Linear Programming and its Applications to Computer Science

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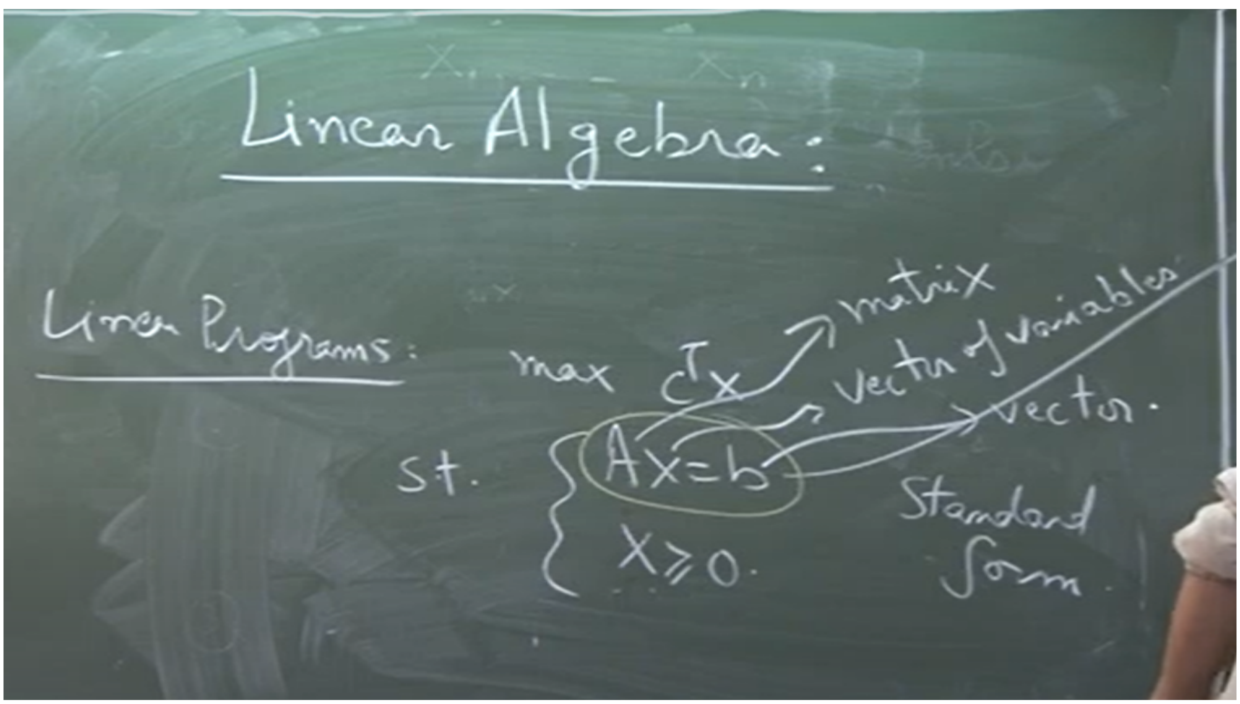
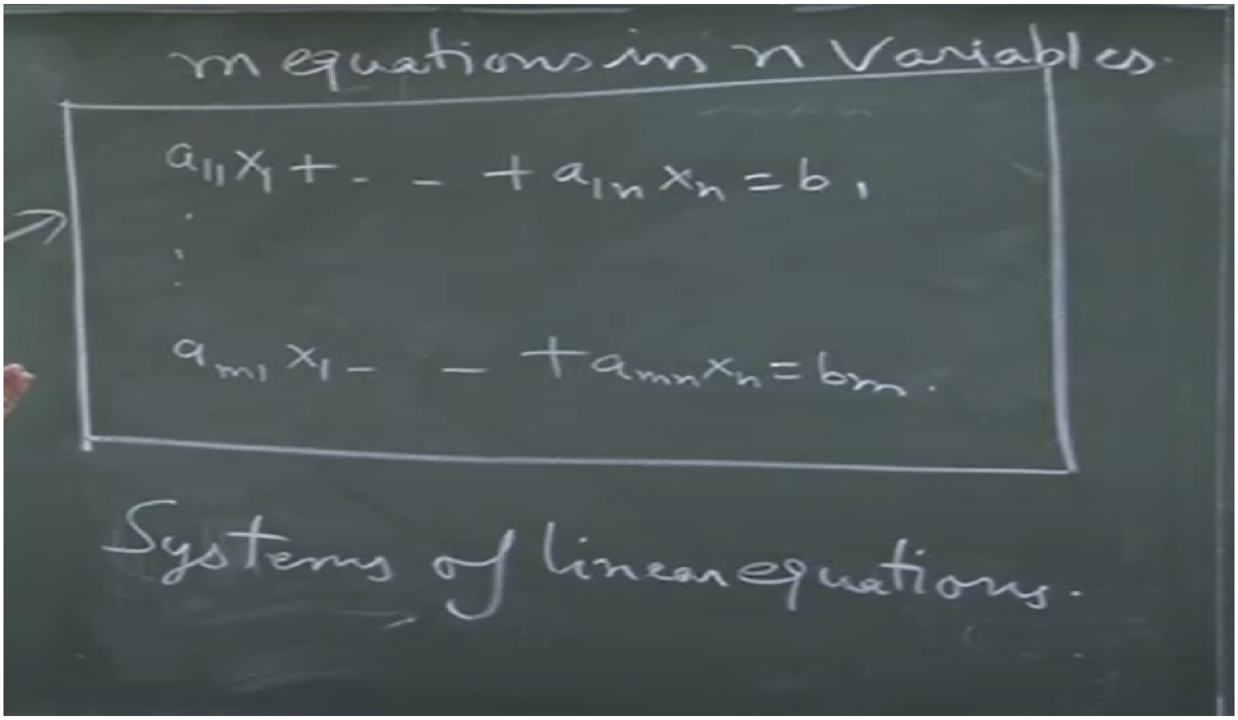
Lecture – 03
Gaussian Elimination with Examples

