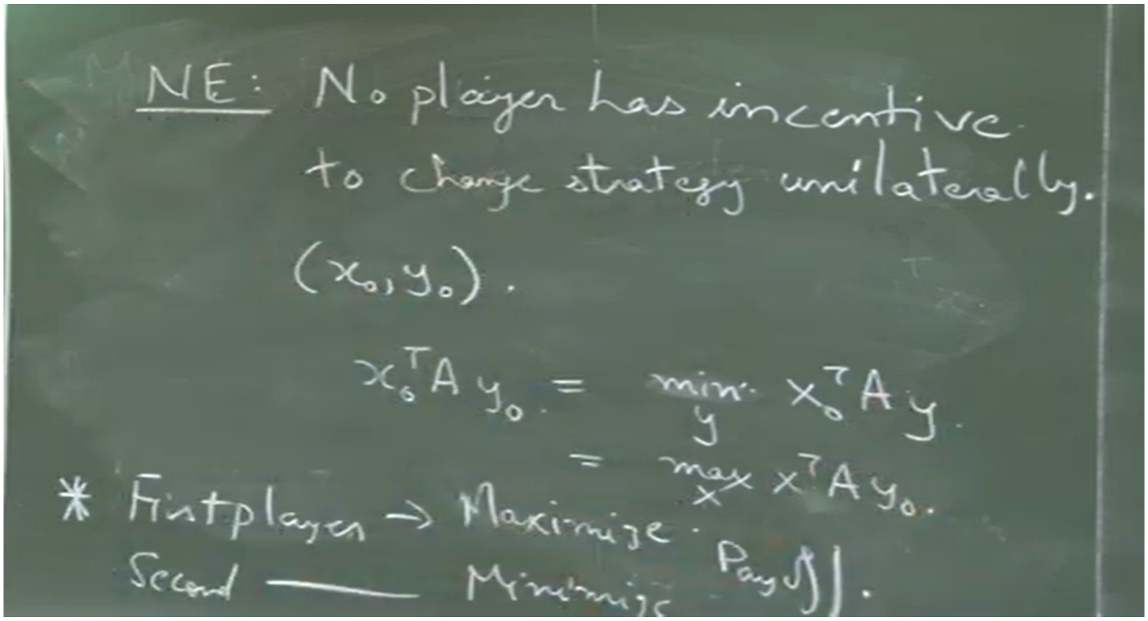
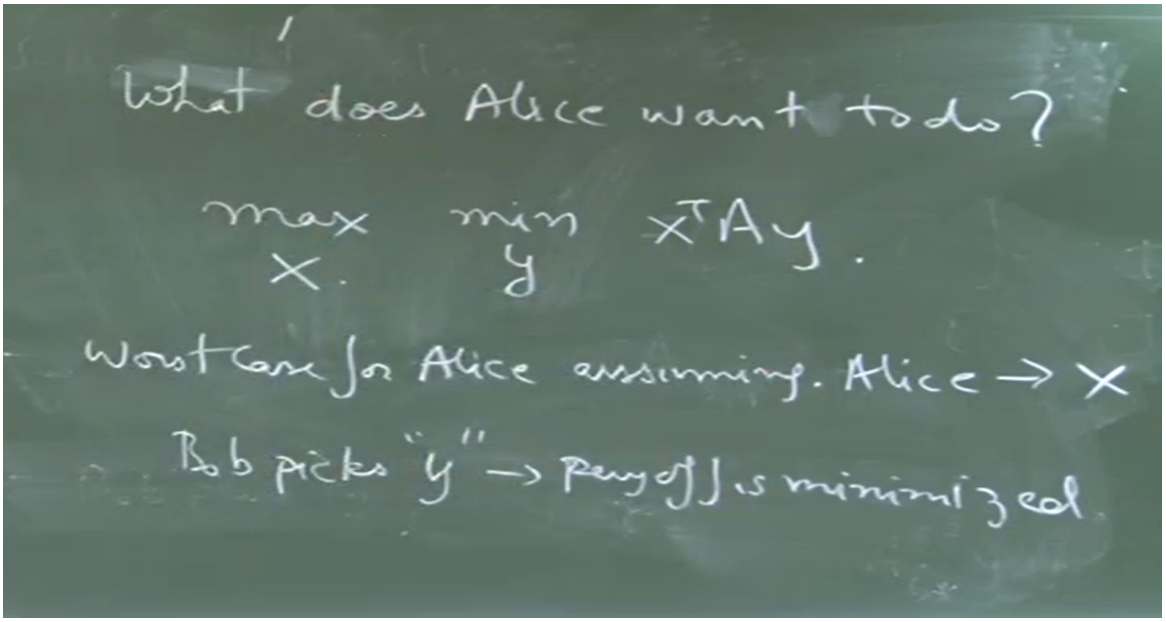
So, then there is no confusion then it is right good. Now, the next question is what if Alice is plays with X Bob plays with Y what is the payoff why you guess the form by looking at this. See what is happening when will they get A i j payoff with what probability will they get A i j payoff. So, that means I want to calculate this quantity these are the possible profiles i comma j for each of the profile this is the payoff this is the probability. So, this is the total payoff questions right and then this is equal to this how do you prove this vector matrix multiplication and I am not going to do it.



