Linear Programming and its Applications to Computer Science
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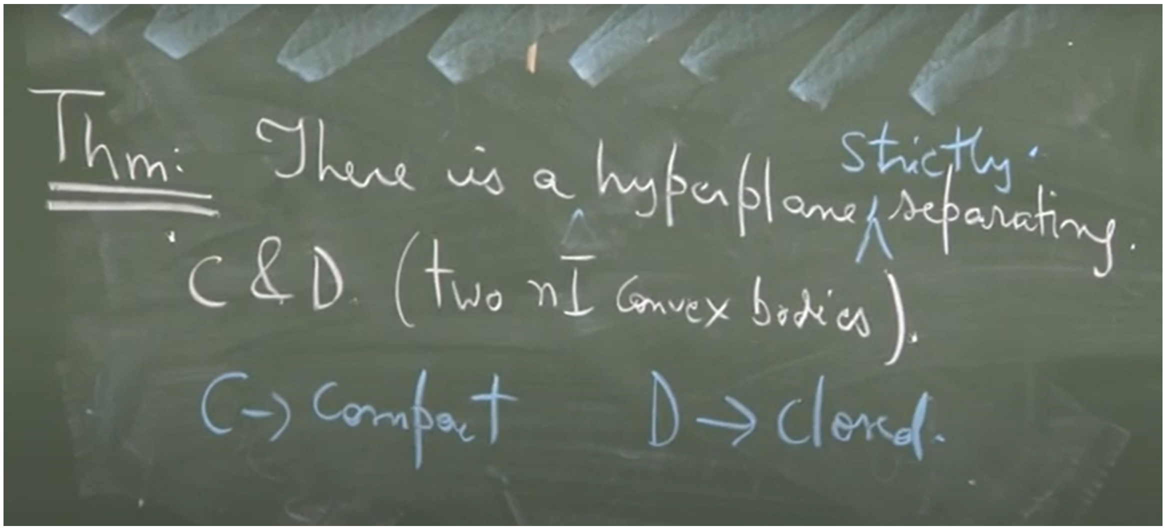
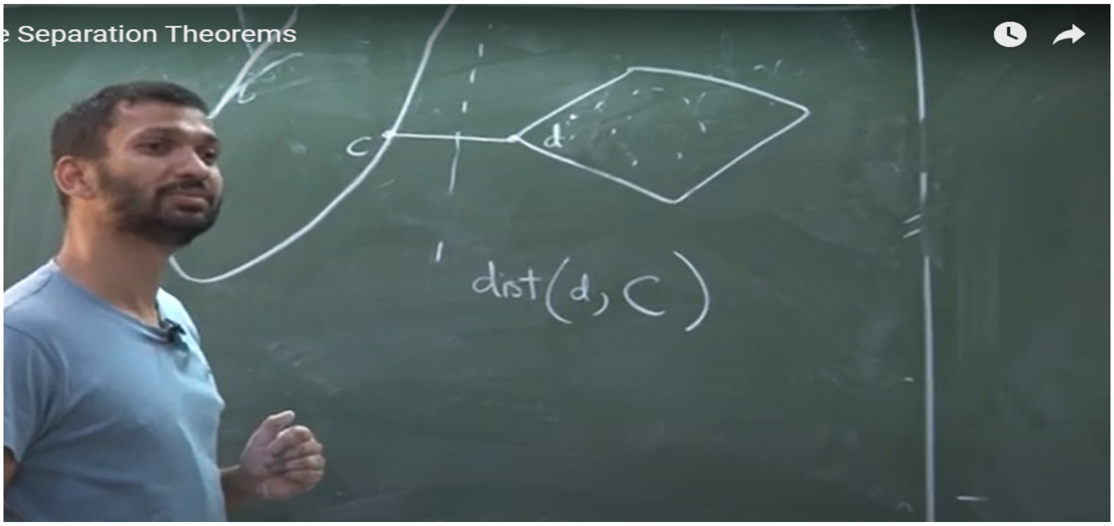
Lecture – 24
Hyperplane Seperation Theorems

So, there is a hyper plane separating C and D, I should say strictly separating C and D if I assume that C is compact and D is closed.