Hello, and welcome to another lecture on Linear Programming. Remember we had talked about what a linear program was and you wanted to study linear algebra. What was a linear program? We said a linear program is something an optimization problem where maximization was a linear function and even the constraints were all linear. That means I can have linear equations, I can have linear inequalities either greater than equal to or less than equal to, greater than equal to 0 and so on and so forth. Again, you have to believe me all of the such constraints can be changed into these two. This is what we called the standard form.



All linear programs can be thought of optimization linear function such that I have constraints Ax =b is a system of linear equations. They look like this with the constraint that every variable is positive. This is what we wanted to solve and we realized that first step in solving all this in the linear program is to actually concentrate on Ax = b and ask can we solve Ax = b. Remember here A is a matrix, X is a vector of variables, B is another vector.



If I expand it out it will look like this bunch of linear equations. When I say Ax equal to b, when I say system of linear equations all this should make this picture in your mind. It means I have m equations here in n variables. My variables are X1 up to Xn, my coefficients are a11, a12, a1n so on and so forth and b1 to bm are constants on the right hand side. Ok. So to take an example remember last time we wanted to solve this bunch of equations.

If I want to talk about it here there are two variables and three equations. The variables are X1, X2 and constants are 1, 3, 1, 1, 7 and 4 in this case b1 is 98, b2 is 40, b3 is 390. So this system of linear equations we want to solve. I will write this as Ax = b. How to solve Ax equal to b? And when I say I want to solve Ax equal to b this should be very happy moment for us as a mathematician because this is what I will call the problem of linear algebra.

What I mean by that going to a bit of philosophy if you look at any branch of mathematics most of them owe their existence to a central problem. There is a single problem which we wanted to solve and that gave rise to the entire Mathematical theory behind that problem. For linear algebra I would say there are two problems one is Ax equal to b and another is Ax equal to lambda x. This takes us to Eigen values and Eigen vectors I am sure you have seen this calculating the Eigen value of a simple matrix all that. For this course let us not worry about this.

What we will start with and we will be focusing on is how to solve Ax equal to b and using that solution to finally incorporate even x equal to 0. So this problem is something which has been solved in many ways. I am almost sure that this was this you must have seen in your basic course of Mathematics in your engineering course. So this lecture is basically a refresher of linear algebra or a part of linear algebra which talks about solving Ax equal to b. If you are not familiar with this if this is new to you I encourage you to look at any standard book of linear algebra my favorite is the book by Gilbert Strang which is linear algebra and its applications.

