Linear Programming and its Applications to Computer Science
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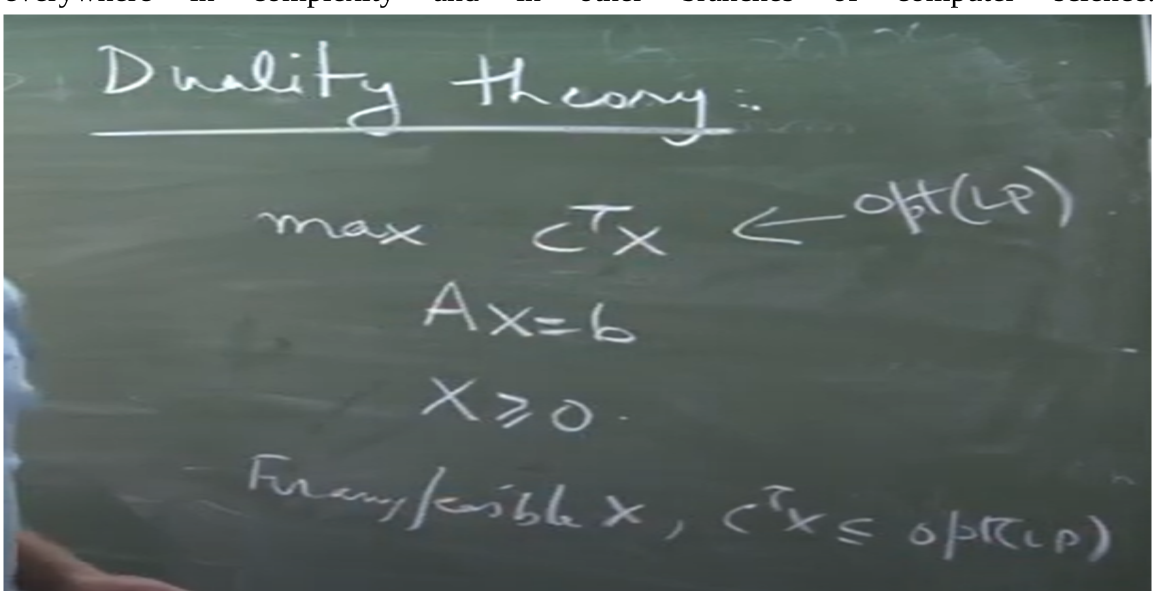
Lecture – 23
Introduction to Duality

Welcome, let us talk about duality theory. So, generally duality is a very well studied concept in mathematics. There are linear functional and everything there are many ways in which people define dual space relationship between the these spaces and functions and all. But I would want to be restricted to linear programs. And for linear programs duality is actually a very strong tool and for me personally the reason why it is a strong tool is because of this. Let us look at this maximization problem correct.

I want to figure out the maximum value here right optimal of this L P. But in computer science generally finding the exact value is pretty difficult right. What we can do is try to give lower and upper bounds on this value. If even if the value is 10 we can say it is more than 8 I am still happy.

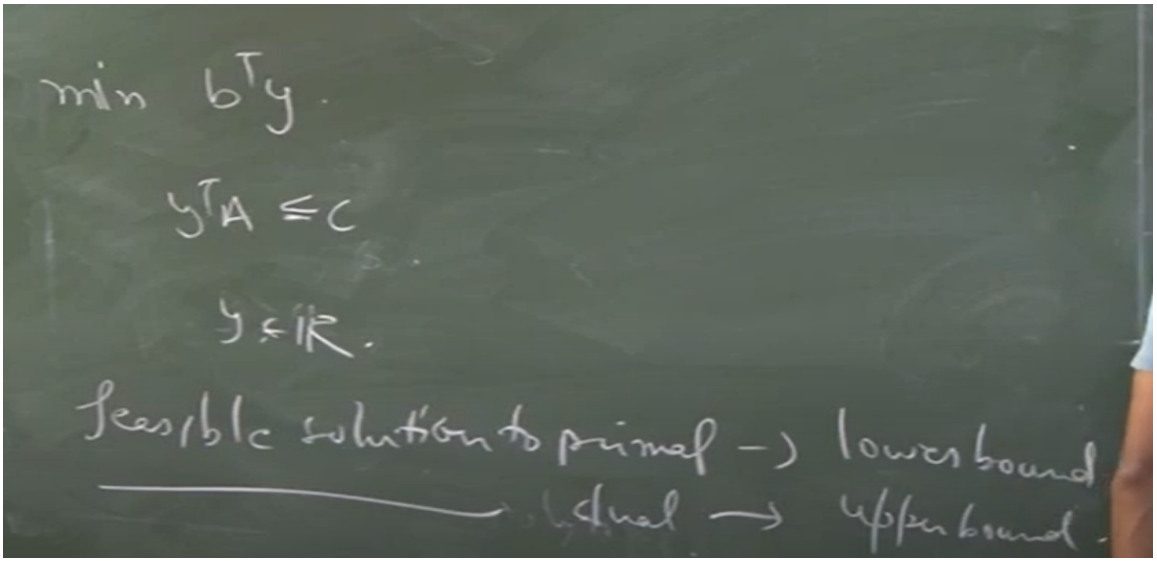
We can say it is less than 12 still happy right at least we give an idea of how far away we are from the optimal. It tells us that this value has to be bigger than 20 at least. You know suppose there is an algorithm whose running time is in terms of this linear program. And if I show you that you know N square is a lower bound on your L P and then you are like that is not a great algorithm for sorting right probably you need something better. So, these of the lower bound and upper bounds are there in almost everywhere in complexity and in other branches of computer science.



