Linear Programming and its Applications to Computer Science
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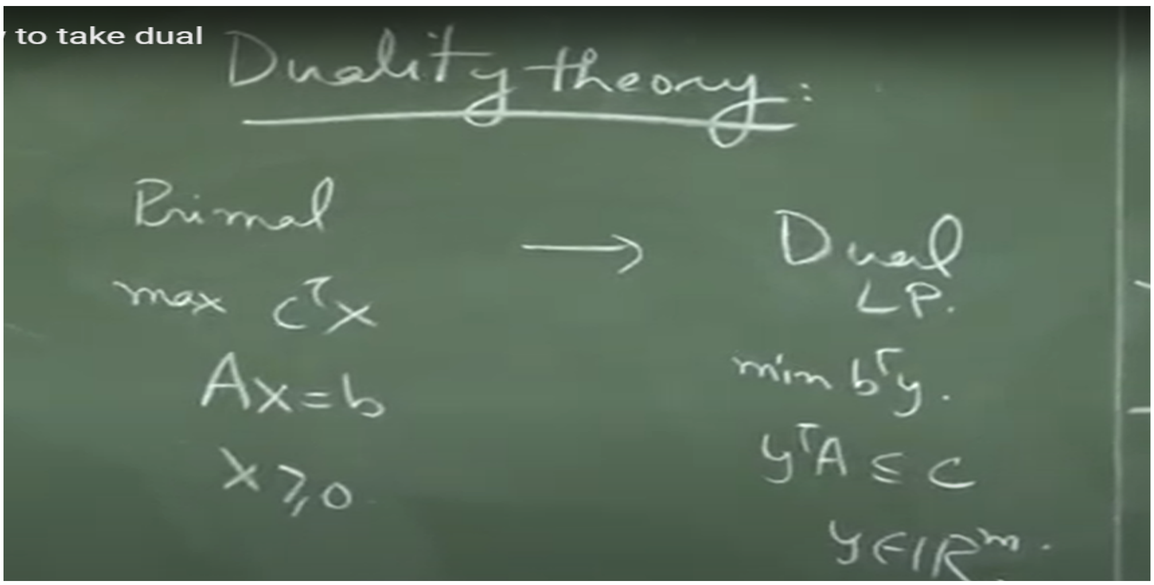
Lecture – 26
How to take dual

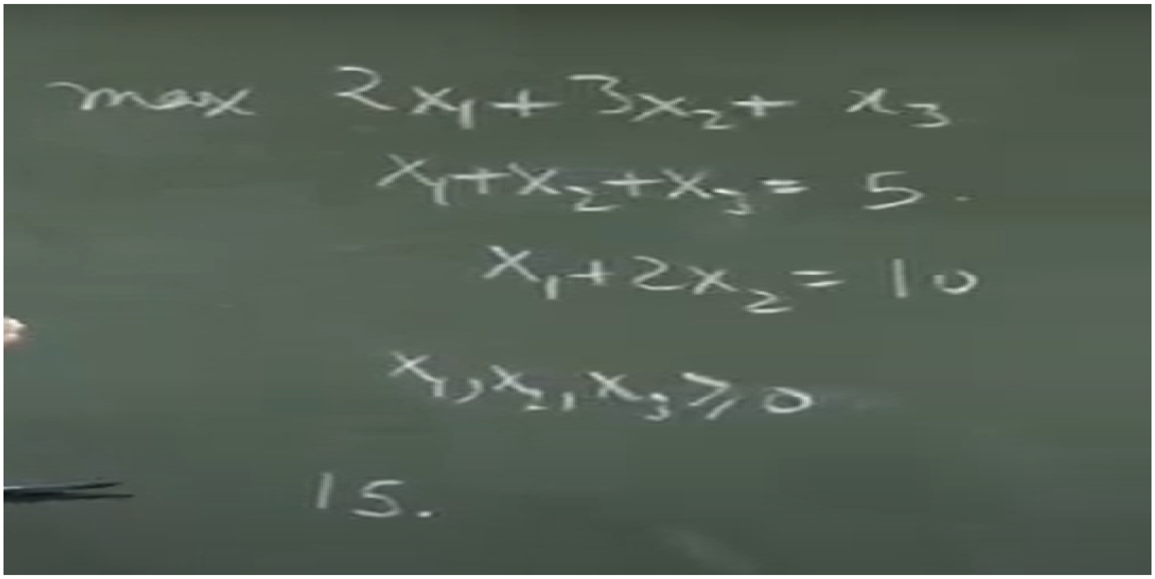
You have a linear program which we call a primal which looks like something like this and duality theory allows us to create a dual linear program. It is also a linear program and I can say what it looks like whatever is the number of equations here.



