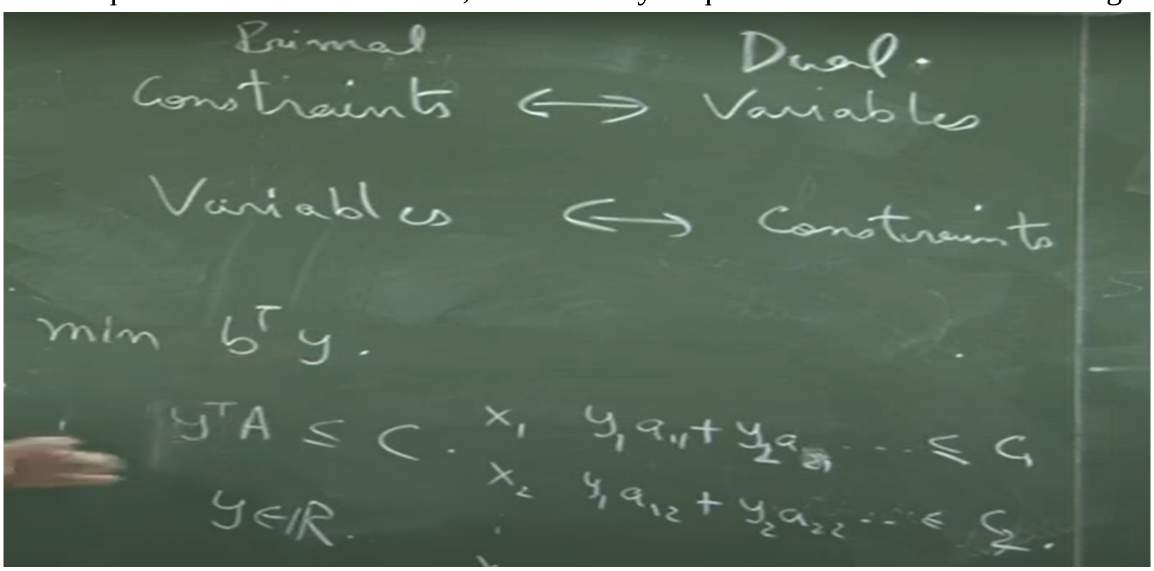
Linear Programming and its Applications to Computer Science

Prof. Rajat Mittal
Department of Computer Science and Engineering
Indian Institute Of Technology, Kanpur

Lecture – 27
Examples of taking dual

Second thing is, now I took this program, took the dual of it, I got this. But now this is also a linear program, there should be a dual of it. If you take the dual of this, you will get this. How you take the dual of this? So one way is you convert this into standard form, then you know how to take the dual and do 1000 things. But the better way and the way in which you should always take the dual is keeping these things in mind. So let us take the dual this way again for this.

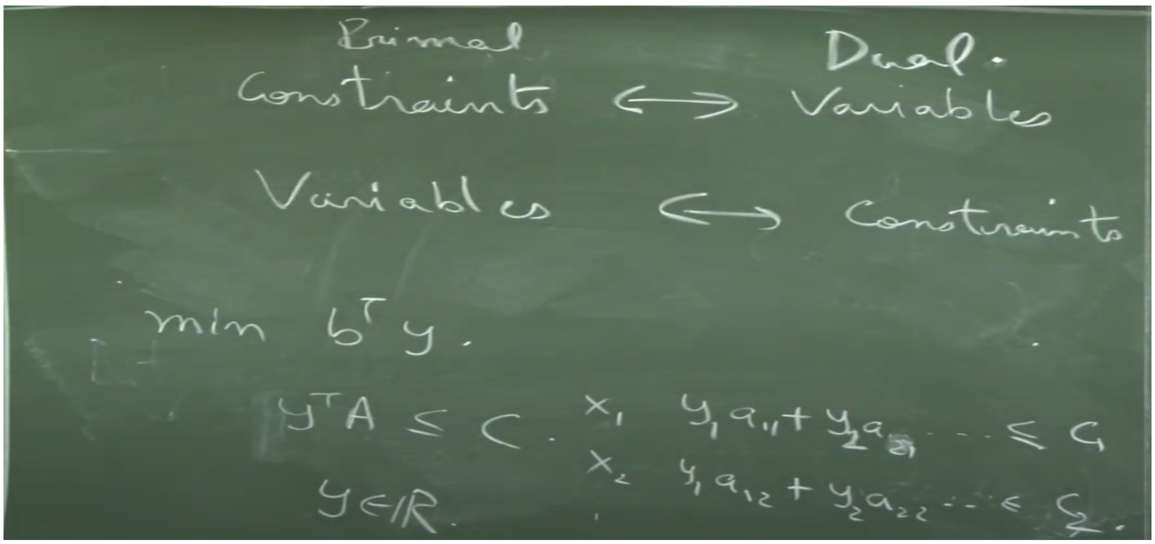
And you should be awake and help me out otherwise we will do some mistake. And then you will be staying here for much longer than you want, ok. So Y transpose A less than equal to C. This means, how many equations are there? N right.