So, we will see the proof of this again there are many hyper plane separation depending on what bounds you can put on C and D.

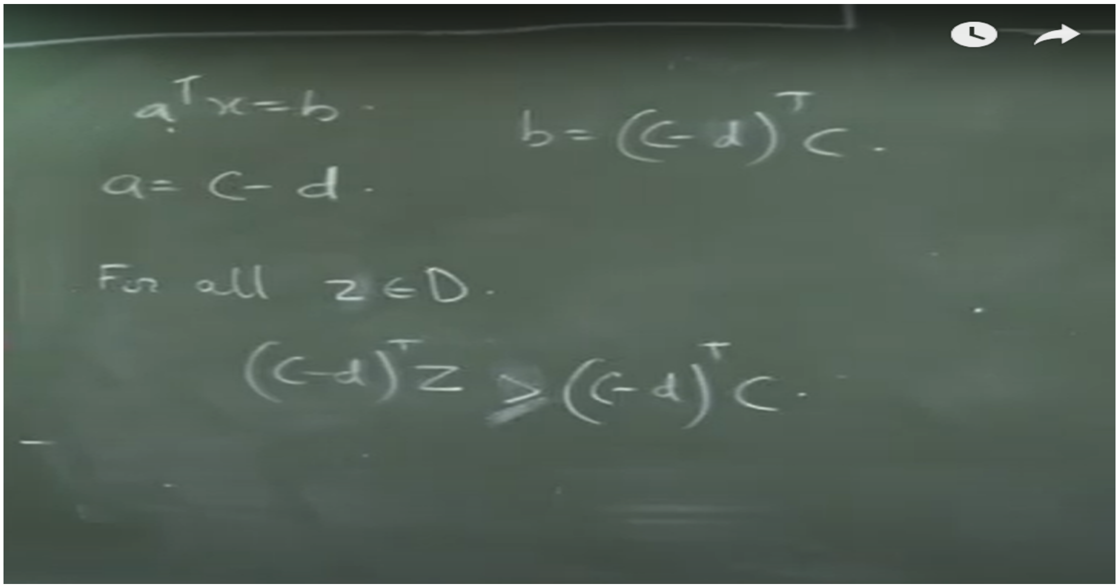


We will just do a few this one is for a just getting a feel of it how the proof looks and then we will do what we call for Farkas lemma is going to be very very useful for our duality. And again you will see what changes is the kind of C and D are what kind of special things we are putting on them and how that changes the kind of separation we achieve. So, what do you think could be an idea you have these points you have this nice probably this is infinite convex right how would you start to construct hyper plane separating these 2 bodies great very good right. So, C and D small c and small d there are like we knew that with the point there is always a point which is closest it sounds believable with 2 convex bodies with compact and everything there will be 2 points which are closest to each other sounds good this is the idea of the proof let us prove it.

By the way I had written this theorem right that for the point and the compact body there are 2 closest points there is a closest point in the compact body just for your knowledge actually there is a strong theorem which says that if you have if you are minimizing or a function a convex function on a compact body you can always find the optimal. So, that if I take this set and for any point here there is a closest point I take this as a function I optimize over a compact set I will find 2 points which are closest to each other. So, again this is an idea from this very strong theorem I am going to prove it, but I am saying remember this that the compact body is nice because the optimum exists if you are optimizing a convex function over a compact body you can find the optimal. The distance from the no no.
So, the closest point in this closed body. So, my for any d I define it to be distance between d comma c which is the minimum over and this from the lemma I studied in the beginning this is a well defined function. And then I take the optimal over this compact set and then things will be fine. Again I am also not very familiar with these things these are all major tools in analysis they are good to know and good to know where their proofs are. Now, so given this now the perpendicular bisector is basically the hyper plane separating them very clear right, but we need a mathematical proof this is not this picture proof is just 1 idea right.

So, how do we prove this? What is the equation of this hyper plane? Remember hyper plane is a transpose x equal to b right. So, what is a what is b? Very good right sorry. So, let me make at a slightly simple I know that this distance is not 0 why is this distance not 0? Non intersecting. Let me look at this hyper plane or wait let me see which 1 do I want to look at this 1 they are parallel you do not see it, but they are b is correct by definition c lies on this hyper plane. So, that is my b.