But duality theory allows us to create a dual LP such that any solution here is an upper bound here. Do you see a relation between these? It could that you do not. What is going to happen now is that I am going to talk about duality theory in a very different way in terms of upper bounds and lower bounds. We will come up with these programs.

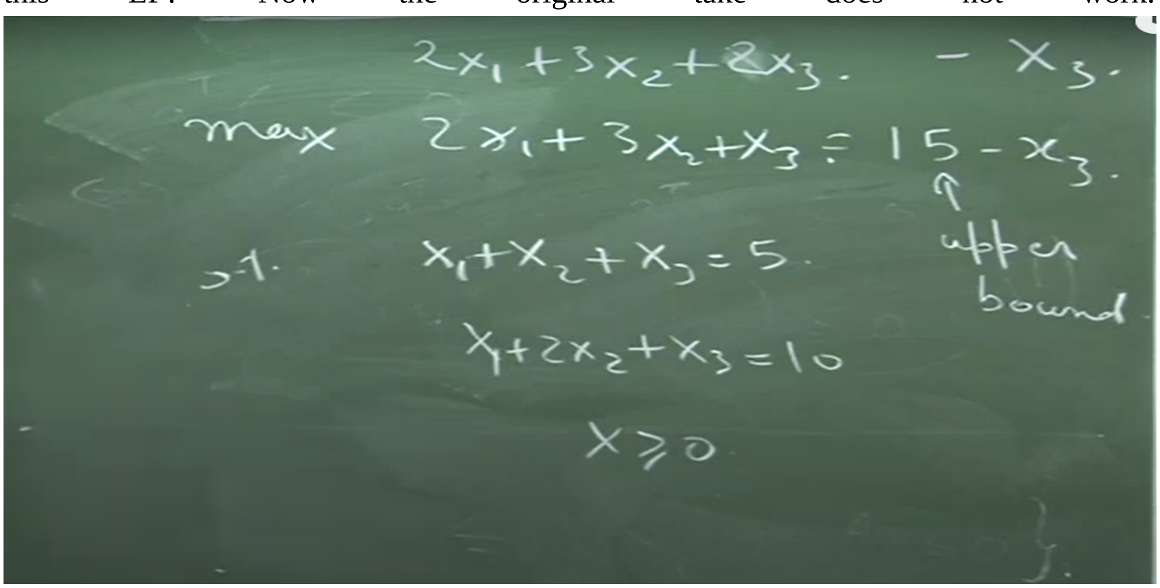
After that we will see this relation. Objective is clear. If I am given a maximization problem it is easy to give a what kind of bound lower or upper for a maximization problem what is easy? A feasible solution will give a lower bound right because the value of any feasible solution will be less than the optimum right. Now we want to create an LP so that we can give upper bound also.

T for few minutes forget about lower and upper bounds. Look at this great LP. This is not a great LP but starting LP. LP is clear. What is the value of this LP? Exactly it turns out that the value is 15 because I can just add these two and get the objective function.



Sounds great right. My claim is it is what proof you have given me it need not be 15 small technicality not relative duality, but there is a small technicality and since we are talking about linear programs it might not be 15. What have you assumed? There has to be a feasible solution. If there is a feasible solution then the optimal value is 15 correct and this technical case will keep emerging keep troubling us in duality theory. So you will remember this right feasibility is important.

If it is feasible then definitely it is 15, but there might be a case when this is I am not saying about this particular LP. That is why I knew that someone will say oh this is feasible that was not the point I was trying to make. My point was the proof you have given me needs one more thing. You guys are good at solving LP. What is the value of this LP? Now the original take does not work.