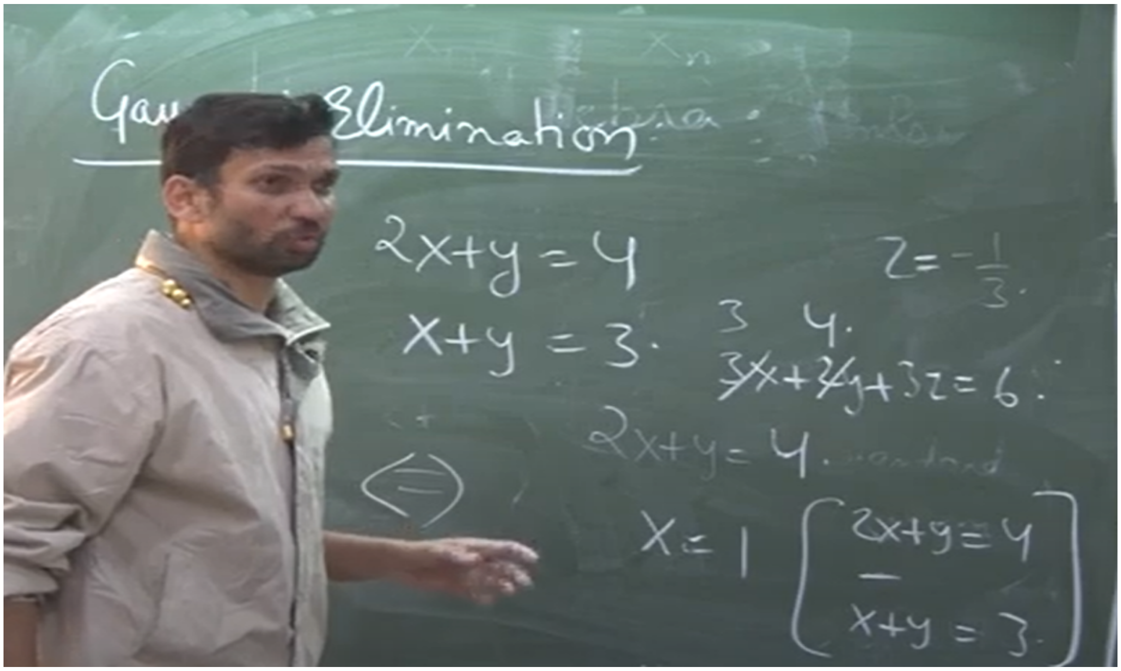
But any standard textbook on linear algebra will be good enough for these constraints these are central basic concepts of linear algebra. So next two lectures we will talk about Ax equal to b how to solve Ax equal to b. In generally when we have such a kind of a general task of solving such difficult problems or such set of problems the first step is to actually look at examples simple examples and see can we at least solve some simple examples and find a way. And let me give you a very simple example. Notice that I have replaced x1 with x and x2 with y I do not think there should be a cause of concern this is just change of variables.

If I give you this set of linear equations you will immediately realize oh this is very simple. Right the solution is x is equal to 1 and y is equal to 2. So by doing some manipulation you can I am sure almost all of you can solve this set of equations. Right this is not the only case there are complications here. What about this set of equations? 2x plus 2y equal to 4 and x plus y equal to 3.

And if you think about it for a little bit you will realize that this set of equations cannot be solved. In some cases we have a solution in some cases we have a no solution it will turn out there are some cases where we have infinite many solutions. So even this innocent looking bunch of equations can give rise to many situations. How are we going to solve? Now once again you can say oh you gave me the equation I looked at it and I could solve it. That is great when you had small number of variables and small number of equations here, but like a linear program we want to do it when we have thousands of variables and thousands of equations.

Can we still design an algorithm can we still design a program to solve all such equations and not such simple equations. Though if you want to even solve such general equations let's take some hint from how we went about solving this equation. What we actually did to come up with this solution here and that should probably give us an idea of solving this big equation. If you already know it then this is called Gaussian elimination. If you do not know it you are going to learn it in next 30 minutes.

What is Gaussian elimination? We are going to redefine our thinking and see that there is a standard way to attack any such Ax equal to b problem. If you wanted to solve this bunch of equation I think a very natural way to look at this is and say oh there is an extra x here. So what I can do is I can subtract this equation from this equation this will tell me that actually adding one extra x gives me one more. So this bunch of equations are actually equivalent to notice this is obtained by doing this right 2x minus x is x y minus y is 0 4