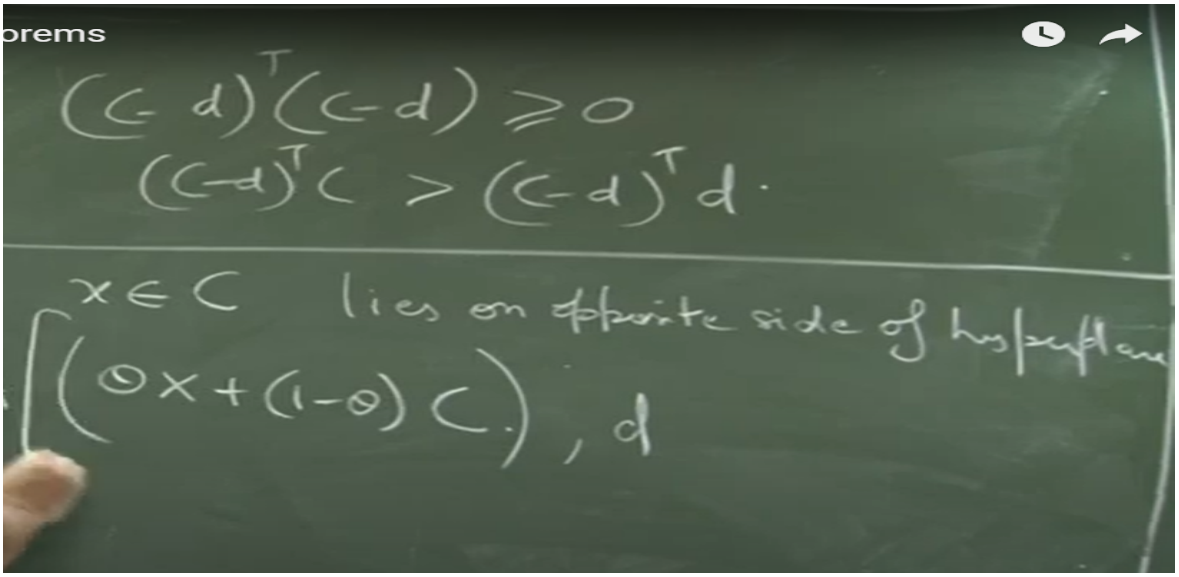
So, now I want to show that for all d in d sorry I should not say d I should say z transpose z this is going to be less than or greater than you can figure out the direction greater than equal to then you move it to perpendicular bisector and then this will become strict or something. So, this is already strict let us see 1 direction 1 of the direction will work great. So, sorry you just check the sign of c minus d transpose d and c minus d transpose c array you make me do painful work. So, this is correct or not correct? Incorrect. So, now you know how to check this or just decide which side will delay this is positive this is greater than 0.



So, c minus d transpose c is greater than c minus d transpose d. So, everything in this side has to lie on the less less side take care. But again what is the idea it is not by picture because we needed so many assumptions right. So, we need convexity what does it mean suppose there is a point here why it is a why is it a problem what are we saying we are saying that no exactly. So, if there is a point of capital c which lies here then I know that this entire line has to be inside c and then the perpendicular or whatever that is going to be a problem there is this small thing where this distance will become less that is the idea.