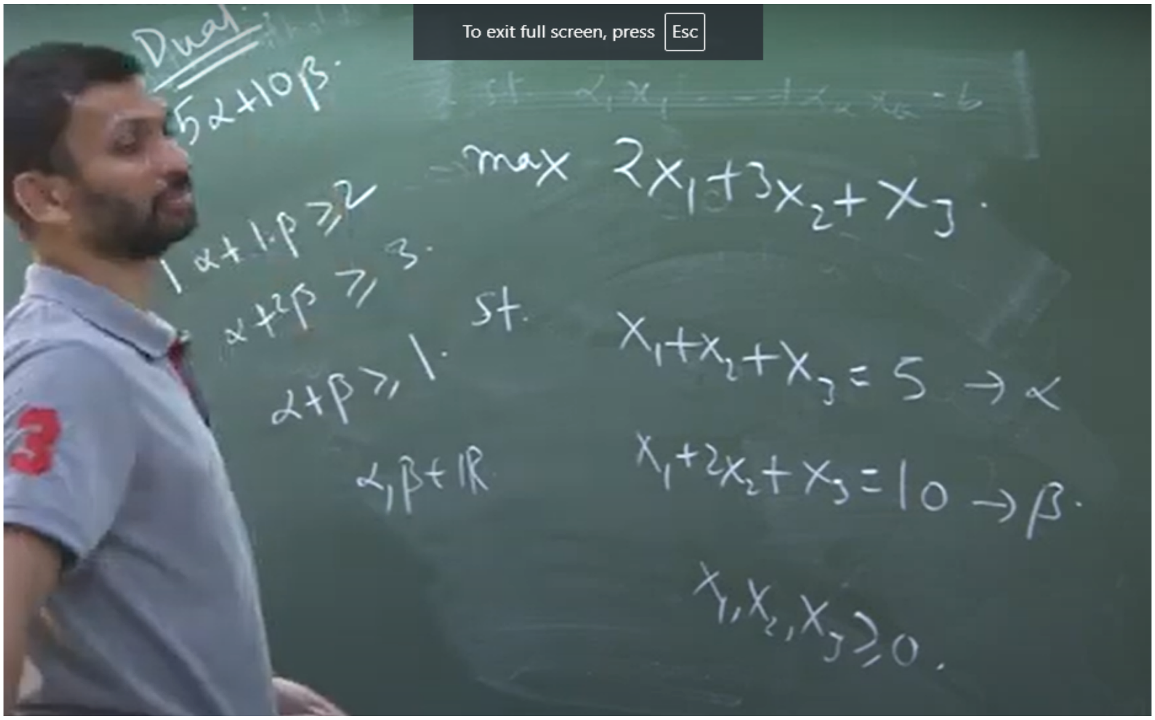


What was the trick? This plus this was equal to this right. Now if I sum them up what do I get? I get 2 X 1 plus 3 X 2 plus 2 X 3 correct. So my objective function is it is this quantity minus X 3 correct right. So now what do I know? Yes you are trying to solve it, but that is not the point I am trying to make. My point is we know that 15 is an upper bound.

We have started creating upper bound right. So now can I create more upper bounds? What is the general technique now of creating upper bounds? What did I do to create an upper bound? Because you know interested just in this LP right. You want to take an take a dual for general LP. So what was the idea here? So constant in some negative terms or other way I want to construct a linear form which is entry wise bigger than the quantity here right. And how did I construct this linear form? It was added, but do I really need to add only or can I generalize it more? I can take any linear combination of these two.