And this is I think a very useful form this is called actually a quadratic form X transpose A X called a quadratic form this tremendously useful specially when you talk about spectral graph theory and quantum computing and all those things. So, now we know what expected payoff is right and remember what was Nash equilibrium no player has incentive to change strategy unilateral right. So, it might happen that both of us change strategy then the payoffs can rise, but if I keep my strategy fix you change your strategy you are only going to lose this is the Nash equilibrium this is unilateral you do not want to like both players. Obviously both players change then it is very very difficult equilibrium is when I am going to play with this strategy now whatever you do this is the payoff I am going to get sounds good. So, that means if let us say X naught Y naught is a Nash equilibrium right then I know this is the payoff for the first player correct.



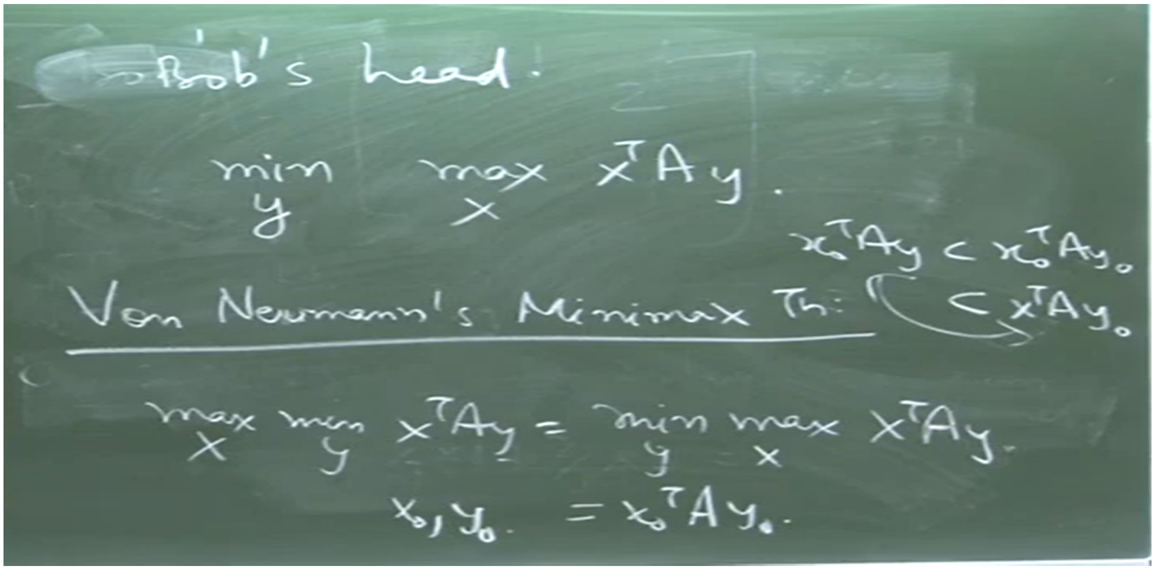
Now this sorry min because if at Y naught this is not minimum then second pair has the incentive to change the strategy from Y naught to Y. Second player is minimizing the payoff right second player wants to minimize the payoff first player has to maximize the payoff. Let me just write it because yeah once you think about it is obvious, but yeah first player wants to maximize right. Similarly this is also equal to if not first player will change strategy from X naught to X. So, this kind of formulation is nice, but now let us intuitively see what Alice and Bob actually want right.