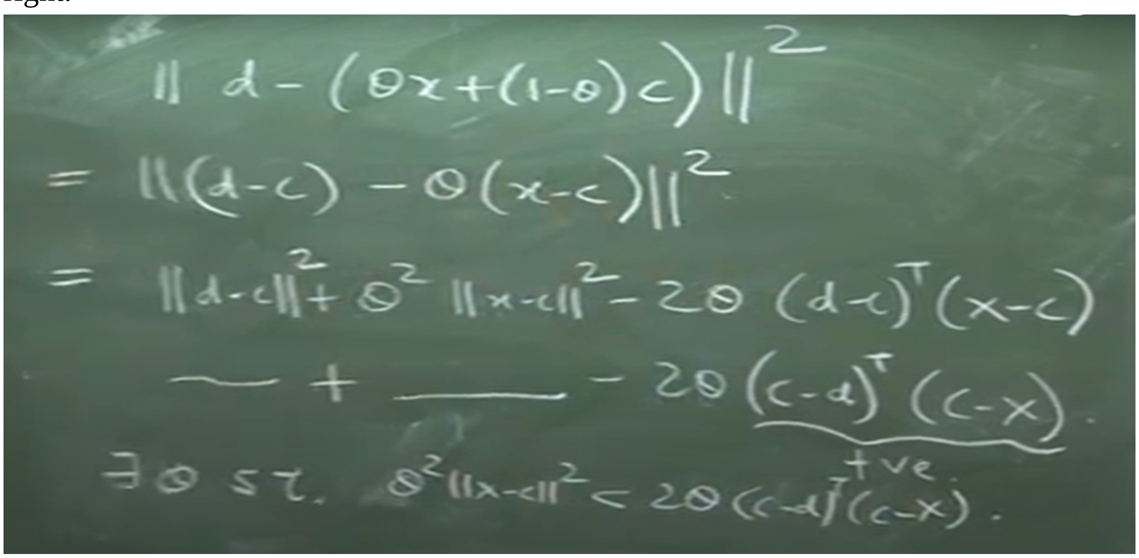
So, you see it in picture if you do not see in picture I can make a better picture. But the idea again is if there is a point in c or actually you know what I think this picture is nicer because this is parallel or something. So, notice from here that if suppose this set was extended and the point came here. That means this entire line is in d and then this is an issue I will formula I will do it mathematically also. So, let us say x element of c lies on opposite side of the hyper plane I think I am going to switch all the signs I think I am going to show this.

But you know where this thing is going right by from picture basically what I want to show is that no point of c can lie here and again the z. So, that means you are understanding everything great. So, if x is element of c it lies on the opposite side of hyper plane on the wrong side then I am interested in this line right. In other words I am interested in this I want to say that there is a small theta positive theta such that this distance is less than the distance between c and d. Let us analyze this distance.

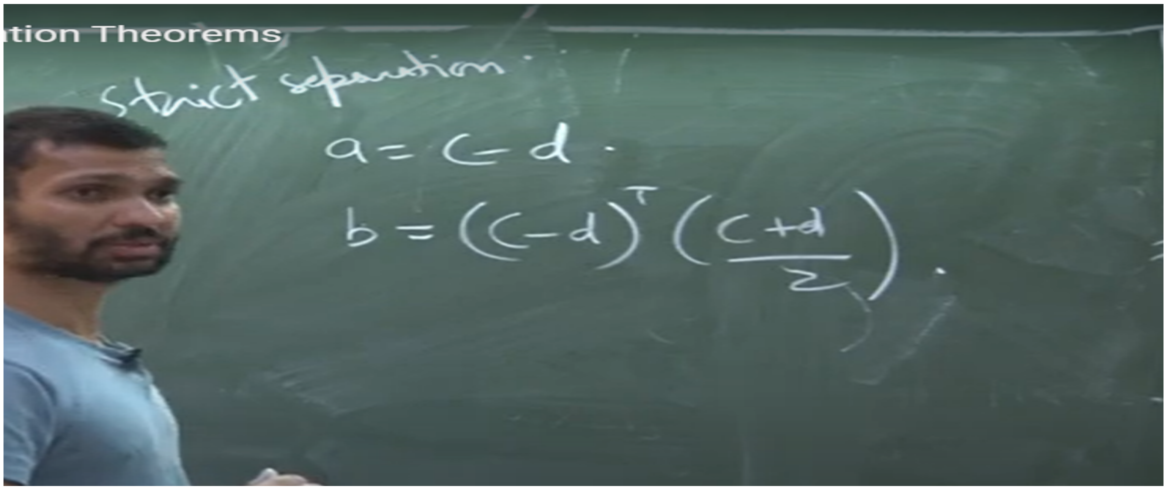


What is the distance d minus, let me out here minus plus or minus good. What do I know about this? No I am talking about this quantity I said x lies on the opposite side of the hyper plane right. So, that means this is going to be positive again you can see the sign from all that, but basically if x lies on the. So, let me just do it right. So, what do I know? So, d is in the smaller side and I said x lies on this side right.