Now I need to multiply each of these equations with a constant. So that I get something which is, sorry, depends is your C a column vector or a row vector. So let me do this. Sounds good? y1 a11 plus y2 a21 ym am1 should be less than C 1. That you cannot, I hope argue.
Now so if I want to take the dual of this, how many variables do I need? Because there were M equations. So I want to maximize something which is going to be a function of X. There will be such that and then condition on X.

So first thing, this is a minimization problem that means I want a lower bound. So the linear combination should give me something which is less than the linear form here or bigger than the linear form here. Exactly so that means we need equality. We need to get this linear form exactly. If I get a higher value than this, it is not going to happen.

I will not get a lower bound. If I get a linear form which is smaller than this, still it is not going to tell us anything because Y 1, Y 2, Y 3 are all reals. So I should get equality in the constraints. Depending upon what is your variable, the sign of the constraint will be decided. So I will get and now remember this is less than equal to C.



Right? So what will be the value? If I am multiplying everything by X, I will get C transpose X on the right hand side. So I want to, my lower bound is C transpose X. The value of the linear like it was B transpose Y right because X 1 multiplied by C 1, X 2 multiplied by C 2. This will be my 15 or whatever I came right. That would be X 1 times C 1, X 2 times C 2.

So C transpose X. What is my constraint? X 1 times A 11 plus X 2 times A 12 plus X n times 1 A n should be. This one small thing. What should be the sign of X? X is a variable in my primal.

That means its sign will be decided by the constraint in the dual. So now if you think about this, if I multiply, so what will happen? If I multiply with the negative thing here, the sign of the equality will be reversed. I want whatever I get, I want C transpose X to be less than D transpose Y and that will only happen if I multiply the inequalities with the positive thing. So I want X greater than equal to C. Lower bound? Because you want a lower bound.

That is the whole premise. Now this is a minimization problem. It is easy to give upper bound. You want to give a lower bound. So how do we get a lower bound? And the easy proof that it is upper bound, lower bound or whatever is this chain of inequalities.

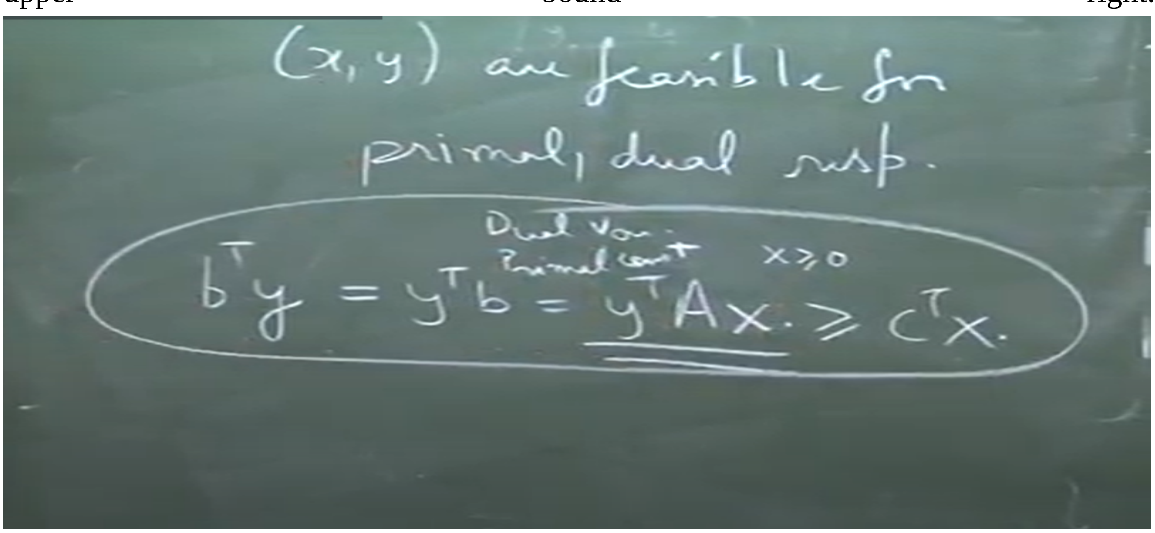
Let us say X and Y are feasible, dual respectively. So now you have B transpose Y or this is just fine right and then B is AX for some X and this Y transpose A is greater than C and this inequality will only happen because X is greater than equal to C. I am saying for this to work. And here also if I had multiplied by something negative here, then the sign of the equality had to be reversed right. Then whatever linear form I am getting, it would not be less than C transpose X.

