Linear Programming and its Applications to Computer Science

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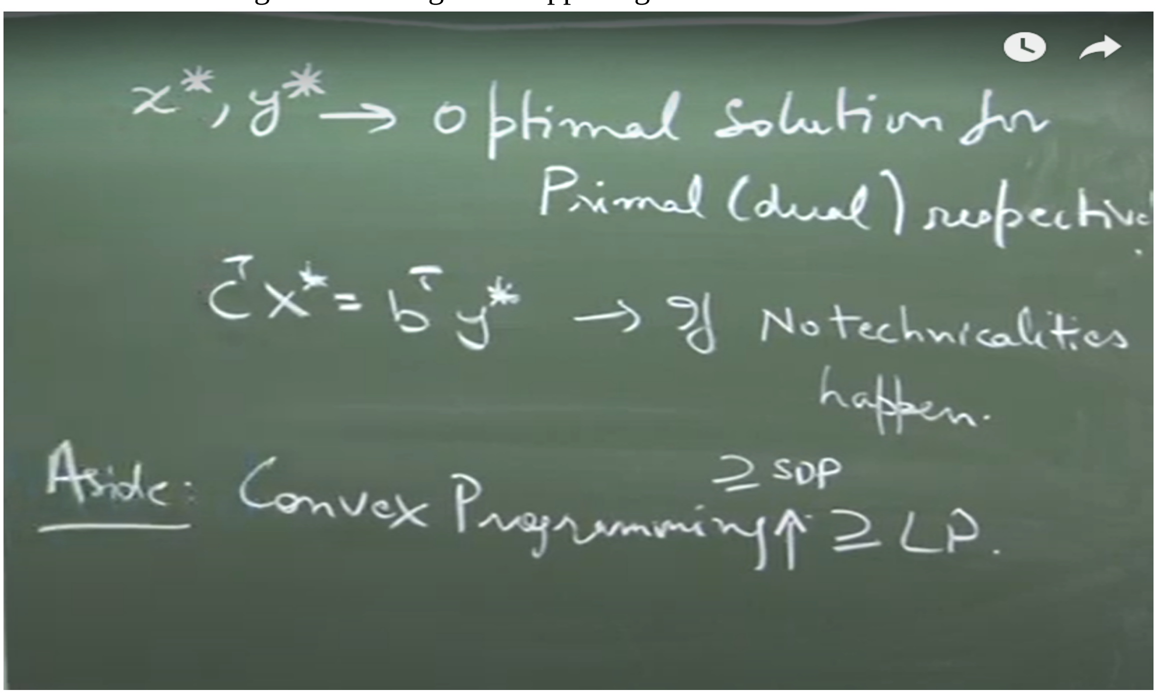
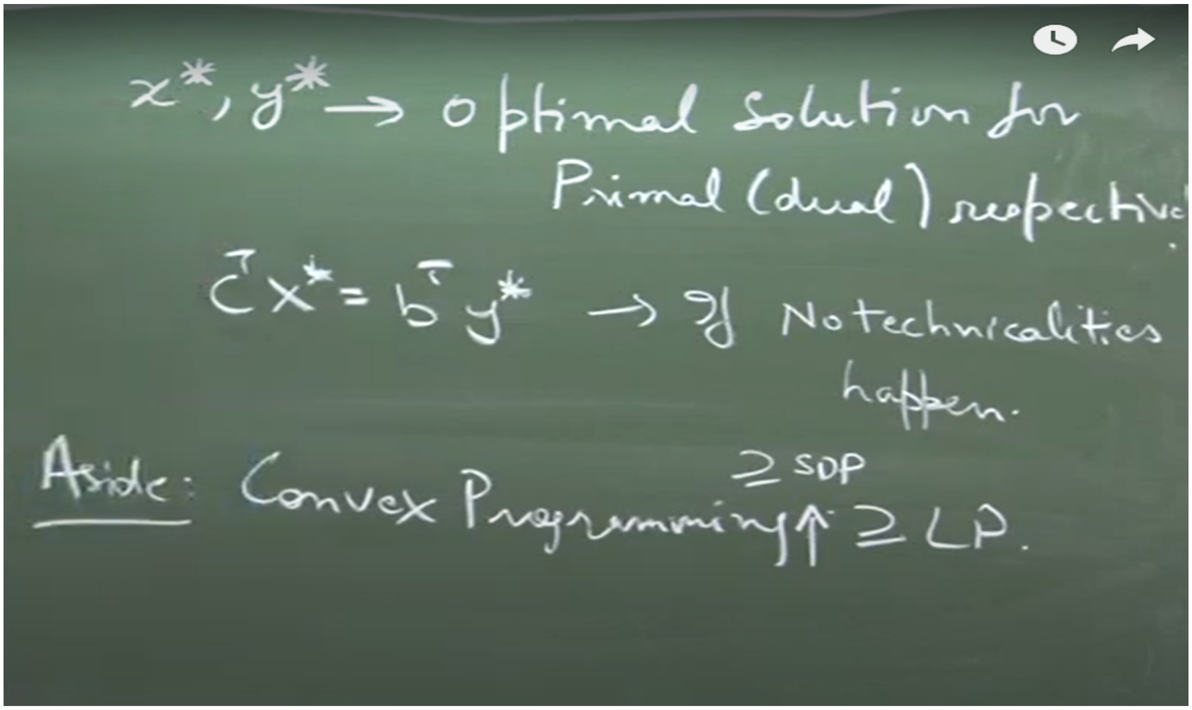
Lecture – 29
Proof of Strong Duality

We see the proof first. Let us fix up our notation. So, that we do not get confused p and d we already know let us say x star and y star are the optimal solutions for dual respectively correct. And what we want to show is that if no technicalities happen right. And you remember technicalities is one is infeasible both are infeasible and things like that. So, if none of those strange weird things are happening both are feasible then this is the case.