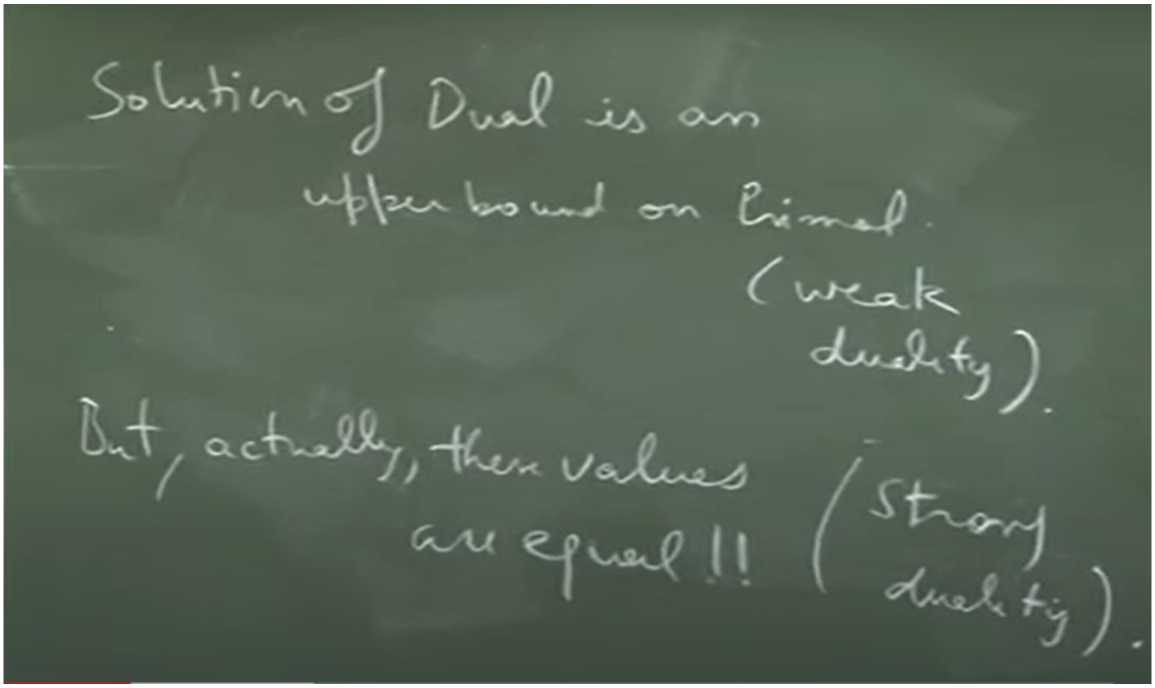
I take any linear combination of it such that I get something which is bigger than the quantity here right. Then right. This is probably a bad example because I think this two will show that x 2 is 5 and then actually 15 is the best value, but anyway. So the idea is you take alpha and beta such that 1 times alpha plus 1 times beta is greater than 2 right. 1 times alpha plus 2 times beta is greater than I want to make sure that the that when I take the linear combination the coefficient of it is higher everywhere.

So I compare the coefficient of x 1 I am comparing the coefficient of x 2. Alpha plus 2 beta is the coefficient in my linear combination this should be more than. What do I know about alpha and beta? Positive, negative could be anything right. Yes, alpha and beta are the constants I am multiplying these equations with. This is generalizing this strategy of creating an upper bound.



I say that I multiply the first equation with something, second equation with something so that I get a linear form which is higher than the objective function entry wise right. Sounds good? If I multiply by alpha and beta what is my upper bound? If this conditions happen and what is the upper bound which I get right. Should I keep it as this, should I minimize it, maximize it exactly I am creating an upper bound I want the best possible or the tightest possible upper bound. This is the dual of this program you got your first tool. And again the kind of connections you will see are amazing.

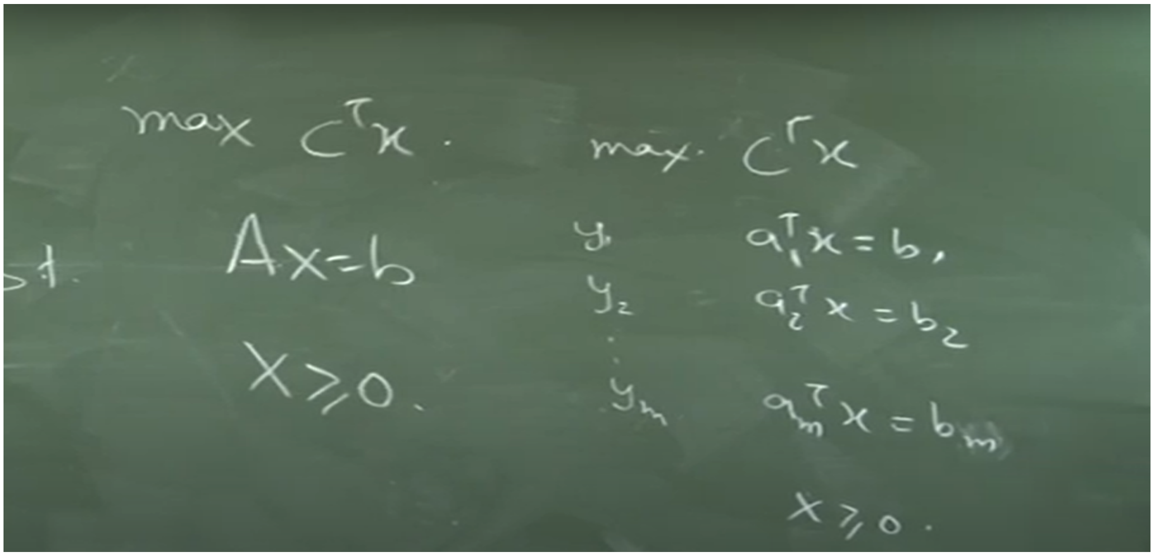
But one of the thing which I want to once again emphasize is it is very easy to see now the way we have considered it is that any solution here will give us that these values are actually equal. There exist an alpha and beta so that you can always whatever be your LP you will always get the tightest possible upper bound. This general you know strategy of creating upper bound you were just thinking oh let us try it this actually the tightest possible. This is the amazing part of duality. It is not surprising that you are getting upper bounds the strategy was designed to give upper bounds, but why is it tight? So this is what we call generally and I will keep repeating this terms weak duality, but weak duality.