See it is the same calculation which is I am doing here. When I am multiplying by X, I want when X multiplied by this quantity, it has to remain less than C transpose X. That will only happen if X is positive and then that compared with B transpose Y. So this is basically the thing which is going on. So again this is because of primal variable or dual constraint.

This is because of dual variable or primal constraint and this was all pretty cool because AX was equal to B. That is not going to happen in the next example. I am going to make your life more difficult. Now even this is going to have inequality or equality. What is the dual? This dual not you are not going to get this dual.

You are going to tell me what dual you are going to get. So now one way is again you can start multiplying everything right. But the easier way is the dual is going to look like this. You have to figure out the sign here. You have to figure out the constraint here.

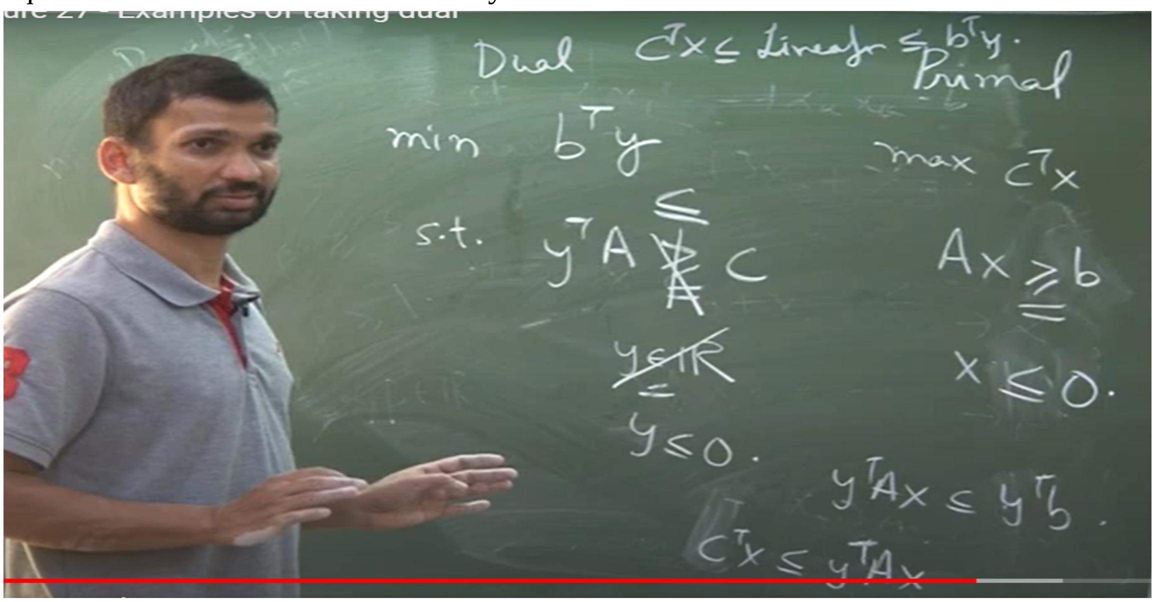
But you can remember by doing this thing that how should the sign change. First let us try to fix up the sign here right. So if I want the sign here that will depend on the sign of the here is the side which what constraint. So it should depend on this right. I want an upper bound right.