Before I move forward I wanted to just as an aside I want to mention convex programming. What do I want to mention about it? Notice that many of these things which we are saying about linear programming the niceness comes because the feasible set is a convex set right. So, there are cases there is this bigger branch which contains linear programming and many other places where my feasible set is convex. And then you want to optimize some nice function it could be a quadratic function it could be a linear function it could be a convex function so on and so forth.

But the idea is that still some of the properties of linear programming can still be used here. And people in optimization actually study convex programming. I remember a case when I went to this optimization person told my program said oh it is not convex we cannot do anything. So, that is generally the feeling they have convex good not convex not good for optimization right. So, this is a major field of study of which LP is a very very small subset.

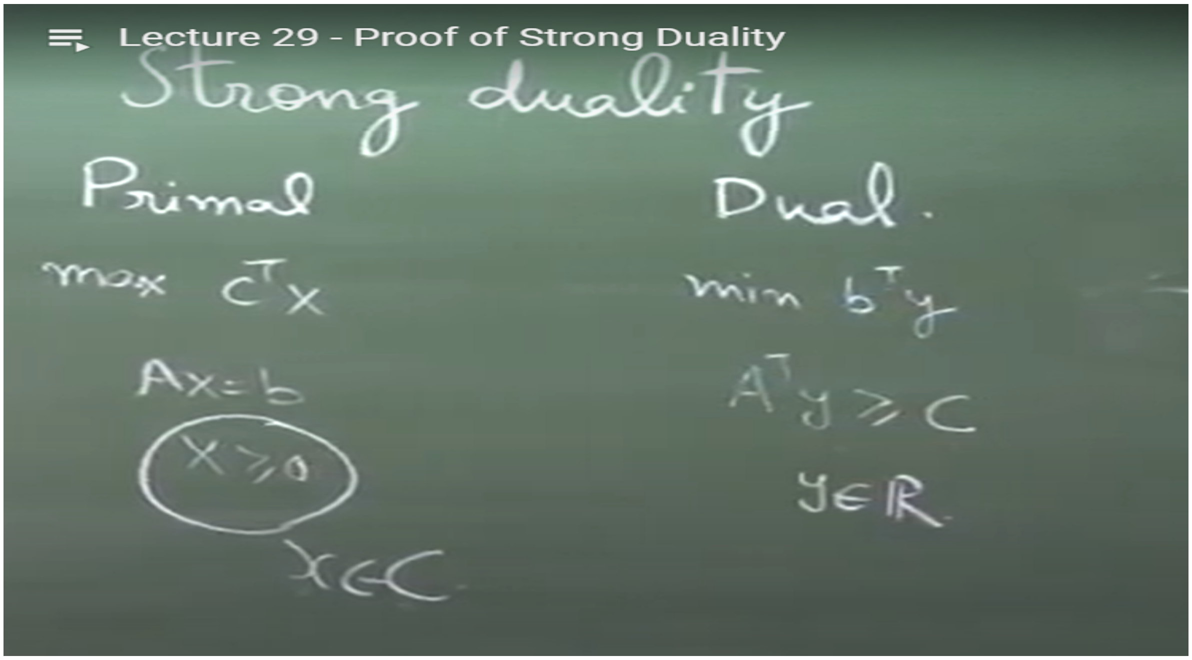


There is another small subset which kind of comes in between which called semi definite programming and I will cover that. That is also used a lot in complexity theory semi definite programming and this is where my research started so kind of fond of it. So, I will tell you about this whether you like it or not. But yeah so LP semi definite programs convex programming and what you realize is that this is the strongest. That means, very few programs can be written like this very few in the scheme of still a lot.

But few compared to others though it has strong properties. As you move this side the set increases, but then the properties decrease. Duality is one thing which works out for convex programming. You can define what we call dual of convex cones. So, instead of this we can say x is part of some convex cone and then the dual will be all the things which have positive inner product with everything here.

Why were we saying x should be positive? Because x positive is the elements which will always give you positive when multiplied by something which is positive. Similarly, you can define this idea of dual cones and this entire duality theory can be extended to convex programming. But then strong duality does not come for free. Weak duality duality theory works out strong duality you have to be careful. Here it is automatic here it is almost automatic here you have to work for it.

So, in the big scheme of things it is nice to remember yes LP is something we are studying, but there is a bigger class of problems. So, if your problem is not LP not all hope is lost probably it is semi definite programming then you are still very, very ok. If it is convex programming you are fine. If it is not convex programming change your problem does PhD on something else. No I am just joking there are other techniques too, but this is the general idea.