And if you want to feel the intuition what do you think is the reason of strong duality hyper plane separation. Yes it is basically saying given a convex region a point outside however close that point is, but if it is outside then you can always find a hyper plane separating it. That hyper plane is your dual solution and I will when we do the proof of strong duality you will exactly see this. It is just in you know one more dimension than the usual dimension that is why you do not see it yet. But the idea is this is coming from this exact point that if you have given a cone there is a point outside it does not matter how close it is you can always find a hyper plane separation.

So you have this cone of primal you have this upper bound if this is not tight there exists another dual solution whose value is better. You can keep doing it till you find the best solution that is the idea. But we will see many examples of this weak duality versus strong duality frame work. So now we have learnt the technique let us see how to apply it. What was the technique? You take the constraints multiply them by linear constant you have take the constraint multiply them by linear multipliers and then try to give an upper bound.

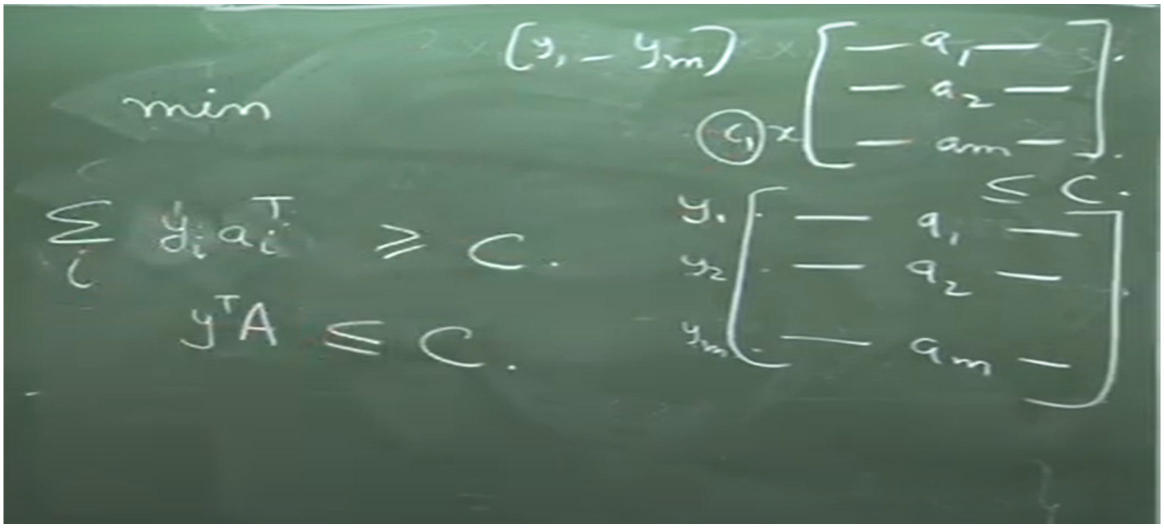
So now if this was the equation and generally now given the state you are in nobody will give you a linear program like this probably mean one of the exams just to iterate you like simplex method. But otherwise most of the time you will have these big linear programs with you know variables abstractly and you still have to write the dual in terms of these variables. So what is the dual of this? How did you come up with it? Good. Let us do it clearly there are m equations. So now I am going to have m dual variables these are the once I am multiplying by.

