Linear Programming and its Applications to Computer Science
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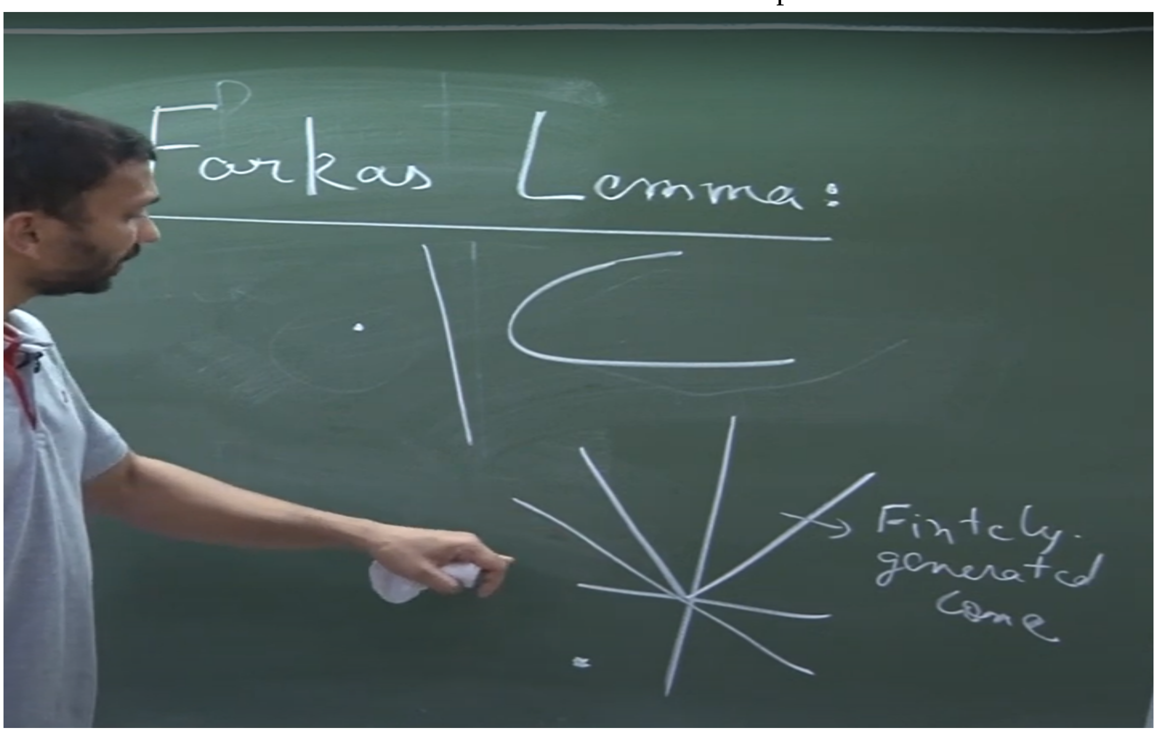
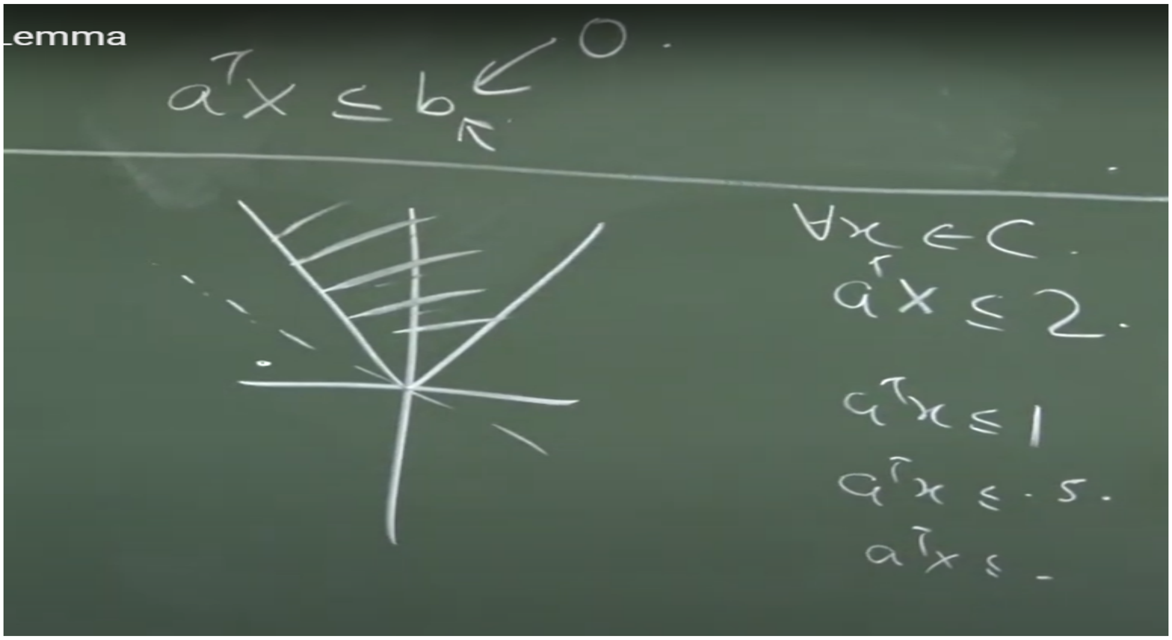
Lecture – 25
Farkas Lemma