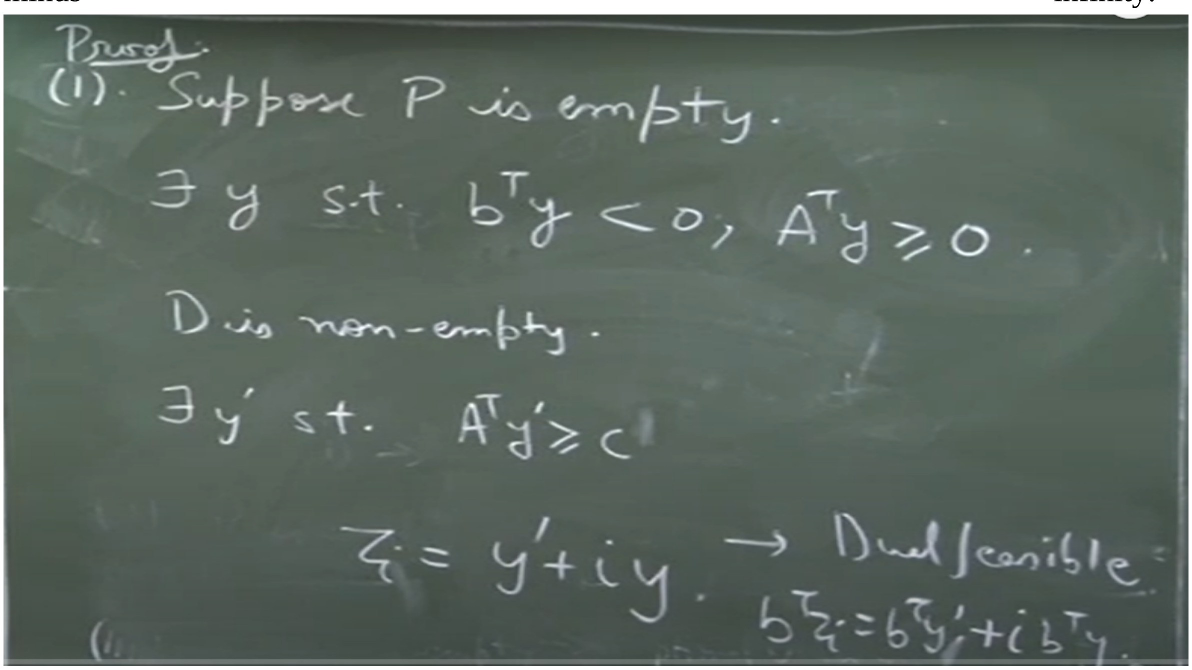
So, one of the nice property of LP is that strong duality almost always holds. And if we have this convention of minus and plus infinity then it is kind of always there. Now, we are ready for the proof I will erase this you will remember this actually let me just keep it I think it is important.

So, let us take the first case suppose P is empty what does this mean that we have to prove that is not again the wait proof. So, we cannot say that means, this is infeasible that is what we want to prove.

So, now, but from before hand before knowing the statement of strong duality if someone tells you this what do we know there is no x greater than equal to 0 such that A x is equal to b. That means, I am not looking for English equivalence and you are a mathematical equivalence. You should be able to tell me this I told you strong duality is relies on what they are going to elementary, elementary before that there was an elementary version of duality. Yes. So, what does it mean b is not in the cone of vectors of A, A x is equal to b x greater than equal to 0 this is saying that b is not present in the cone finite convex cone generated by the columns of A. So, then what does Farkas lemma tell us there has to be a hyper plane separation there exist a y such that what do I which sign do I prefer in the original thing if I had written the opposite signs do not worry you can multiply y by minus 1.

So, there is a hyper plane separating the cone and the point right this was the hyper plane separation theorem great. Now, d could be empty or d could be non empty right if d is empty there is nothing to prove both of them are empty we never said anything. But suppose d is non empty that means there exist a y prime such that sorry A transpose y prime is greater than equal to c right. So, what should I do now? Now, I am not doing this as a cone I am doing this as a constraint here and what is my task I want to prove that this is what do I want to prove t is empty dual is feasible. So, I want to prove that this is unbounded right.