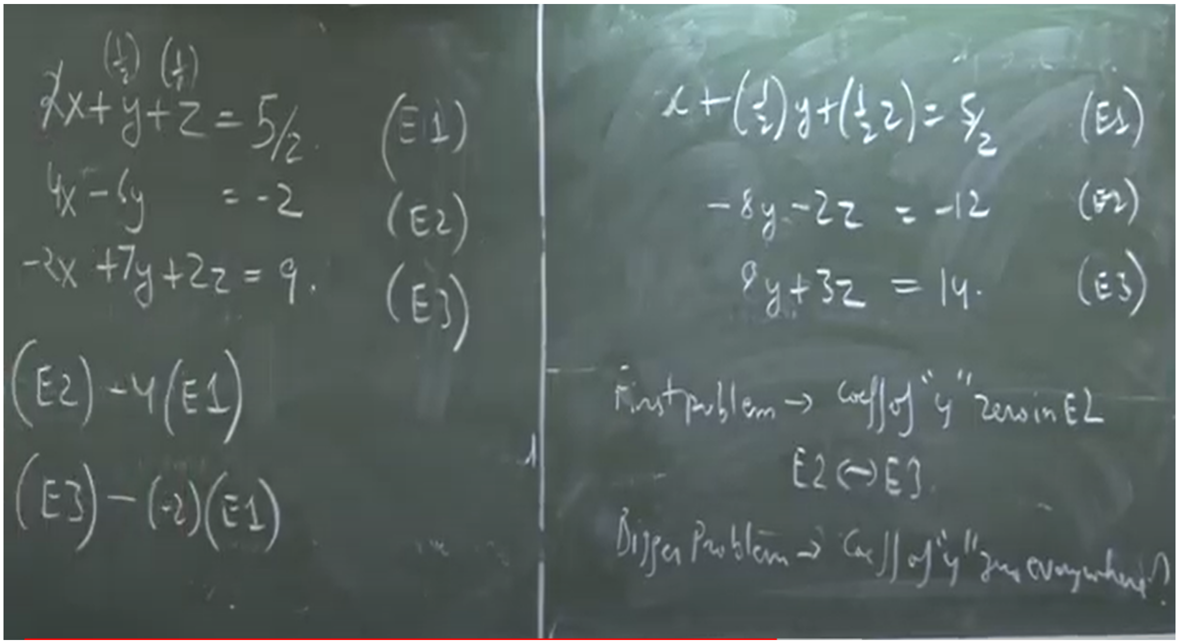


minus 3 is 1 and once I did this step my life became easy if x is 1 I can substitute this value and this will give me y is equal to 1. So if I put x equal to 1 in this equation 2 4 y equal to 2 great.

So I found x equal to 1 and y equal to 2 as a solution. Now this is what I will call a good form you know why is this a good form x is equal to 1 I get the value of x then I can substitute x in the previous equation to get y and probably if there were more equations it might have happened that the next equation was an x plus y plus z. So I will substitute the value of x and y I will get the value of z. So to give you an example let us say if I had these 3 equations x equal to 1 y equal to 2 and then right. So I get z is equal to minus 1 by 3.

So I can back substitute and get my entire solution. So my objective would be to convert take this general set of equations and convert them into good form. You can notice that this is trying to make a upper or lower triangle. This is what I want to but I had a set of equations which were very simple our life is not going to be that simple all the time. We take more examples and we see more complications.

So let us take more and more examples and it will tell us what can happen in different such situations. Now this is a slightly bigger example which has 3 variables now and I want to solve for x y and z. Now looking at the idea of this is it possible that I remove x from equation 2 and equation 3 that is my first objective because remember here I made the coefficient of y 0 here I will first make the coefficient of x 0 in these 2 then coefficient of y 0 in this equation that is what I want to do. So to make my life easy I can divide by 2 so that I have a single coefficient of x and then I can multiply this by 4 to make this 0 I can multiply that this equation by minus 2 to make this 0. So 3 what I am going to do is I am going to do 2 minus 4 times equation.