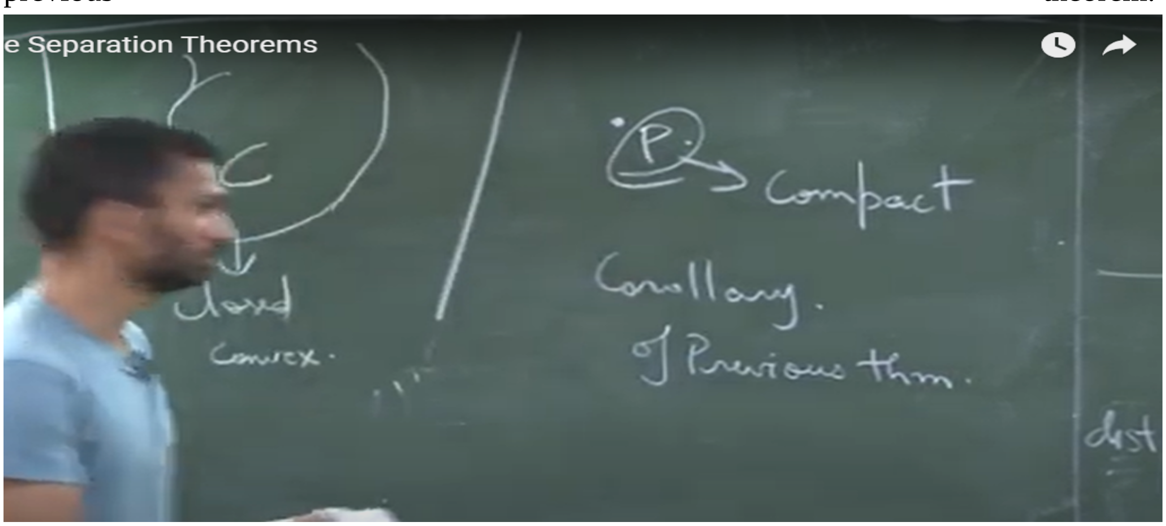
So, then c is bigger than x, x is the smaller side d is smaller than this. So, that means and x lies on the d side. So, that means c minus d transpose s c d right. That means c minus d transpose c minus x is positive. Now, I am fine great now this is positive why am I excited about this? Exactly that means this and this quantity there is some fixed ratio right.



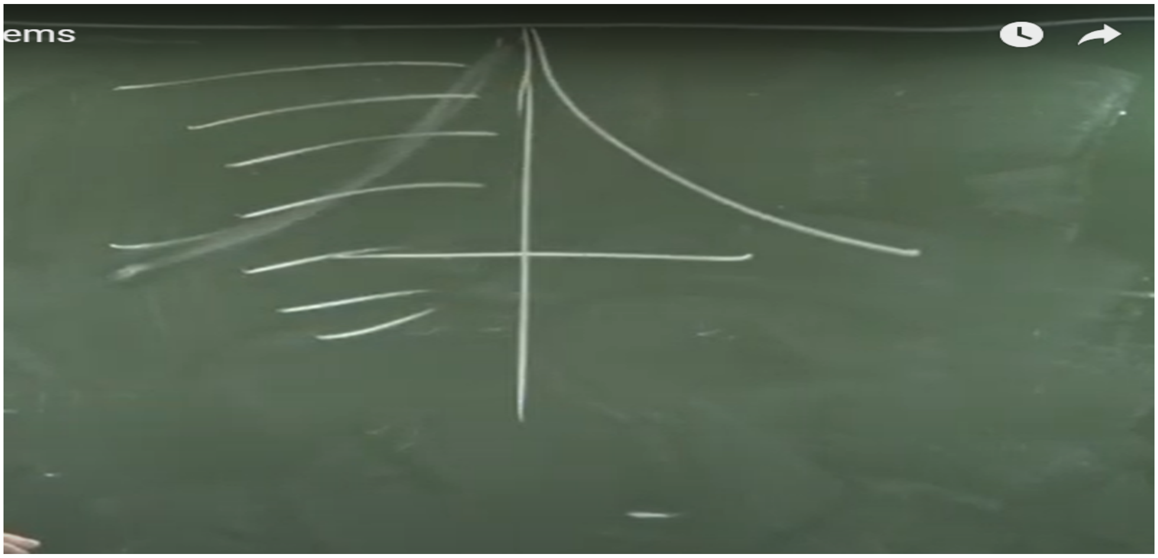
If I make theta extremely small right this is going as theta square this is going as theta. So, this will become smaller than this there exist a small theta such that this quantity such that whole square becomes less than 2 theta. And if this is not clear to you go back and read about asymptotic O notation theta notation omega notation theta square is small theta is big that is why we like asymptotic. There is a small enough theta such that this quantity will be bigger than this. That means this distance will become less than d minus c square which is a contradiction.