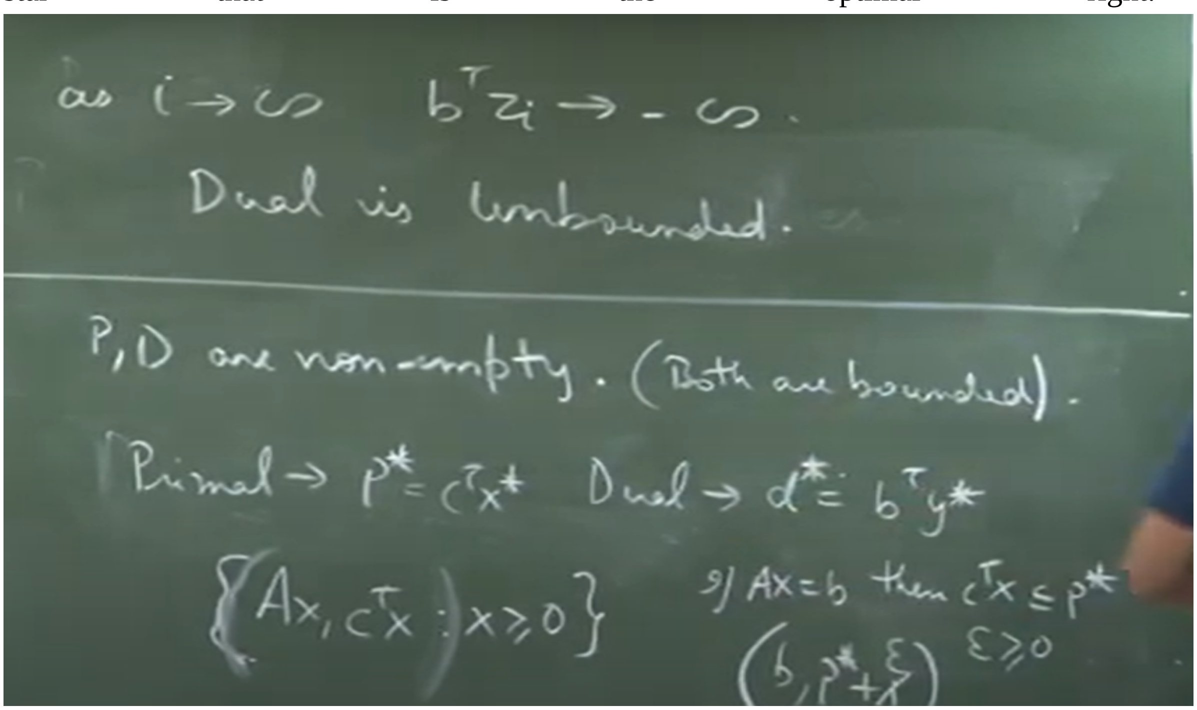
So, you look at these equations and tell me why it is unbounded you should be able to see this exactly if I take y prime add multiples of y what happens is this constraint violated it cannot be right. So, look at this solutions right they are all dual feasible everyone agrees this is also dual feasible because A transpose y prime is greater than c A transpose y is greater than equal to 0. So, A transpose z i is also greater than equal to c take a positive quantity bigger than c you add a positive thing it will still remain bigger than c again this is vector inequalities by the way right, but it is fine. But what is happening to the objective value B transpose z i is B transpose y i plus sorry B transpose y i this is a fixed quantity B transpose y i is negative as i tends to infinity this becomes minus infinity.



Ok right because I have constructed feasible solutions of dual which are taking my value to minus infinity? So, that means dual is unbounded clear and by the way if dual is unbounded then clearly primal is infeasible right why if dual is unbounded and if there is a feasibility then there I have a lower bound, but I know lower bound cannot happen because this is unbounded right. So, this unbounded feasibility all that we can now leave and we know dual of dual is primal. So, we can say that if you know P is empty then dual is unbounded or right. So, we leave out everything now we are in the final interesting case which is P and D are non empty.

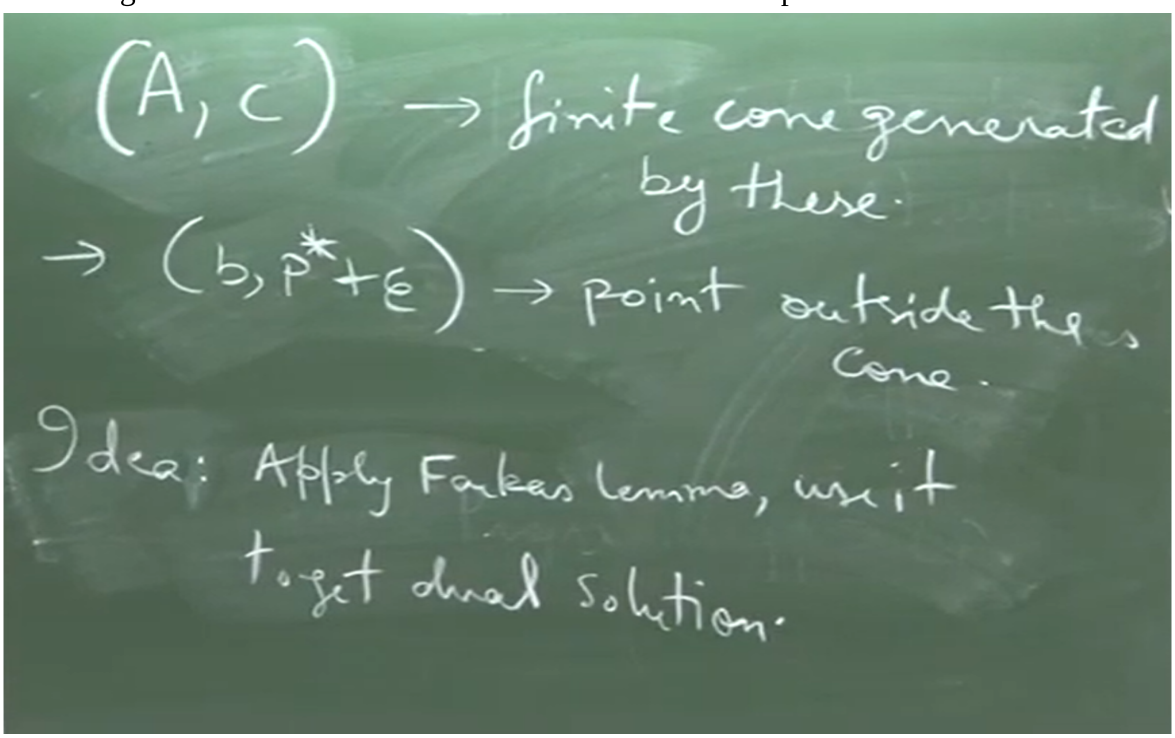
So, both of them are non empty now I want to prove that both the values agree I get this is my target this is the interesting part of the problem. Technicality is over this is the main part. So, let us focus and again the idea is to apply for Farkas lemma right, but what to apply Farkas lemma right I do not see any like A X equal to B is feasible A transpose Y greater than equal to C is feasible right. So, there is no cone infeasibility here on which to apply Farkas lemma, but let us say. So, are non empty and both are bounded I can assume that because if any of them are unbounded then I know that other has to be empty right.