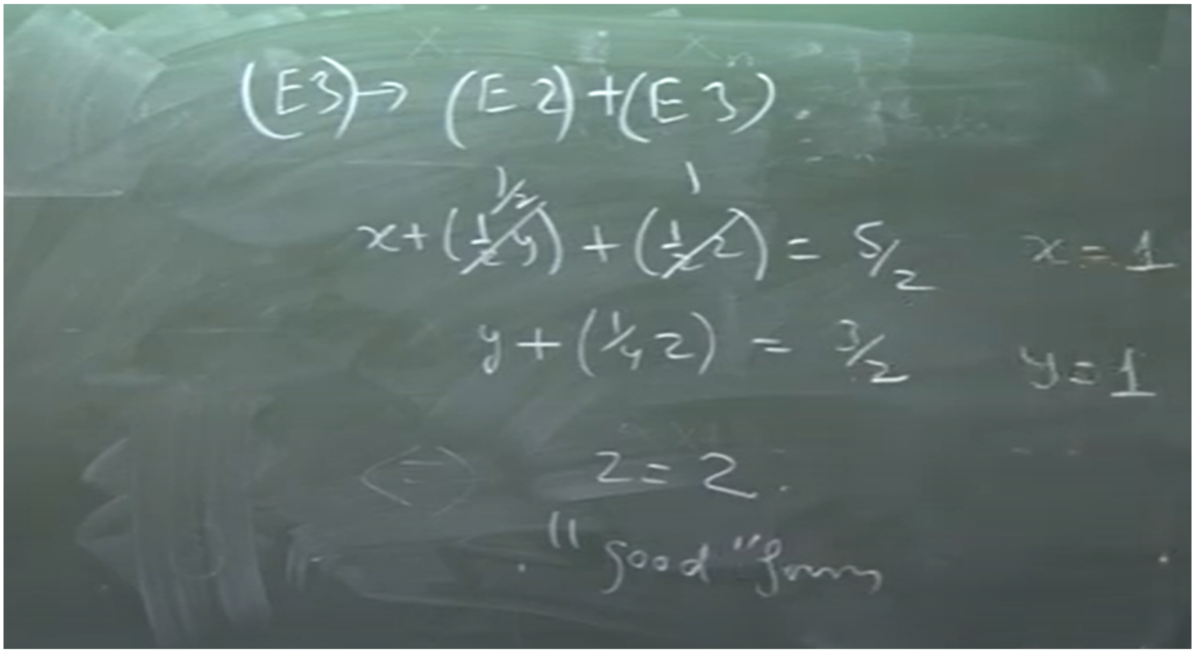


So let me just make a better notation so that you do not get confused sorry for this let us call it E1 equation 1 equation 2 equation 3. So what I am going to do is I am going to take equation 2 and subtract 4 times equation 1 that will make the coefficient of x 0 and to making the coefficient of x 0 in equation 3 I am going to do 2 times equation 1. You can make these calculations and what you will get is x it is important that you should be able to do these calculations on your own. So make sure that you can come to this step and now I can add these 2 equations to make the coefficient of y 0 in equation 3. Notice it requires a formal proof it is there in any little book, but this operation when I add this and write the added equation this and these are equivalent system of equations.

That means the solutions here are the same as the solutions here. These operations do not change the set of solutions that is it is easy to prove it. Now just to give you an idea there could have been complications here what if the coefficient of y was 0 here. If the coefficient of y was 0 here instead of minus 8 how can I remove it from equation 3 that is an easy thing right very good? So as Tufan said one way is I say that oh z is my second variable or other way is I say that oh I will just interchange E2 and E3 and then I will have a y coefficient and I can make sure that other way y becomes 0.

This is not a big problem. So first problem could have been coefficient of y 0 in E2, but then you can exchange E2 and E3. But a bigger problem is if coefficient of y is 0 everywhere. There are multiple such cases where our standard procedure will get stuck, but there are solutions out. We will talk about them, but for now let us proceed here.

In this case life is simple. I replace E3 with E2 plus E3 and then Once again we have things in the good form z is equal to 2. If I put z is equal to 2 here this will show that y is equal to 1. This will be 1, this will be half. So by back substitution I can solve everything.

