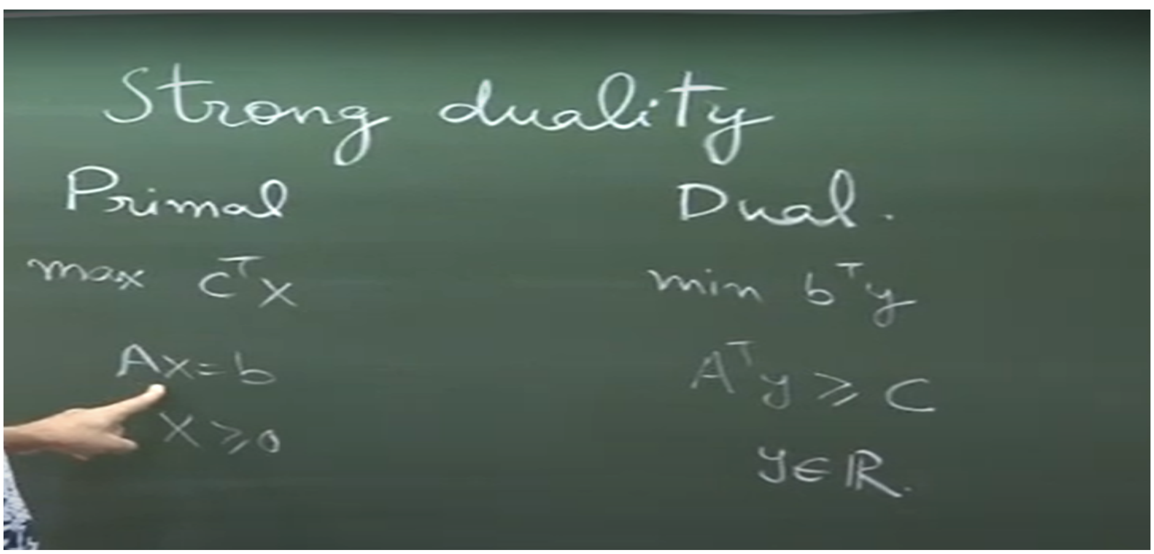
Linear Programming and its Applications to Computer Science

Prof. Rajat Mittal
Department of Computer Science and Engineering
Indian Institute Of Technology, Kanpur

Lecture – 28
Strong Duality

Welcome to the another lecture on linear programming. We were looking at duality theory and the perspective we had was giving bounds. If a linear program is given to us in maximization format, one way to give lower bounds is by showing a, how do you show a lower bound here? A feasible solution on this side right and then the dual allowed us to construct another linear program such that any feasible solution is an upper bound here. And this is kind of symmetric in the sense that any solution here is a lower bound here, any solution here is an upper bound here. And in an ideal world we would want that the best solution here is actually the best upper bound. In other words this optimal, this optimal is equal.