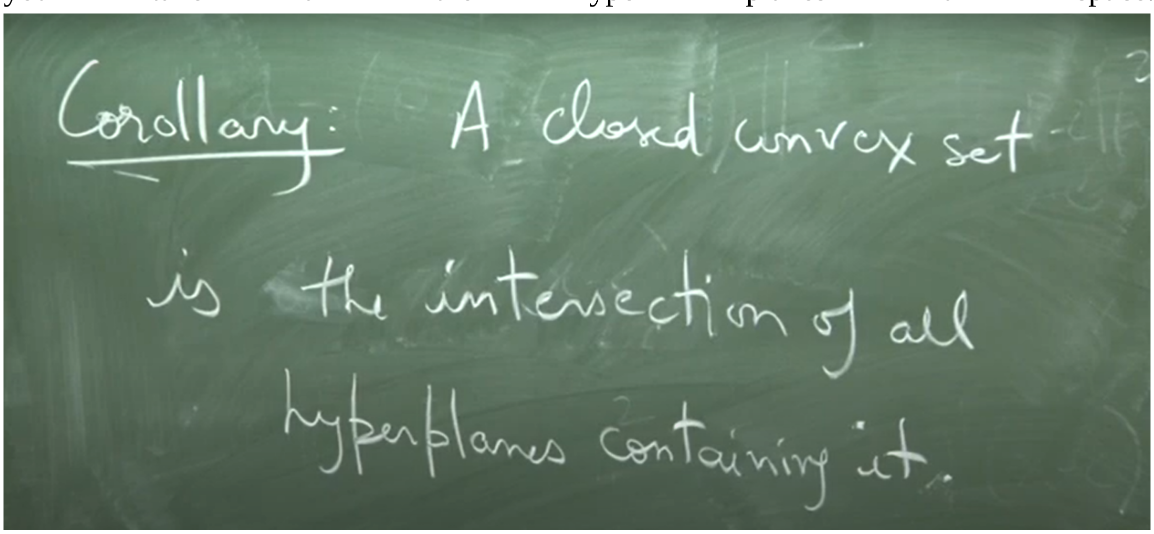
So, I will finish now with few more separations by the way I think this was already said. But if you want to make a strict separation in this case your a will be c minus d and your b would be because you know that c plus d by 2 is the midpoint right. So, this at c it is a middle separation at c plus d by 2 it will become strict I do not need to prove this. So, now suppose I have a closed convex at c and a point p which is not in c what can I say weak separation middle separation strict separation this is closed. How many for strict how many for middle and we always know majority is always right this time it is there will be a strict separation because this is compact right this is simple corollary of the previous theorem.



Sorry. Because compactness for which one for d like in the proof because otherwise we cannot find 2 closest points give me 2 closest points on this both closed and convex.