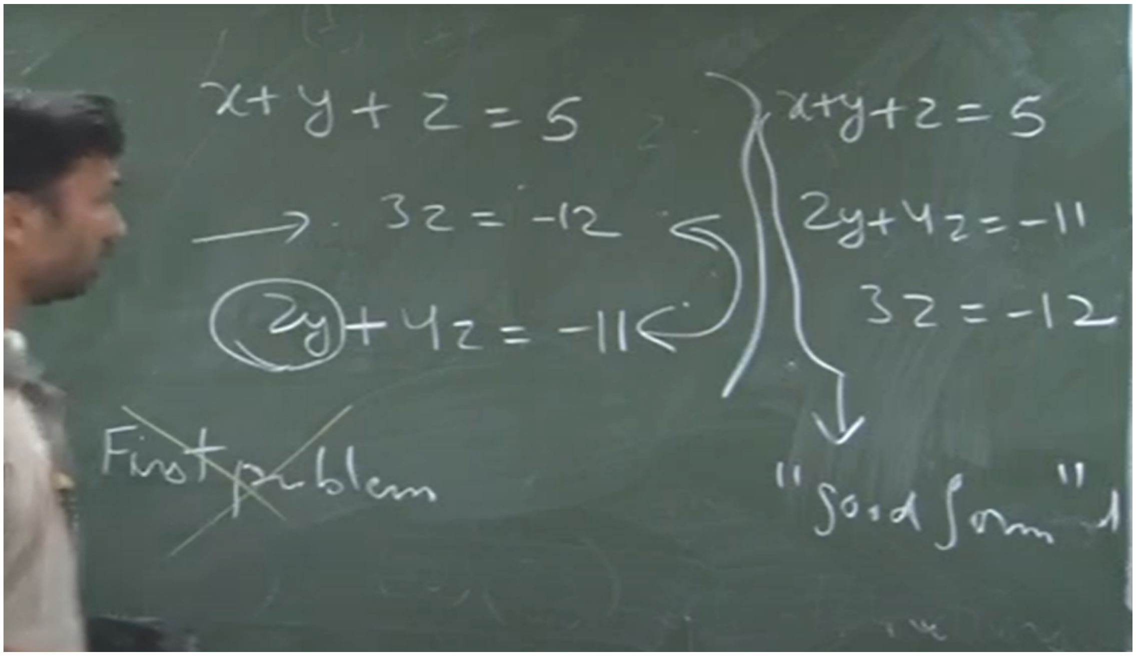


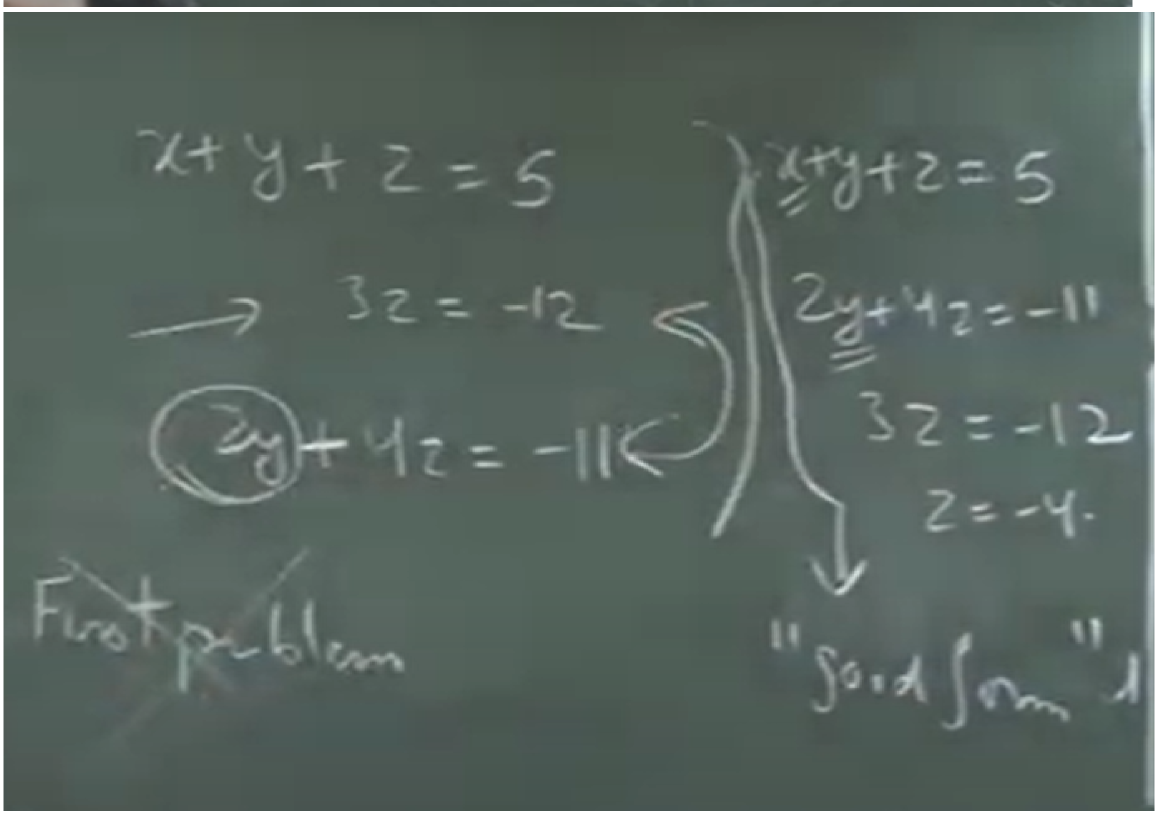
You can convince yourself that even if there were more equations and if my process does not get stuck into these problems I will always get into a good form and my life will be ok.