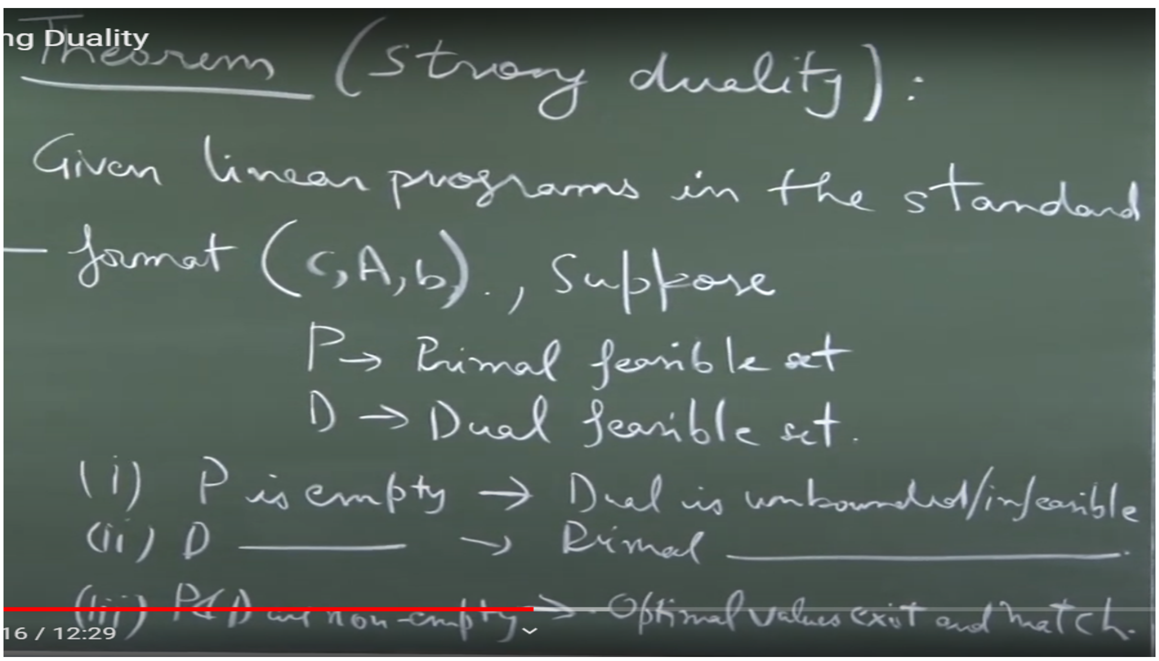


And since we live in an ideal world these two things are going to match. We do not live in an ideal world, but these things are going to match right. So, this is the this can give you the strongest upper bound possible right, but there was a technicality right. The technicality was about feasibility, infeasibility.

It might happen that this is infeasible right. What we know is if this is infeasible, if this is unbounded that means there cannot be a solution here. And if this is unbounded then this is cannot be a solution here. By the way small obvious question if this is unbounded that means the optimal value here is minus infinity. If this is unbounded here then the optimal value is if this is infeasible then the optimal value here is we will call it plus infinity.

So, as just to match again this is not mathematically matching right. Plus infinity is not equal to plus infinity that is not a mathematical statement. Just to make the convention similar saying that the values match generally for a minimization problem if it is infeasible we say that the optimal value is positive infinity. For a maximization problem if it is infeasible then we say that the optimal value is negative infinity. In some sense intuitively it makes sense right that as soon as you start getting solutions here it is gets it is it starts rising from minus infinity it can go up and up and so on and so forth.

So, now with all this actually I can write the strong duality principle very clearly. So, this will be our formulation of strong duality theorem. Then assume that you are given linear programs in this format. That format is this that means you are given C A B that is all right. You are given these suppose P is the primal feasible set similarly d is the dual feasible set.