Welcome, we are covering linear programming and we were looking at what we called hyperplane separation theorems. In pictures the hyper plane separation theorem was you are given some two convex bodies. And if they were non intersecting then we in some sense either weak sense or strict sense we could always draw hyperplane between them that was the hyper plane separation theorem right. Specifically one thing we did was if there is a point outside a convex body then there is a strict separation with the hyper plane right. And we said that there are many versions of this hyperplane separation theorem depending upon what kind of convex bodies we are considering whether the separation is strict or not strict. One of the one separation which is very useful is Farkash lemma which talks about a cone and a point outside the cone.





What it mentions is that given a point outside a cone there always exist a hyperplane passing through the origin which distinguishes between the cone. There are two ways in which it is different right. One is we have a special convex body which is a finitely generated cone or a convex cone. The other thing is that the separation is not strict in this case one of the point of the cone will lie on the hyper plane, but the hyperplane is passing through the origin.

So the b when we write for the hyperplane a transpose x less than equal to b this b ok. I will do a different proof of Farkash lemma, but actually it can be obtained from this theorem. Can you think of can you give me an intuitive reason why if you have a convex cone you have a point outside why should there be hypeplane passing through the origin which separates the two. So remember what will this give us it will give us a hyperplane with strict separation, but it might not pass through the origin. So given a hyper plane from this theorem which is a strict separation, but which is not at origin how do you convert it into a separation with a hyper plane which is at origin.

By how much? What is theta? Ok hint the answer is very simple. Suppose for a cone you are given this equation a transpose x less than equal to b. Do you think this b matters? What is the definition of cone? Suppose there is a convex cone for which you have this equation a transpose x less than I say that for all x element of cone a transpose x is less than equal to 2. My claim is for all x a transpose x less than equal to 1 a transpose x less than equal to 0.5 a transpose x whatever you want y.

What do we know about cone? Equation formed of the 0, but here I think it is a clear simpler reason which is from the basic definition of cone right. Suppose a transpose x is less than equal to 2, but there exist a particular x naught such that a transpose x is less than 0.0000. Then what do I know about x naught? What can I create a new point such that a transpose x is less than equal to more than 2? You can multiply x naught with whatever damn constant you want multiplied by 1 million multiplied by 1 billion whatever you want right. So, then that means there exist.