This is the good news. In Ax equal to b if I do not get stuck into this problem I can always convert into good form and I can find solution. Good. Where will there be a hiccup? And these are what I call weird cases. There are some set of equations which can actually create problem for us.

So let us see. I am not going to write every step in detail now because you have seen how to make it work. Once again if I want to remove coefficient of x I will take this equation multiplied by 2 and remove it from here. So once you do this you will see that x and y both are eliminated here and again important point is these two system of equations are actually equivalent. They are exactly the same. Instead of solving this I can now solve this.

But now this is our first problem. It is not really a problem. Tufan has already told us how to solve it. But let us say even at the first glance it seems like oh my god I wanted to remove coefficient of y using this equation but the coefficient of y here is 0. So first problem. But I would say even this is not even a problem at all.





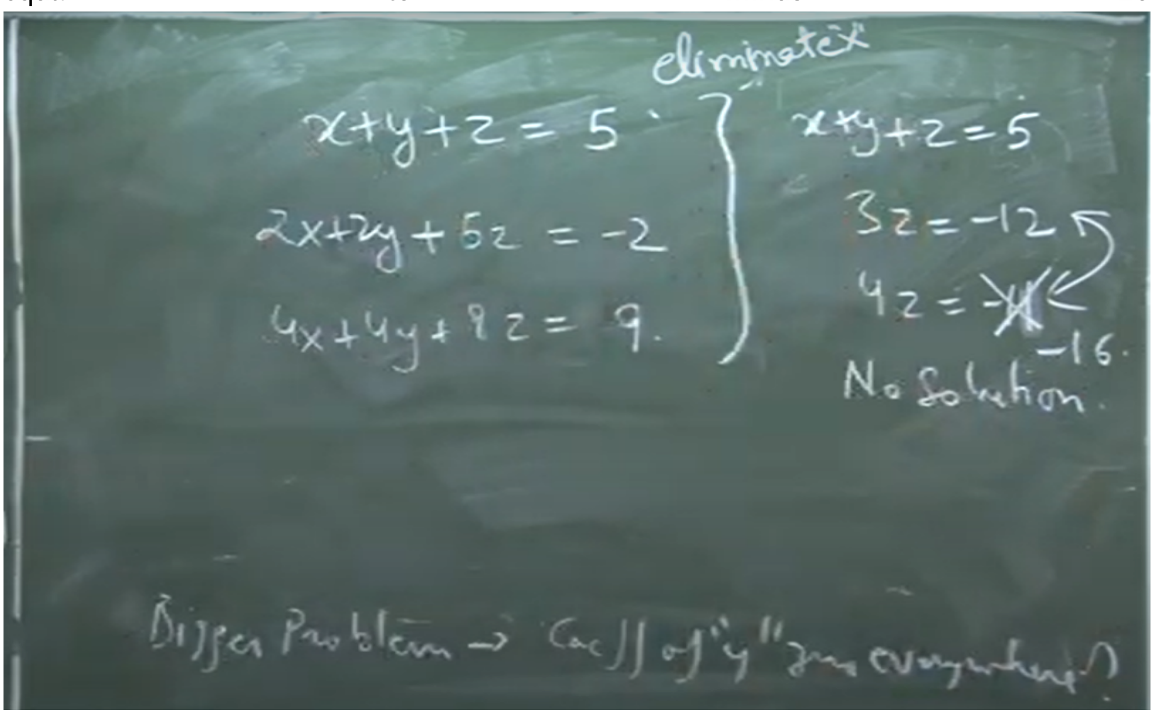
This is very easy. Instead of this I can just interchange this. Obviously if I change the order of these equations the set of solutions do not change. So, then I get this. And I do not need any convincing to tell you that just because I write this equation first and this equation later I do not change any solution. So now once I have gotten this notice this is in good form.