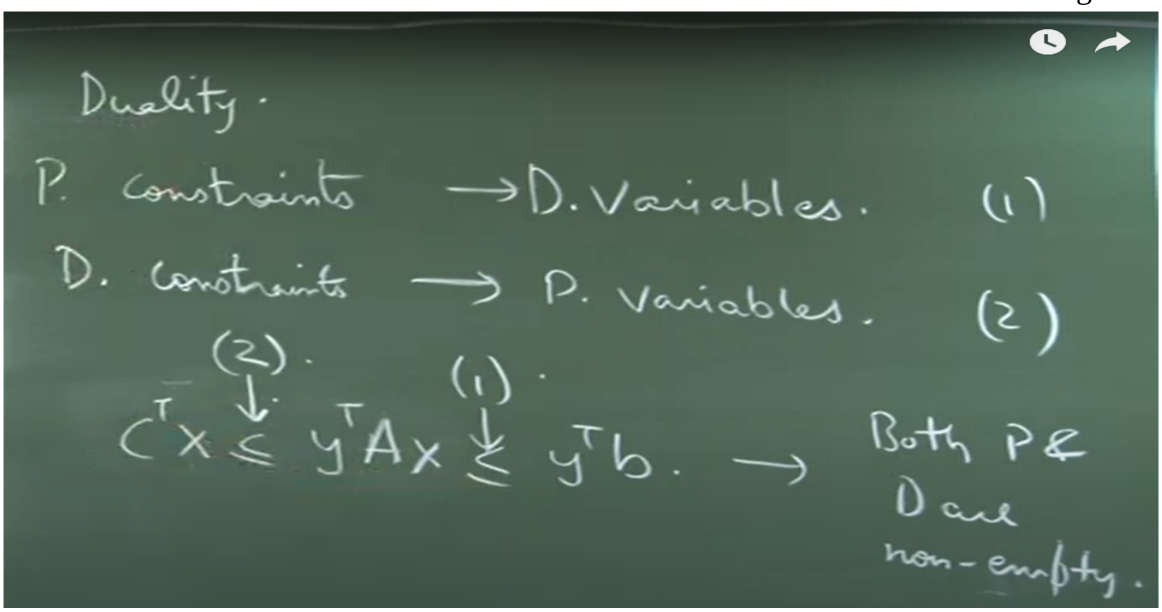


So, this is the notation again I am writing this for this particular standard linear program, but you know it does not matter. I can choose another standard linear program here get a dual for that and a similar theorem can be written like this. This is just to make it concrete, but the strong duality obviously holds for any linear program. And we saw last time how to take the dual right we do not have to convert into standard form memorize this and do it right we can take it directly right we will talk about it, but yeah. So, if P is empty this implies dual is no dual is unbounded or infeasible.

Similarly if D is empty these are the boundary cases and the interesting cases P and D both are non empty and this is mostly going to be the case. And if this is the case then optimal values exist and match sounds good. And if you remember how did we take dual. So, just to remind you this equivalence was very important primal constraints corresponded to dual variables and dual constraints corresponded to primal variables right. And then we think of what elements should I multiply here.

So, that I get a lower bound here right and this dual format will always look like this. This will depend on dual constraint or primal variable. So, this is because of. So, when you are given primal your variables the sign of variables are specified you want to figure out the sign in the dual constraint this will tell you how to. If X is positive then Y transpose A should be more than C if X is negative then Y transpose A should be less than C if X is no constraint then Y transpose A will be equal to C.

There is no other way otherwise you cannot write this right you want to impose constraints on your dual constraint and primal variables. So, that this is always true. Now, when you look at this here I see primal variables will be given that will already always be given to me. Then I can use the dual constraint I can specify the dual constraint such a way. So, that this is specified similarly this is done using B my primal constraint will be given.



Suppose it is given that AX is equal to B then whatever be Y this will always be true. This is the case here since AX is equal to B I have no constraint on Y. So, this you if you know this framework you can figure out the exact constraints by yourself you do not have to memorize this is the dual no just the idea is you want to take linear combinations here. So, that we get something it looks like this what are the exact things they will follow from this standard set of inequalities this will determine and this is very clearly this specifies this correspondence why this correspondence is there. So, using this you can always take the dual then again, but if you want to write this equation in some sense you always assume that both P and D are non empty this one equal here.

So, yes this will be equal in this case, but I am writing this equation in general when you will be taking dual and trying to emphasize the point that depending upon what is the constraint between X and B I will keep Y's. So, that this is satisfied it might be strict there are cases when this might be quality this might be quality I do not know it is, but general format is this. So, this is this meta inequality which will you use to figure out exactly what will happen in your particular primer questions. So, it is always risky when I ask any questions and you do not because then I am going to ask questions, but that I will leave for some time, but in the end I will ask questions if you are all about this, but did. So, now we understand the intricacies generally primal and dual value will agree except both could be infeasible it could happen that one is infeasible other is unbounded these are the cases.

So, and I have probably told you before that and today we will see that the proof is basically based on Farkas lemma right and you will see how why Farkas lemma is like elementary duality and then elementary duality is row rank equal to column rank or the null space of A transpose and column space of A something which you did in the exam right. So, all these are some formulations of duality.