That means B transpose whatever comes here that linear form should be less than B transpose y right. Because I am going to say C transpose x less than my linear form less than B transpose y right. This is going to be my argument. I am going to multiply these get something which is higher than this. And then my argument would be C transpose x is less than equal to that term and that term is less than equal to B transpose y.

So B transpose y is enough. Now for this to happen y should be I want y transpose A x to be less than y transpose B. That will happen for which y? Negative. y transpose A x less than y transpose B because you are going to give this argument for your duality. This is your argument to duality right. This is a condensed form of duality which you were doing right.

Writing down equations multiplying this is the linear form which you get. You multiply your constraints with variables of the dual and then you apply this argument right. So, then you want y transpose A x to be less than y transpose B. So, this tells us y less than equal to 0. Now you are worried about this side.

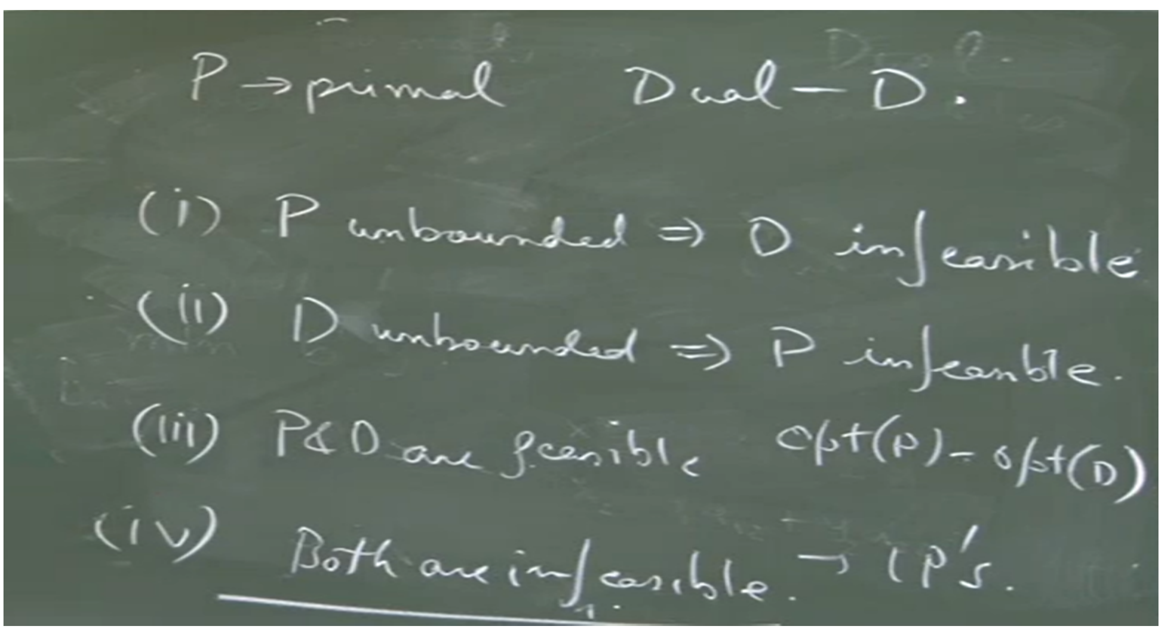


Now you want to say C transpose x should be less sign will be determined by the sign in the primal variables right. This sign with the primal variables. So, if I know x to be greater than equal to 0 and if I want this condition to be true what relation do I need for y transpose A and C? So if x was greater than equal to 0 then I would have want y transpose A more than C, but x is less than equal to 0. Minimum B transpose y y transpose A less than equal to C y less than equal to 0.

So this gives you a generic way to take duals. Write something like this write the maximization problem, minimization problem the questions are signs here and here and they that sign you can determine by looking at what you want to do. This is what you want to do you want to create a linear form which is an upper bound on primal value which is a lower bound on the dual. A linear form which is a primal I should not say it a linear form because it is a linear form with respect to x if y is a constant. It is a linear form in terms of y if x is a constant.

So I do not know quadratic form you want to create this quadratic form which is less than the dual value, but more than the primal value. And if you want to keep these that will easily tell you what signs to put, but again the amazing part is there exist a y such that y transpose B is going to be equal to C transpose x strong duality. But wait that is not true. So yes so let me talk about the technicality, but we know this right primal dual how to create these the constraint here corresponds to variable here the variable here corresponds to constraint here. Now let us talk about the technicality.