Once again if these equations are in good form I can back substitute. Z is going to be minus

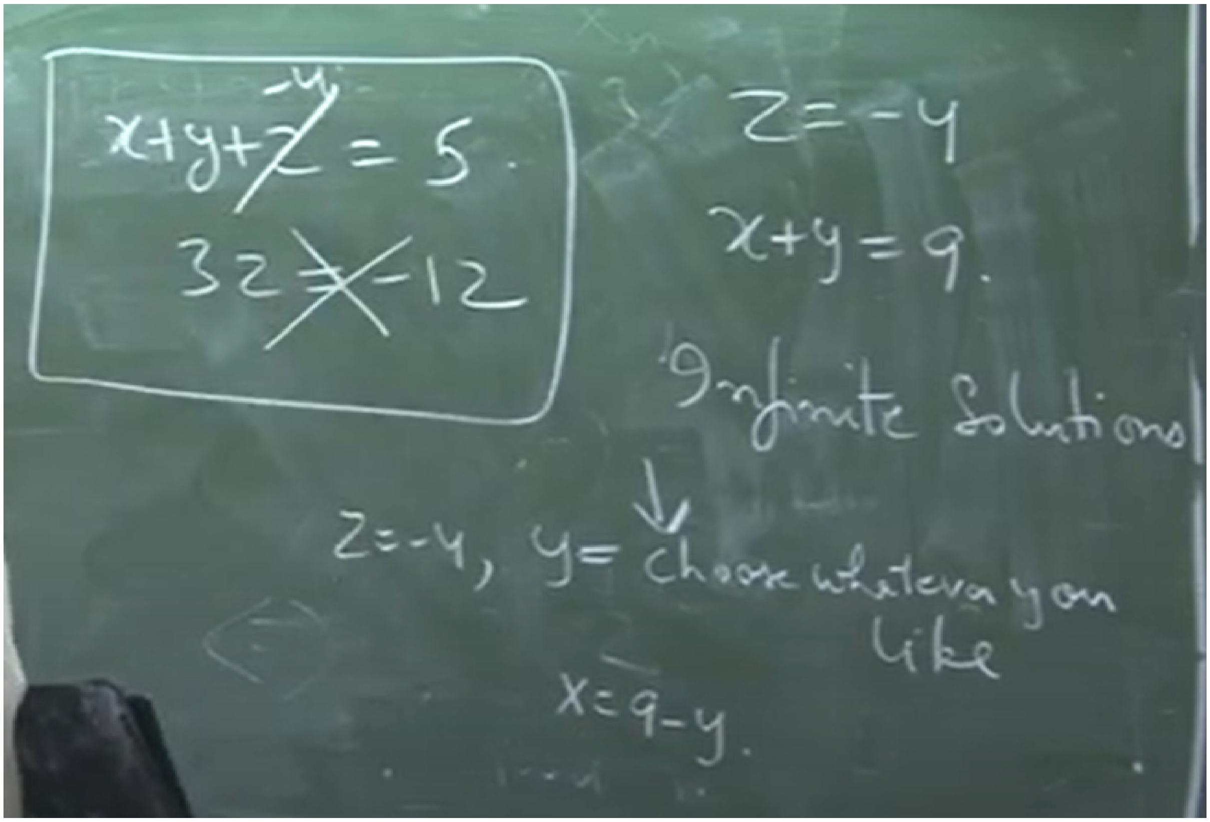
4. You can calculate value of y. You can calculate value of x. The first problem is not really an important problem.

Even the bigger problem is not going to be a very big problem. You will see that there is a simple resolution to go about it. Let us say we had this set of equations. And now once again if I try to eliminate x I get right what should we do now? We are in a bigger problem where coefficient of y is 0 everywhere. Though there is a small thing here that if you ignore this equation look at these two equations it tells us that we have no solution.

If 3z is equal to minus 12 then 4z should be equal to minus 16. So we can forget about y we can say that oh there is no solution possible just because of E2 and E3. No solution. But what if this was actually the correct number? Let me just do it correctly. If instead I replace it by minus 16. Now what do I know? I have x plus y plus z equal to 5, 3z equal to minus 12, 4z equal to minus 16.



I know that equation 2 and equation 3 are the same equations. So why worry about this futile equation? I will just say my original set of equations this box is same as this box. And now I can even replace this I know z is minus 4. So I get x plus y equal to 9. This is the case when we actually get infinite solutions.



In this case your z is minus 4, y you can choose y or you can choose x, y you can choose whatever you like. And depending on that your x is 9 minus y.