So, these are non empty implies both are bounded. So, now suppose the optimal value of primal is P star dual is D star this is just notation and this is equal to C transpose X star A transpose Y star and I want to prove that these are equal. So, now the infeasibility or the fact that you know I cannot write something as a cone is that let us look at everything in one more dimension and this probably something which I mentioned this is going to be for kash lemma or this is going to be hyper plane separation in just one more dimension. Now, let us look at all these points this such that X greater than equal to 0 right. What do I know about this? If A X is equal to B right then C transpose X is less than equal to P star that is the optimal right.

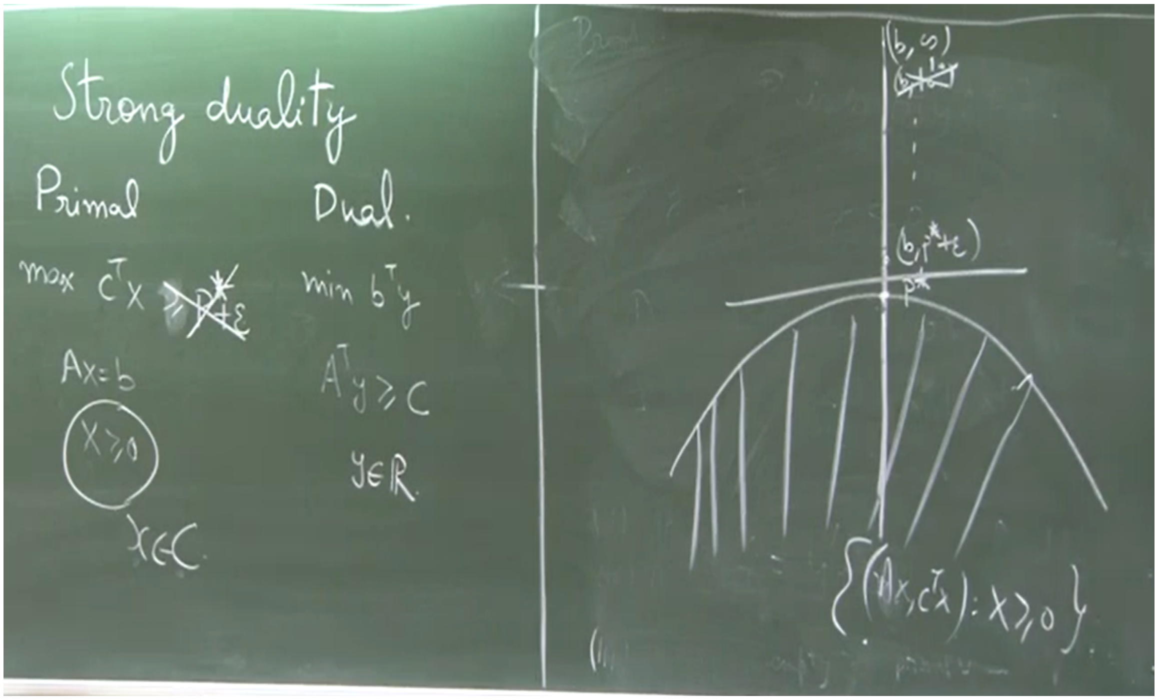


So, what do I know? If A X equal to B then C transpose X is less than equal to B star. Other way to say it is can you find a point which is not part of this not in terms of X. Very good B comma P star plus 1 right this is not part of this agreed though this is a good question if you have done a discrete structure course in previous semester, but if you are looking at linear programming you do not want 1 here you want epsilon greater than equal to 0 right Y 1 this is a continuous domain. So, we want to say that any B comma P star plus 0.0001 B star B comma P star plus 0.0001 is not there in this cone. And now if I show solution for any of the epsilon here then I am done. So, this is the plan once I know that this point does not exist here I can apply Farkas lemma and using that I will construct a solution for the. So, let me just write down the idea. So, yes when you look at it see here it was quite clear how this was hyper plane separation question.