So, now my claim here is that giving lower bound on such a thing is actually easy. I have a easy way to give lower bounds here. Can you see one way in which I can give lower bounds? Sorry right. So, if I can come up with the feasible solution here greater than equal to 0 here. Then for any feasible x, C transpose x is less than optimal of L P it is a lower bound.

It is as simple as that right any feasible solution gives me a lower bound here. And this method of giving lower bound is actually tight. The best lower bound on optimal L P can be given by a feasible solution the optimal solution right. I am not saying anything great it is just an easy way in which I can give a lower bound. What is striking is the upper bound side.

What do I mean by that? It turns out that for any linear program you can write another program which is a minimization version which is again functions of this only.



I do not remember exactly why. So, there will be no and this would be I think. So, I am not sure this is less than or greater than we can figure it out, but there is a program which looks like this which is derived from this program and it is a minimization problem. Also the 2 values are exactly the same.

What it means is now I have the same tool for giving upper bounds. I take a feasible solution here it suddenly gives me an upper bound here. So, in some sense there are 2 things which are important there is an L P which gives me an upper bound which is not hard this is called weak duality. The more striking part is actually turns out that this is the best way to give upper bound. There is whatever closest upper bound you want to give there is an optimal solution which gives that.

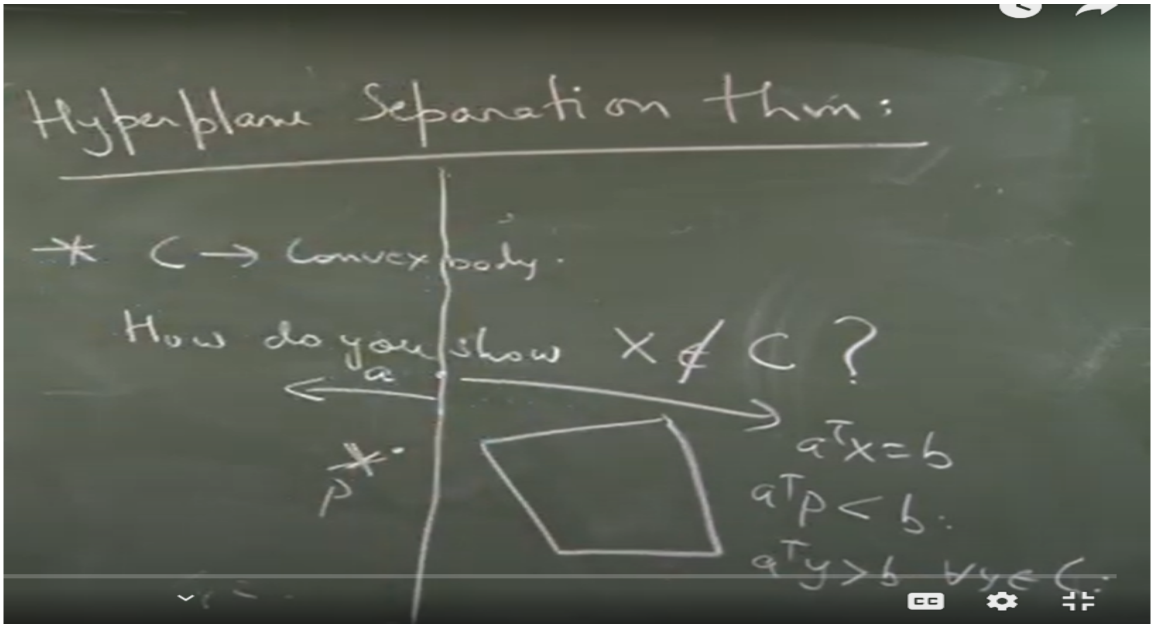
Then there is an solution here which gives that upper bound. So, this duality this way of creating another linear program which is so related to this original program is nice. And there are I do not know how many numbers I have read where this upper bound is given by finding a feasible solution. So, feasible solution to primer is a lower bound and feasible solution to dual gives me an upper bound. And obviously, the roles can be interchanged this is the dual of this if you have a minimization problem I have a natural way to give upper bound this gives me a natural way to give lower bounds.