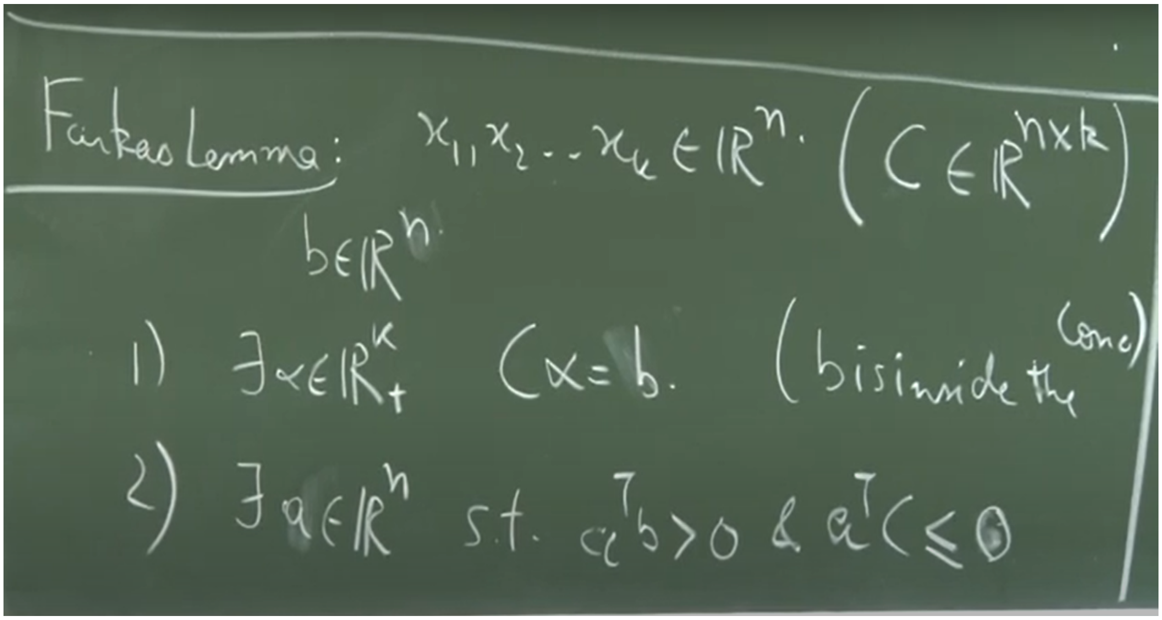
So, this implies there exist y such that element of the cone such that a transpose y is sorry this was equal to 0.01. So, if in cone there is anything which is positive I can attain any positive number. If it attains any negative number then I can attain the entire negative real line right. So, for a cone this does not matter.

So, what you are going to do and it is given as an exercise for you. Suppose according to this lemma this theorem between a point and a convex region. Suppose this gives you some hyperplane which separates this point and this convex cone just may be equal to 0 that is all. That means you just shift it to origin that is going to be your separating hyperplane. It is as simple as that and that you can convince yourself pictorially, but definitely should hit by equations.

Sounds good. So, look at the one that the hyperplane is given by this theorem. All the elements in the cone will have the same sign the entire either negative real line or the positive real line. This is going to have opposite sign obviously right. It is a separation theorem.

So, that you can do it right great. So, we know one proof of it. Let us do another proof of it and which probably comes out more naturally if we look at a Farkash lemma in a slightly different way. So, what does Farkash lemma say? Suppose you are given points x 1, x 2 up to x k right. So, these are some columns these are going to be something for which we are going to make linear combinations of and you can think of this as a matrix C or a cone.

So, what are you doing? You are just putting all these things in a as columns. So, you can make a matrix out of it I am calling that as C. Now, what is the cone? It is any positive combination of these things. So, C multiplied by a vector which is entry wise positive. And then you are given a point in the possibly in the range sorry n.