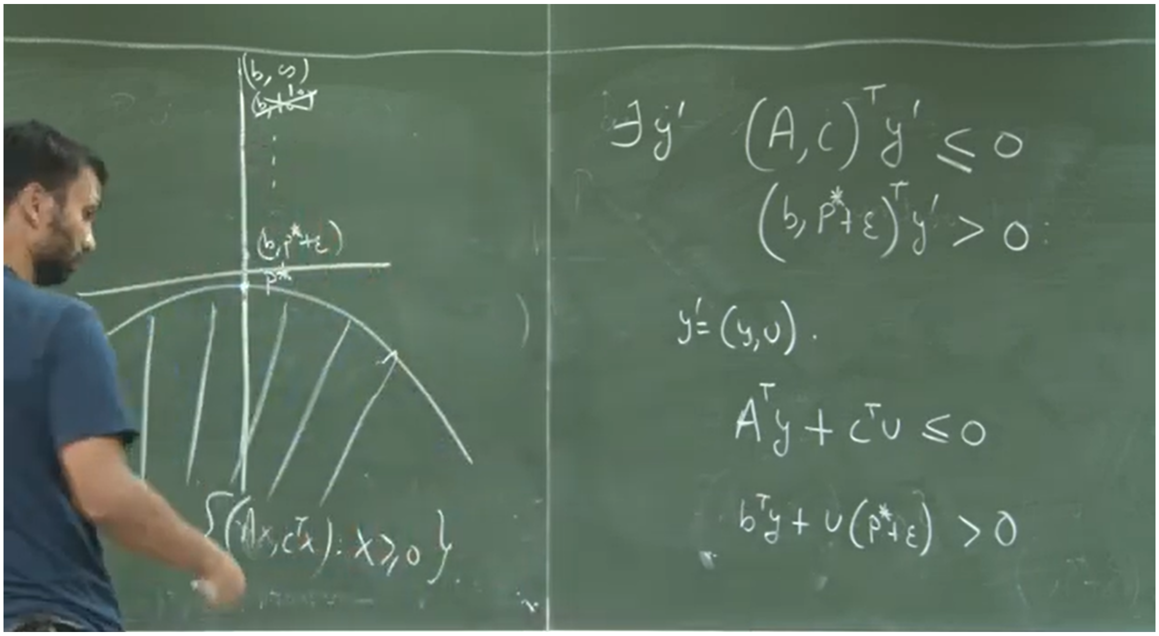
What you have to realize is that you can actually write your objective function also as a constraint you can write or is this feasible right. So, you can apply you can look at this in one dimension and apply Farkas lemma, but let us not worry about this let us see the straight direct proof we are just going to apply we are looking to going to look at this cone. Now, ideally there should be vectors here, but it is understood this is a matrix I am telling about the columns of this. So, columns of A and C they generate a finite convex cone right. And we also know this is the point outside the cone.



So, one way to kind of understand it is my y axis is going to be this last axis. So, I have this region this is this is my feasible region not feasible region, but x greater than equal to 0 what do I get here right. Now, this is the line where let us say this line represents B and this is taking different values at the last coordinate. So, this is in some sense B comma infinity this is B comma 10 to the 10 or whatever you know and this is B star. So, I have rotated my feasible set.



So, that I am aligned with this axis right and here the feasible region is going to be this line, but that does not matter what I am going to say now is that if you look at. So, I do not know probably 10 to the 10 is here I do not know. So, I do not want to write 10 to the 10, but B A comma very very large value, but. So, now if I look at this point is going to be outside the feasible region and if that was the case there would be a hyper plane separating it, but that hyper plane separation will give me a dual solution. So, that is why there cannot be any gap between it all the feasible solutions of dual were giving me upper bounds, but if the upper bounds were not tight then there is a gap here, but the gap allows me to give a better upper bound that is the idea.