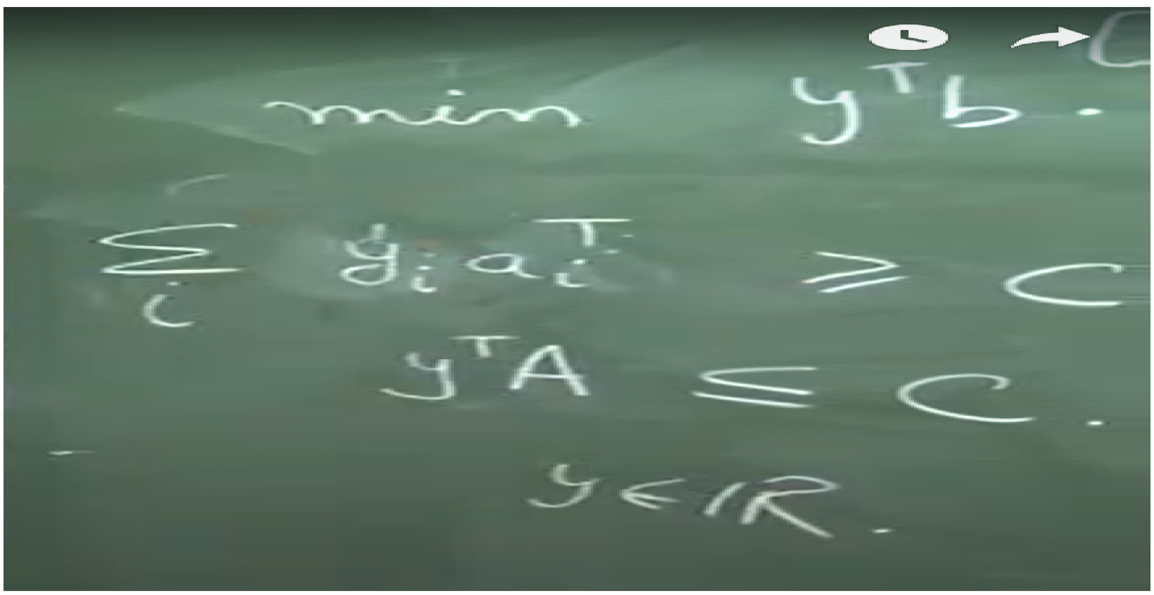


Now when I multiply by them when I multiply y 1 times a 1 y 2 with a 2 and so on they should become bigger than c. Why bigger than c? If x was less than equal to 0 then you would have come up with the thing which was less than c. You want an upper bound right? If I want to approximate this if x is positive then I will say oh it is and now this is an upper bound. If x1 x2 were negative I would say oh whatever be the value of this there is a negative thing added to it will always be this 2x1 plus 2x2 right? So then what you get is I should write it in this way how do I write it? This is a vector multiplication right? So this is a column vector everything is multiplied by y right? So this is a column vector with m entries right? Did I do it correctly? Yes and this has to be greater than here. So you want it slowly look at this matrix this entire row is multiplied by y1 y2 yn right? So what do I want? I am looking at and this is the coefficient of x1.

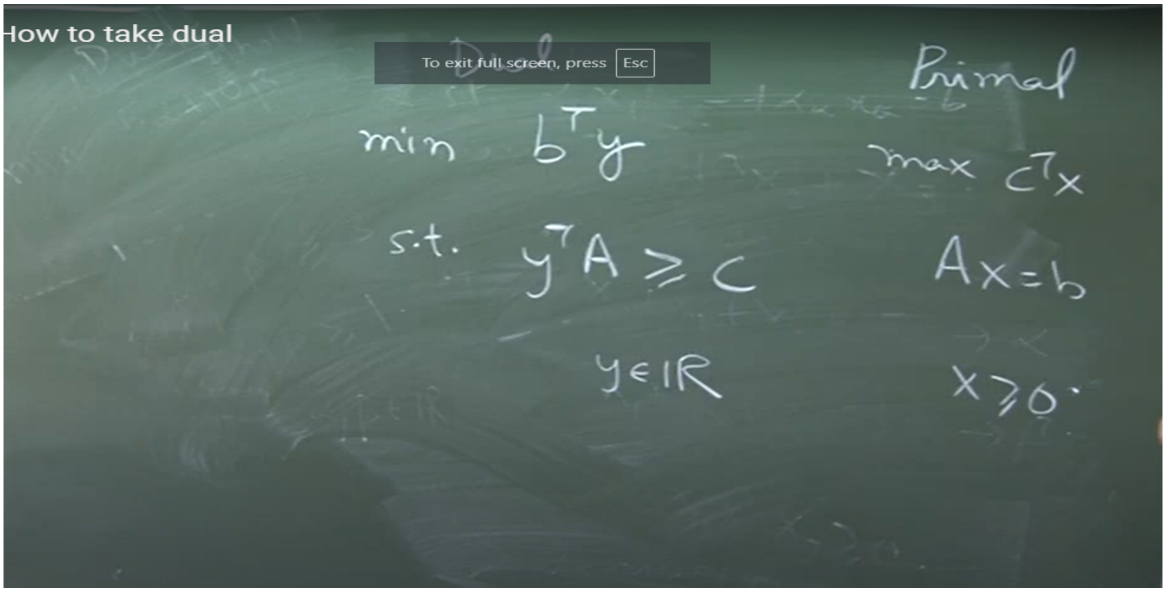
So I want to make sure that y1 times this first entry y2 times this second entry yn second third entry yn times the mth entry is more than c. Then y1 times the second entry y2 times second entry yn this is more than c2. I have just taken these row vectors and made them column vector that is all. Another way to write it is now ai is here. Now y is a vector column vector with entries y1 to yn.