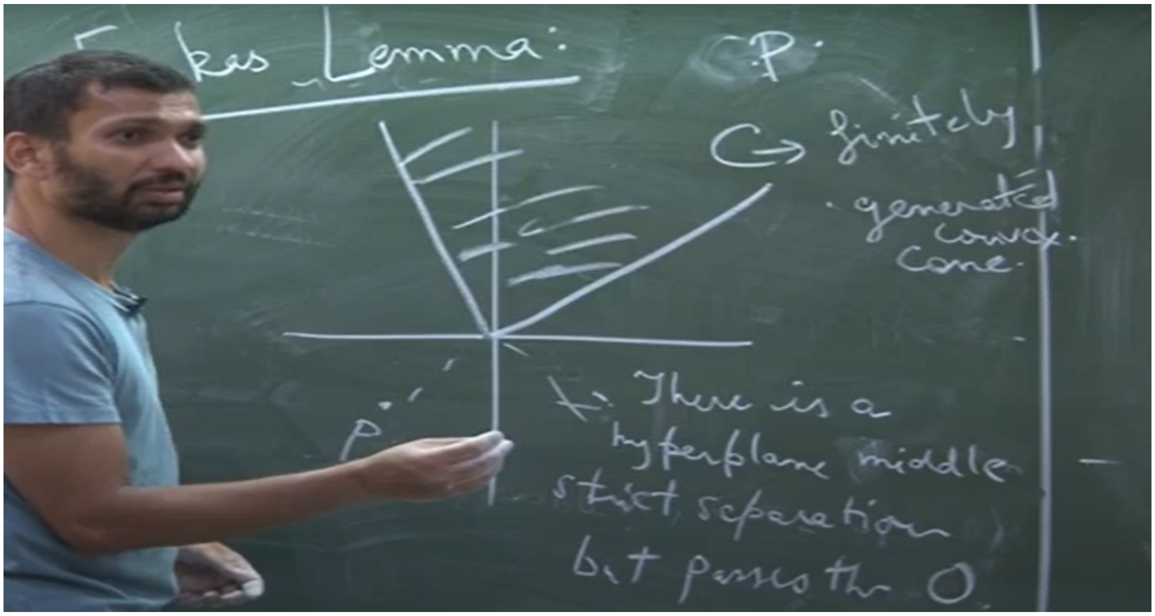
I am not saying that 2 closed convex sets cannot have a strict separation I am saying for all closed convex sets we need not a theorem would say that given these conditions you always have right. So, I am giving you a counter example in this case I cannot find the closest point there is a strict separation here by the way, but our proof study does not work. Actually yeah actually this is not even true right take this there is no strict separation, but actually this theorem gives you a nice description of a convex set. A closed convex set is the intersection of all hyper planes do you see it given a convex body you take all the hyper planes half space.



So, if I want to be correct about the nomenclature the hyper plane is A transpose X equal to B this is the half space this is the other half space are you trying to divide infinity into 2 parts and asking me is it always half I am not going not going to take that bit no dealt with infinity often right. So, do you see this why does it follow from this theorem. So, there was this theorem that a point can always strictly separate a point and a closed convex set. So, how can you say that a closed convex set is the intersection of all hyper planes containing it. So, clearly the closed convex set will be contained in the intersection suppose the intersection has an extra point.

If the intersection has a extra point which is not in the set I can find another hyper plane and C lies on the other side of it right. So, this is a might seem like a very fancy version this is a direct corollary of this theorem good and by the way if I am really interested in using this then I will only use the supporting hyper planes right. I really do not need to have all the hyper planes just the edges faces ones are the useful ones. So, I think my time is almost up I just want to give you another at least the statement of another hyper plane separation which is going to be very useful for us for duality for strong duality. What it does is you say that you are you have a point P let us say and you have a C which is the finitely generated cone convex cone.

So, given a point and a finitely generated convex cone how does it look like finitely generated cone looks like this right a point looks like this. Farkas' lemma says there is a hyper plane which separates them and passes through the origin clear. So, now it says there is a hyper plane middle strict separation, but passes through origin. So, now a closed convex cone is a closed body right and we actually know by the this previous theorem that there is a strict separation Farkas' lemma is another version where we restrict our closed convex body. We get a worst separation we get only middle strict separation we do not get the strictest separation, but hyper plane is nice it is of the form a transpose X less than equal to 0.



So, there exist a now I do not need b such that a transpose P is less than 0 and for all X element of C and again this looks believe. Notice a cone would not have been good enough right I need convex right because even this is a cone. So, this would have been a problem right. So, I need a closed body convex and then I can get this kind of a separation. So, Monday not this you give a quiz on Tuesday we look at this and how it helps us give dual programs.

What was the dual program great way to give upper bounds on a maximization problem or great ways to give lower bound on a minimization problem.