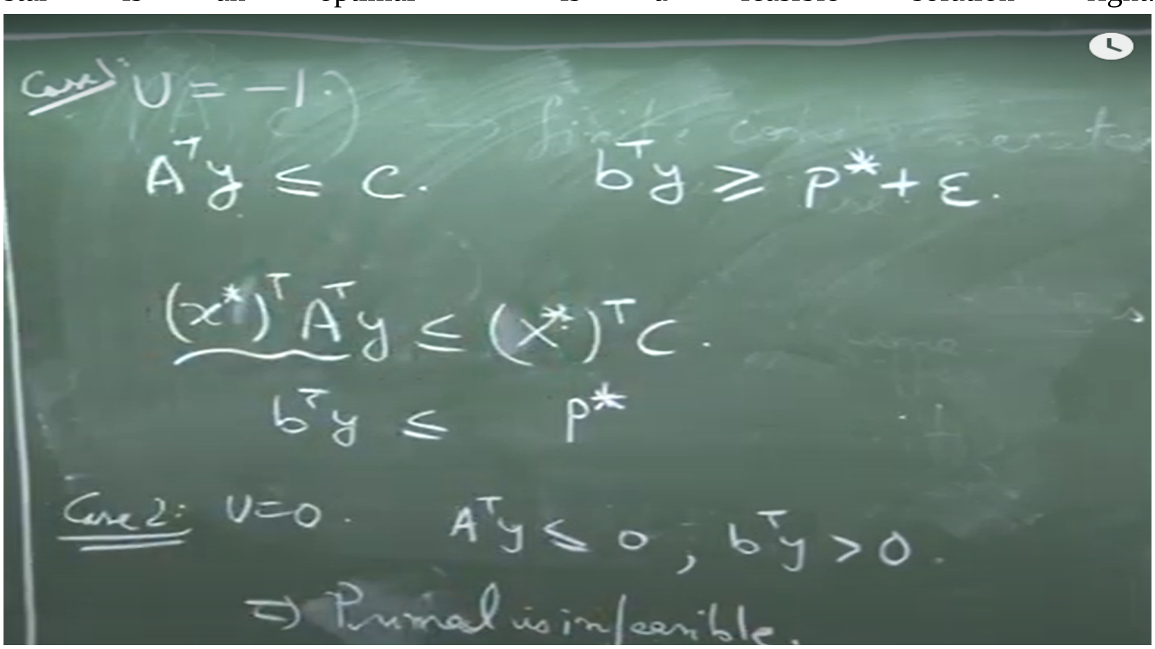
So, let us apply Farkas lemma since you guys are sleeping you will apply Farkas lemma I will not wake up what does Farkas lemma says there exist a y such that or there exist a y prime such that. Let me just see if this is the sign I want or I want to this is fine and obviously, if you want to reverse the sign you can multiply by minus, but there is a y prime right and now obviously, 1 of this coordinate is special. So, I would split it with this special coordinate. So, let us assume y prime is y comma u sorry y is any general I am just writing the name I am calling the first n coordinates of y prime as y the last coordinate as u yes you can check that it is just matrix multiplication and you can check that and I am going to write it for you this is going to say that A transpose y plus and why did not I apply a transpose here it is a number u is a number remember this is 1 dimensional multiplication p star plus epsilon is a number this is a matrix this is a vector inequality this is a number oh sorry this is also finally, this is a number A transpose y plus C transpose no sorry this is a vector inequality this is a number inequality right because b is a vector u is not right ok good. And now obviously, just to make things simple now even if I multiply y by any scalar right still these inequalities will remain true by a positive scalar right.

So, that means I can assume that my u is equal to is that sorry very good it is ok right. So, since I can multiply u by any scale or actually I multiply all these equations by any scalar constant that means I can make sure that my u is either 1 or minus 1 or 0 does that trouble you right you have some number it will be either be a positive number then you can reduce it to 1. So, negative number you can reduce it to minus 1 if it is neither negative nor positive then the only possibility is 0 you do not need to multiply it by anything right. So, you can restrict it to this case and why I am looking at this case is remember if u is equal to 1 right sorry I probably want to switch these. So, u equal to 1 or u equal to minus 1 this starts giving me dual feasibility oh I am just simplifying my analysis by saying that.

So, now, I am instead of saying that u is some any number and then keep dividing by it ultimately what do I want I want a dual feasible solution right. So, you see that this is a dual feasible solution of that form with u extra this is almost a transpose y kind of greater than equal to c right. So, if the things are right u would be good by the way there is no transpose needed here u is a number. So, you so probably it is the column of A transpose I was talking about one of this, but you will those will match I do not want to do the hard work, but basically yeah. So, you have to just see whether this implies this is the cone generated by the rows of A or the columns of A that you have to see or yeah it is c transpose or whatever.

So, you have to see, but this will match like this is not it has to kind of one more dimension. ok, great. So, the neat of the problem is here how do I use these to give me a dual feasible solution ok good. Let us say U is equal to minus 1 then I get that A transpose y is less than equal to c sorry c is a vector this is a vector in quality. So, probably it is c transpose, but again you have to match dimensions right.

So, A transpose y is less than equal to c, but from here B transpose y is greater than B star plus epsilon correct. This is dual feasibility, but in the opposite direction and you remember our Meta thing the Meta equations which we had written. If the dual feasibility in the opposite side then we should not get upper bound we should get lower bound, but this is giving an upper bound there is some issue. So, this is going to give us a contradiction. We just take this multiplied by x star correct what is this? What is this? x star is an optimal is a feasible solution right.