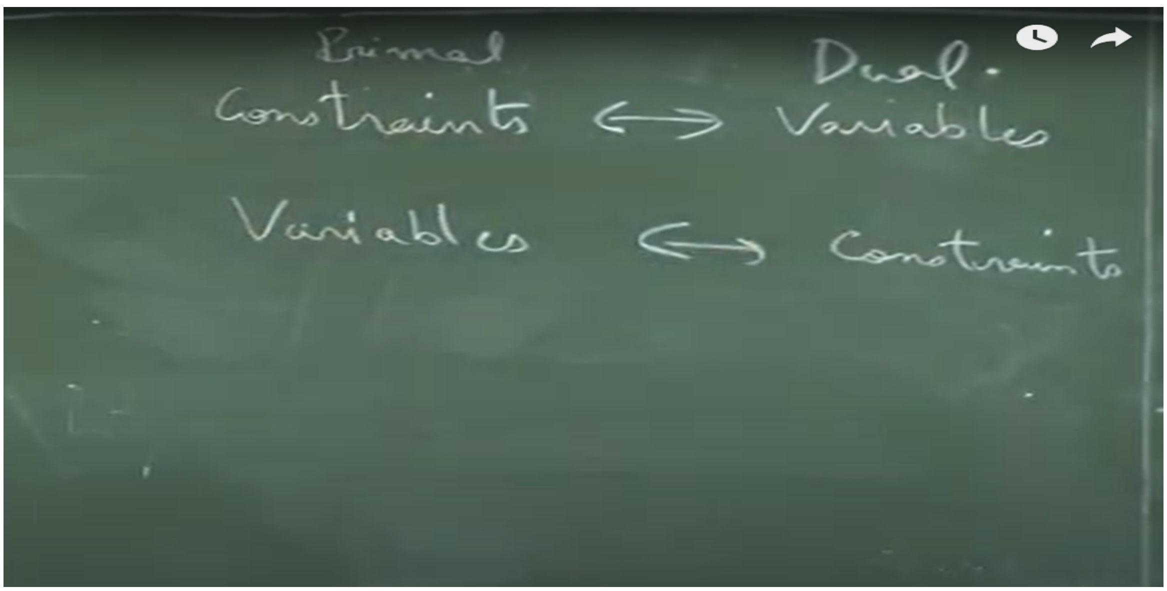
So this is saying remember matrix multiplication this gets multiplied by this this gets multiplied by this. So y1 is always getting multiplied by the first row. So these things are the same. Again since now what about the sign of y? It does not matter we could take anything right? So it is a linear combination it is general linear combination and what were? No, no c I am thinking of it as a column vector. Since ai's were the rows that is why I did a transpose here.





But c I am thinking of it as a column vector remember here it is c transpose this is a row. So this is xn that is why this is c transpose this is a row vector. But even if it was c transpose you can do it. So I would not worry about it right this transpose and getting matching the dimensions that is something which you can do. So why what is the minimization? Sure b transpose y y transpose b whatever it is.





Now there are many many things to notice here. In this combination the most important duality is constraints correspond to variables and variables correspond to constraints. What do I mean by this? By the way we did this every constraint required a variable in dual and every constraint that dual arrived from a table in the primal right. And this duality is going to be very important you are going to see it in many ways.