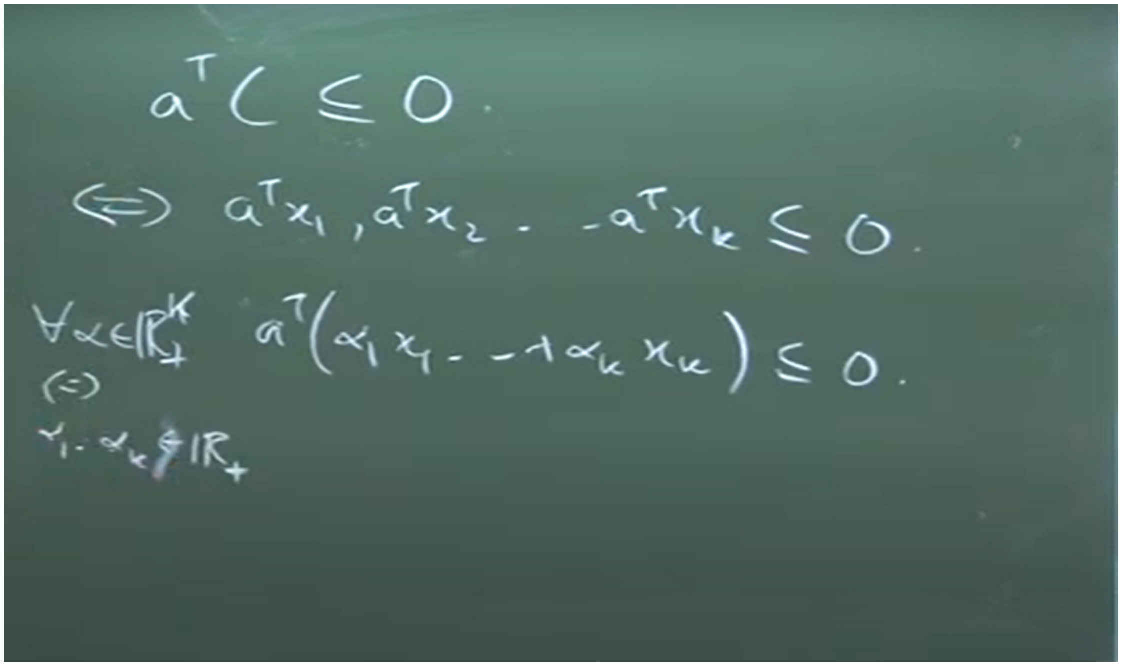


So, exactly one of the two conditions are satisfied either there exist a linear combination of these things. That means, B is in the cone right this is saying this line is saying B is inside the cone. Let us make meaning out of it right does this look like a hyperplane separation theorem? What is the hyperplane here? a transpose x equal to 0 that is the hyper plane here right. So, if this is the case exactly one of them right this is your hyper plane separation either the convex bodies intersect or you have a hyperplane separating. This is exactly the same statement either B is inside the cone they are intersecting or there is a hyper plane.

What is the hyper plane? a transpose x equal to 0. But why does this say that hyper plane separates with the cone in the point? What does this mean? So, notice what this is saying this is saying this is same as saying right. So, this is the vector in quality. What is the first coordinate of it? a transpose x 1 second a transpose x 2 and so on. So, you are saying that the inner product of x 1 x 2 x k with all a's is negative.

So, now if you take even a linear combination with positive things the inner product is going to be negative. So, that means, so for all alpha this is going to be positive. So, even though this might look like a bunch of equations this is the hyper plane separation theorem staring at you. Questions? Sounds good sorry alpha. r k these are the coefficients of x 1 to x k.

k is the number of vectors n is the dimension of the space in which they reside. So, which one do you want to be k plus 1? This alpha 1 to alpha k. Alpha 1 to alpha k are constants. So, k of them I can think of them as a vector in r to the k. That is why I just did not know alpha just a short hand.

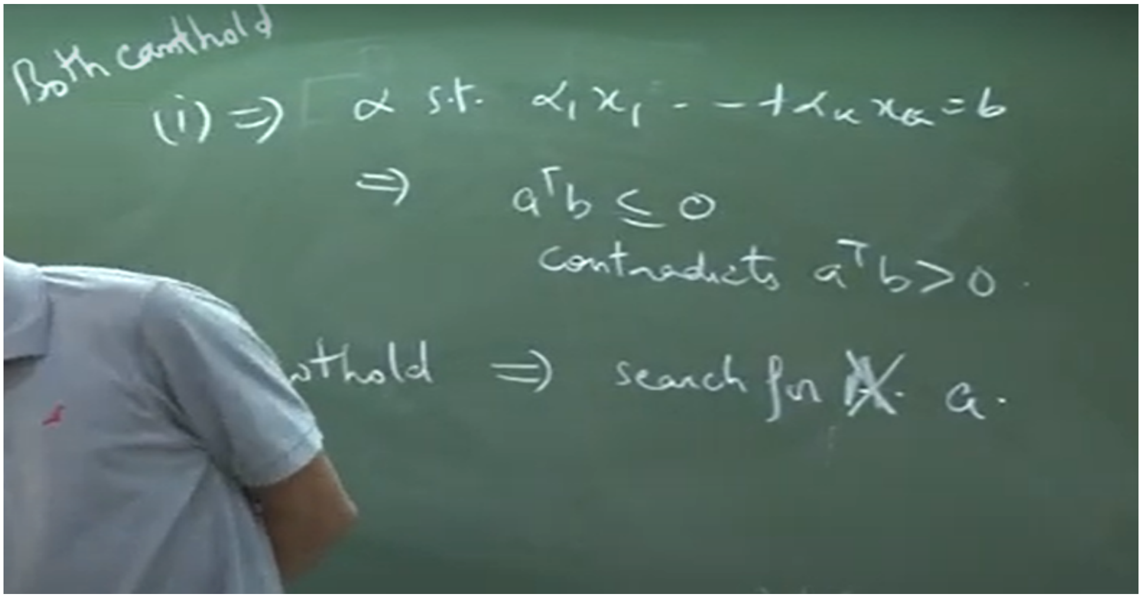


This is same as saying each of them are element of r plus k just wanted to be lazy. Sounds good? So, now we know this is a hyper plane separation theorem between a cone and a point. Let us prove it. Which direction is easy? So, what you want to show is both of them cannot hold simultaneously. If this does not hold then this holds.