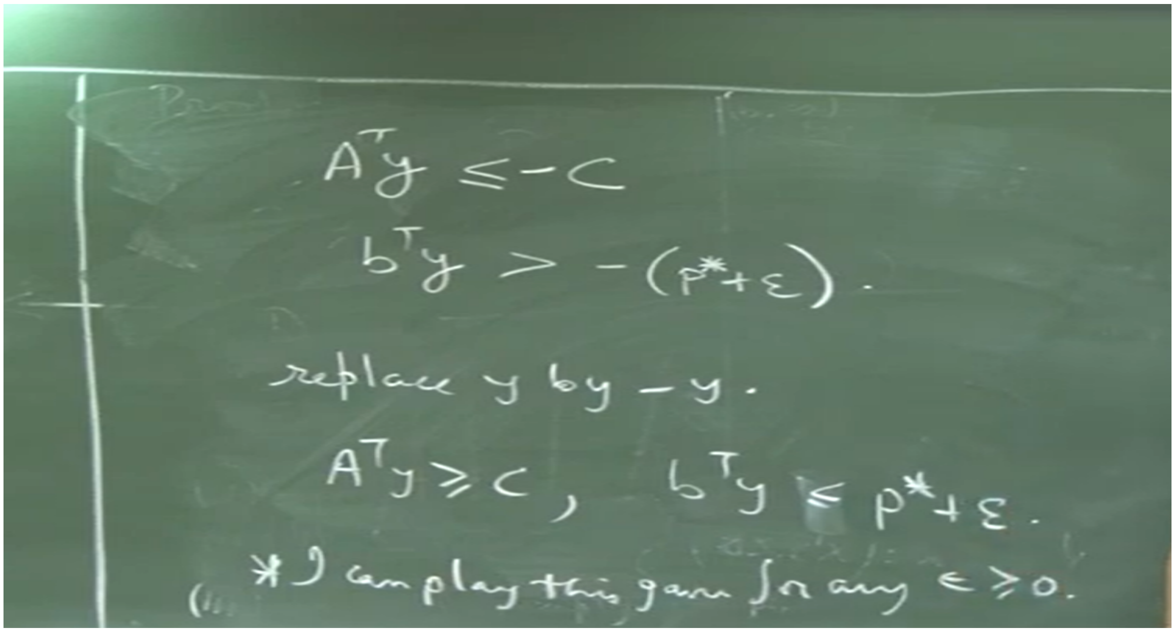


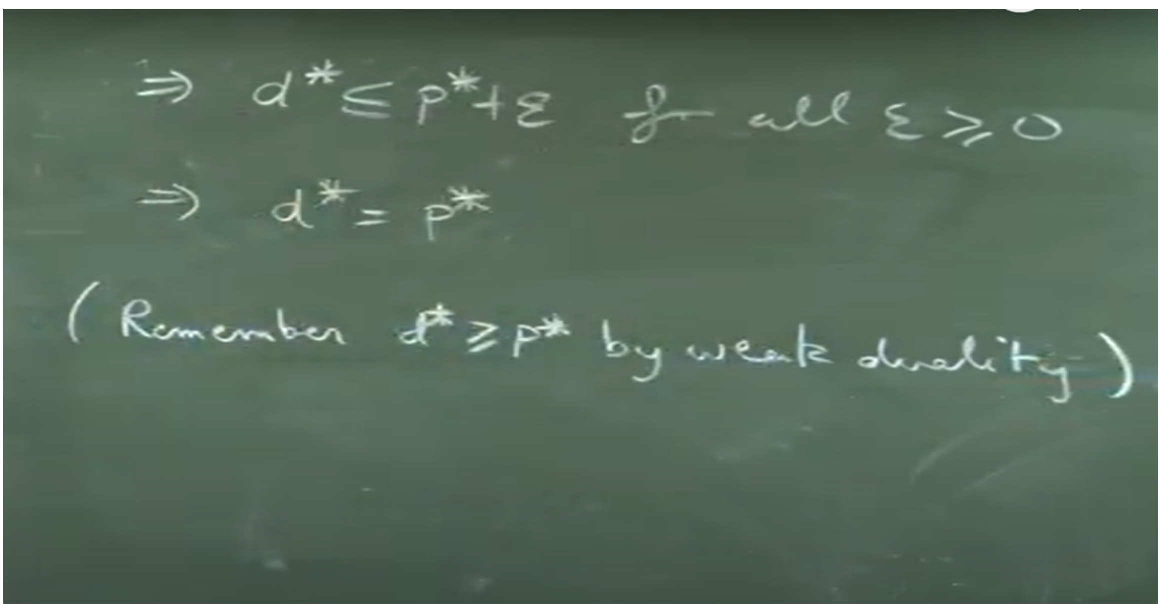
These are basically I have applied the Meta inequality here right. So, in some sense this is saying that your picture will be oriented in the correct way. I am just giving you intuition, but it is basically saying that u is equal to minus is not possible. So, it is basically saying that this thing cannot go in this way that is a contradiction ok. So, final thing has to go up that is the kind of that is what this all calculation intuitively say ok.

So, case 1, case 2 u equal to 0 you will tell me what happens at u equal to 0 do it on your paper. Pan paper why case u equal to 0 do you get a dual feasible solution? You have these equations I just put u equal to 0 here then, but we have already assumed that primal was feasible right. So, u equal to 0 is also not possible. Again intuitively this means that this has to be a non trivial picture in this dimension. What does u equal to 0 mean? u equal to 0 means that this hyper plane it has nothing in this dimension.

So, it is in this that the picture in this dimension is kind of trivial it is inside that. So, all those weird things are not happening. So, these are again cases which we are ruling out by saying that oh this picture actually looks like this everything is feasible then I am going to have some meet here it is not like these are squished inside n minus 1 dimension subspace or something none of that is going to happen. What should I erase? I can erase this. So, if u is equal to 1 a transpose y is less than equal to c minus c good.

Thank you and u is 1 ok. What do I want to do? I want to create a feasible solution of the dual. Looking at this equation how do I create a feasible solution? Replace y by minus 1. And I can play this game for. So, that means this implies d star is less than. And now there might be people who are very familiar with analysis and you might ask oh what if infimum supremum and all those things look at the node there is an extra reading it basically says we are in the nice world with this linear optimization and everything we know that the sequences will converge.





So, we are ok. We do not need to in linear programming we will never talk about infimum supremum you always talk about min max. What is the first property of duality? Any feasible solution will always give you a value higher than b star. Proving strong duality never forget weak duality. So, we know that d is definitely going to be bigger than b star. The question is is it equal? So, this line is just for the if.