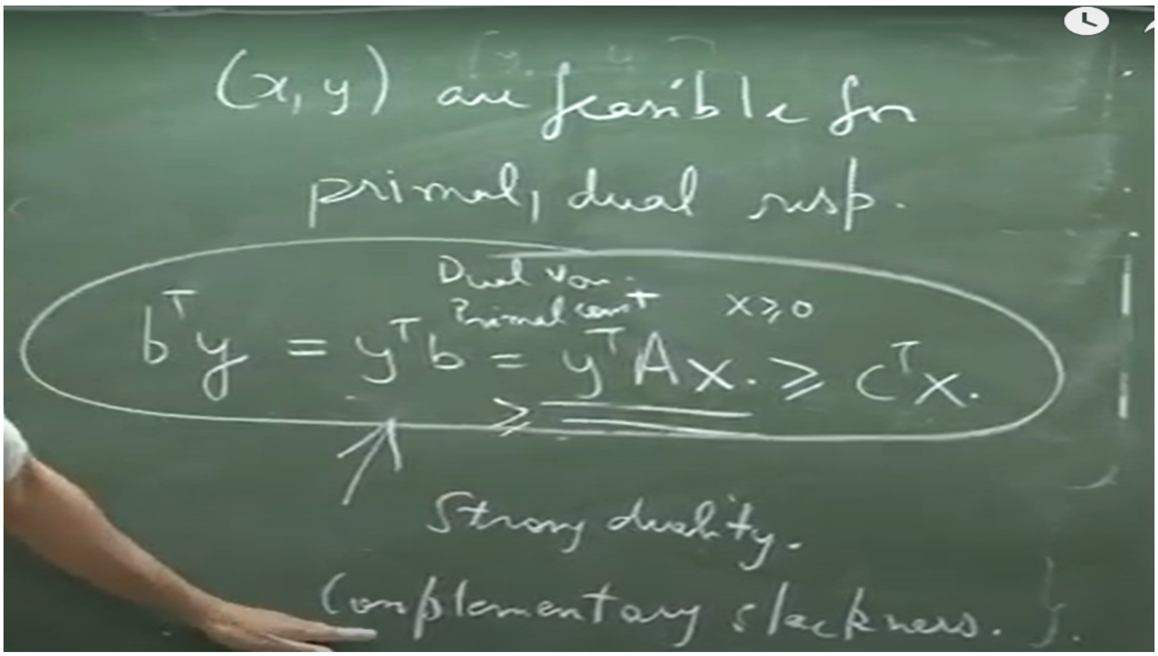
You remember the technicality? What if this is not feasible? So now let us think of it this way if this is unbounded right if this is unbounded then clearly there cannot be any upper bound that means dual is going to be infeasible. If D is unbounded primal is again by the same argument if this is unbounded there cannot be a lower bound there cannot be a feasible solution here. The most interesting case is P and D both are feasible in this case opt of P equal to opt of D. If infeasibility is taken care of then the values are equal this is the strong duality. Is there any case remaining? What do you think? Let me not keep the suspense I do not have any intuition about this it turns out there are LPs where both are infeasible.



If you are very curious do Google search for Rajat Mittal CS 655 course notes duality you will see an example there. So yes both programs can be infeasible, but again 1 2 4 this is the least useful these are the cases which you know needed in the haystack you really have to dive deep and find cases oh my god both are infeasible. For any natural

LP 99 percent probably whatever percent more than 99.99999 percent you will be in this case there are very few cases where these things happen, but this I have seen in my life. This I have constructed for lecture notes, but this is possible sounds good.

So now we are going to see what this kind of equation gives us first thing is equality can hold that is strong duality, but if strong duality holds that means everything is equal here right. If everything is equal here it tells you a lot about the primal and dual optimal solutions that is called complementary slackness and we will talk about it.



So once again in case this is so if there are X naught and Y naught for which this side is equal to this side then everything has become equal and if everything has become equal there is very nice relation there are properties of X naught and Y naught which we can infer from that that is called Complementary Slackness. This is the topic for next class.