Exactly how do you prove that two things exactly one of them holds. Both of them cannot hold simultaneously if this holds then this if this does not hold then this holds which is the easy direction. And I am not going to answer it. I think you need to wake up from your afternoon slumber tell me the answer.

This is not hard at all. Notice that this is where the answer is hidden. Sorry? If first holds then second cannot hold. Or other way to say it both cannot hold simultaneously that is very easy. So, now Arpita has helped you what is the proof others? Why can I clearly say that if the first holds then second cannot if b is equal to c alpha that means b can be written like alpha 1 x 1 plus alpha 2 x 2 plus alpha k x k. Now, if a transpose c is less than equal to 0 then a transpose b has to be less than equal to 0 by the same thing.

So, let me just write the simple idea both cannot hold if one holds implies there exist an alpha such that that would imply a transpose b is less than equal to 0. And why did you take so much time think about it do not think about it this is an equation. What you are saying is that if there is a point inside the cone then there cannot be hyper plane separating should that be hard right. This is what you have proved by equation that is why I am saying this geometric intuition and the mathematical intuition just think of it back and forth. And you will realize that you were like being in tension without any reason right where there is no stress you are creating stress.

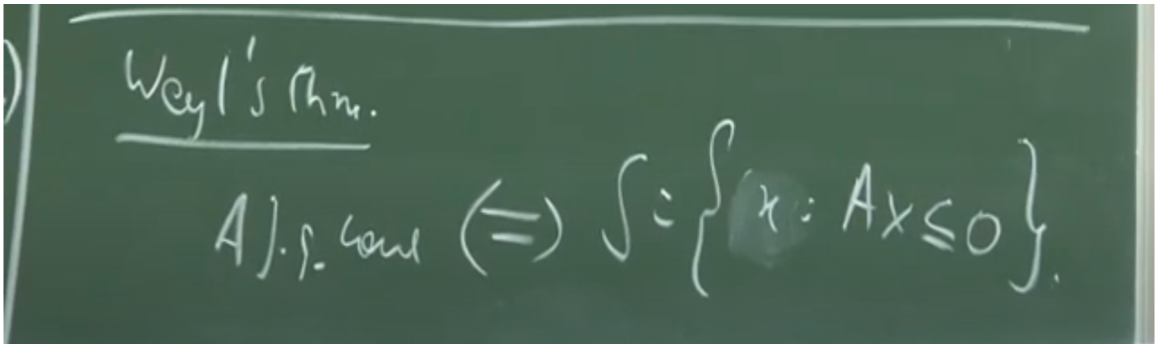


But now is the question of creating stress this is the hard part. Now, I want to show that if one does not hold then I need to search for a sorry that is my mistake. And this will require this old theorem which we had seen three classes before two classes before we know cone in multiple ways. So, finitely generate cone can be thought of as positive combinations of x 1 x 2 x k or remember we had equivalence between polyhedra and sorry Weyls theorem. But it was not the equivalence between polytopes and cones it said polytopes can be described by cone plus convex or bunch of inequalities.

What about cone how do you describe a cone? Equations, equations give me equation let me remind you of something equivalent of Gaussian elimination. No, you are thinking of again mathematical cone wise geometry wise Fourier Motzkin remember we got stuck multiple why did we do Fourier Motzkin? To search for inequalities. Right, so what kind of inequalities did we get a cone is. So, you are probably confused to all this thing, but wales theorem tells you that a finitely generated cone convex cone is equivalent to a set which is it was a bunch of inequalities where b was all 0 is what we did for Fourier Motzkin right. So, again all these things I am not saying you should remember them, but a x less than equal to b can really be can it that b an equal inequality satisfy a cone no because I can multiply x by any constant right.

So, for the cone it is a x less than equal to 0. So, given a quiz right so then these equivalences should flow right given some time should be able to figure out a x less than equal to seems like a cone kind of an inequality a x less than equal to b seems more like a hyper plane right. So, this great right so now that means my cone c also has this description right. Every cone in the world has this description. So, why not my favorite cone c that will also have that description right.

And now I know that b is not in cone c what does that mean sorry what does that mean again I need a small a tell me what is the small a sorry capital A cannot be that A there is a vector capital A is a matrix and that is a genera



l notation we follow right. So, clearly my small a is related to capital A, but how what does this equation mean? This means first row of A multiplied by x is less than equal to 0 second row of A multiplied by x is less than equal to 0. So, if this is not satisfied that means there exist a row of A such that. So, small a is that row of capital A.