For example, what does the first player want to do? Alice wants to maximize the payoff right. So, she wants to come up with an X right such that whatever Bob plays she is not at loss right. So, she will assume that Bob plays the best possible for Bob and worst case possibility for Alice. So, if Alice's strategy is X what is the worst case for Alice assuming Alice has played X what is the worst case? Think about think about it.

Min over what? Right. So, now what is the worst case for Alice Bob picks a Y such that the payoff is minimized right. So, worst case is picks Y the payoff is minimized. So, Alice wants to come up with an X such that no matter what Bob plays she is guaranteed at least this much payoff. Make sense? Now similarly, what is going on in Bob's head? He wants to minimize his payoff such that whatever Alice pays the payoff is maximum whatever pay it is minimized by that Y. So, fix Y what is the worst case for Bob? Alice plays the max X.

Oh because, the proof for 0 sum is much simpler and the case is I do not have to worry about A and B. For now if I have to write Bob's strategy it is going to be max over Y min over X, X transpose B Y and then how I am going to relate these things? It depends on relation between A and B. So, there are different techniques these are things are called fixed point theorems which so, which prove also strong duality, but using fixed point theorems you can prove, but that is a more general more difficult task. In this case these 2 quantities already look related right and surprise surprise Von Neumann's theorem basically says max X min Y X transpose A Y. And I am going to answer every question about it except why is it not called Maximin theorem and minimax theorem.



Von Neumann's minimax is just for this particular thing there are different versions of minimax when this thing is not really linear it is more complicated, but science minimax and you can study. So, this happen in 1940 after that minimax in itself is a different branch. So, we have seen 2 ways in which players can possibly play. One is this abstract notion of saying oh probably there exist a point where both of them are happy. They know that other player is not going to change that is the best strategy for them this is one.

We have done the other way where we are saying that you know what does Alice want to do? What does Bob want to do? And then Von Neumann's minimax theorem tells us that these 2 things quantities are equal. There is some X naught Y naught such that X trans dot this is also equal to. Now, there is a small thing which I have put under the rug I have not told you this is the optimization over X naught this is the optimization over Y naught. So, what is X naught? X naught is the X which maximizes this Y naught is the Y which minimizes this quantity is that Y naught is the same which is going to minimize this quantity. It is not clear it is an exercise you can show it is true the way it is set right you can show.

So, you can show that this indeed the Y which will minimize this will actually be Y naught otherwise there is a contradiction. And it is just simply saying oh suppose there is some other Y then you just compare it with X naught transpose A Y naught and then you will get something like this. So, suppose X is the solution here and Y is the solution here. So, you can get this inequalities, but you know that these 2 are equal this is minimax theorem. So, then Y naught has to be not has to be Y, but this value is going to be same as this.

But yes I am not going to the detail because I want you to do it is not difficult. So that means, Von Neumann's minimax the intuitive approach of solving things is giving us X naught Y naught. It is not necessary that Y naught is the only Y naught which minimizes this. And actually you can kind of show that there will be a deterministic strategy which minimize this I will come to that point. And if there are multiple deterministic strategies any linear combination of them will also minimize this.

So, definitely it is not unique the point is Y naught definitely is one of the minimizers. So, this is the X and Y are strategies now right this is the. It is only for strategies. It is only for strategies yes yes yes yes this is this is a mixed strategy. Sorry I probably if I want to write it as a theorem I should clarify A is an m cross n matrix X is a probability distribution over m elements Y is a probability distribution over n elements.

Then I look at this quantity this is equal to this this is what I am saying. You cannot take arbitrary vectors X and Y see generally you need to have a normalization because otherwise min max does not make sense you just multiply by something it can go to infinity right. So, clearly you have to have some kind of a normal you have to have a bounded region and then you want positive in this case. So, what I am saying is that positive is the only extra constraint in some sense which I am putting you need some kind of a normalization anyway.

No no no what I am saying is that yeah. So, if you want to make sense of the statement you should optimize over a bounded set. Now, if for 1 means min max it works when that bounded set is of this kind I am not saying it works for every convex region no, but there are generalizations of this node I am not saying this is the only one which works.