This is a very very strong way of and we will see lot of examples of this. Yes, so, for any so many of the books they will just write dual in the standard form. So, you what you can do is you can always convert your L P into standard form. But I will show you a method which would not worry about standard form and which will basically be motivated by giving this upper bound. So, I will say that suppose you want to give upper bound can we design another program and then we will see how to come up into.

So, because this is what I have seen mostly being used for this is how I will motivate duality theory. But there are many many interpretations I will encourage you to read other books and see how people talk about this. But this is great this is very very nice and we will see lot of applications. But this very very upper bound lower bound kind of a thing actually emerges from duality in other structures. Mathematical structures which you have already seen I will not tell you which ones they might come in the exam.

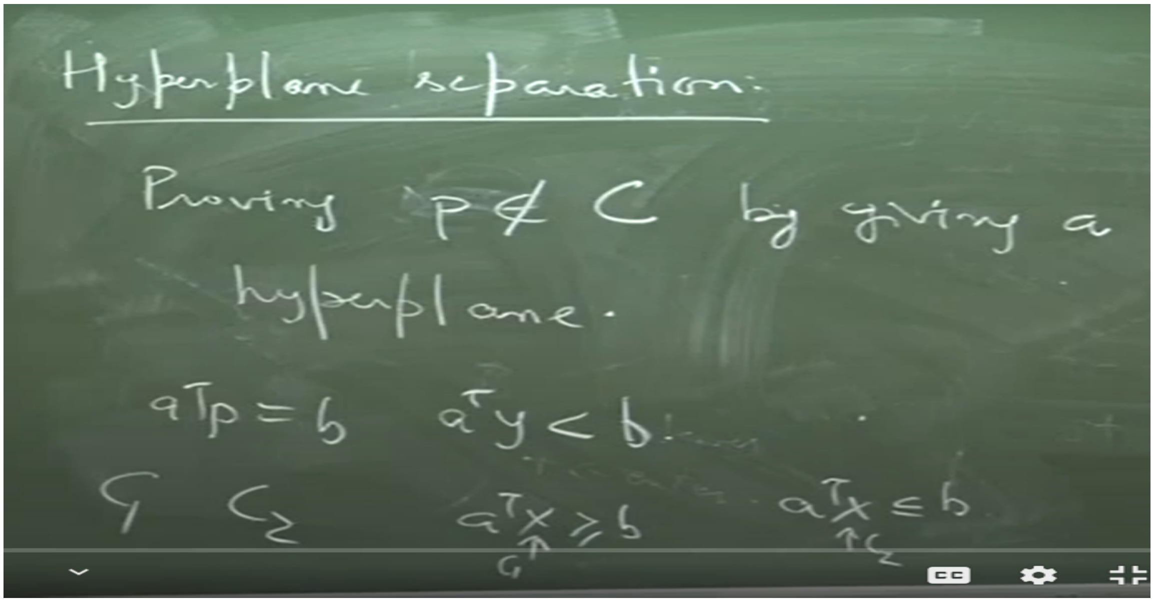
But one place from which you will see a direct relation is called a Hyper plane separation theorem. And this hops back to the question which I asked you in the previous class which was say you are given a convex body. How do you show or prove that some point is not in C not in a convex body. And one way to definitely prove this. So, let us you have an x you have a convex body how do you show that this is not here.

But if you want to try to give a Mathematical proof one way is you might say look at this line all my convex body lies on one side of the hyper plane my point lies on the other side. So, how can my point be inside this convex body. Now this thing is nice because it has a very clear Mathematical meaning. If this hyper plane has a as its perpendicular vector suppose all this hyper plane is a transpose x equal to b. Then I know that for sorry call it p.



Then I know that a transpose p is less than b and a transpose y is greater than b for all y in C right. So, this I can do by constructing an explicit a I do not have to say look at the diagram and see this line is passing right. So, this is a very nice Mathematical way of describing this thing right. So, what is hyper plane separation? Proving p is not element of some convex body by giving a hyper plane. And to do that I will give you a hyper plane separation theorem.

Now in today's class as I said it is going to be fun with geometry what we are going to do is take different kind of convex bodies try to construct different kind of convex bodies and see is it possible to give a hyper plane separation theorem is it not possible to give a hyper plane separation theorem. Now when I say hyper plane separation theorems notice this was a nice separation here both were strict inequalities right 1 point light completely outside thing completely inside this might not be the case. So, we might be interested if you know a transpose p is equal to b, but a transpose y strictly less than b. There might be a case when you want to separate to 2 convex bodies and we say that a transpose x where x is coming from C 1 this greater than equal to b. And obviously in this case it would not be interesting if both the convex bodies lie inside the hyper plane a transpose x equal to b.

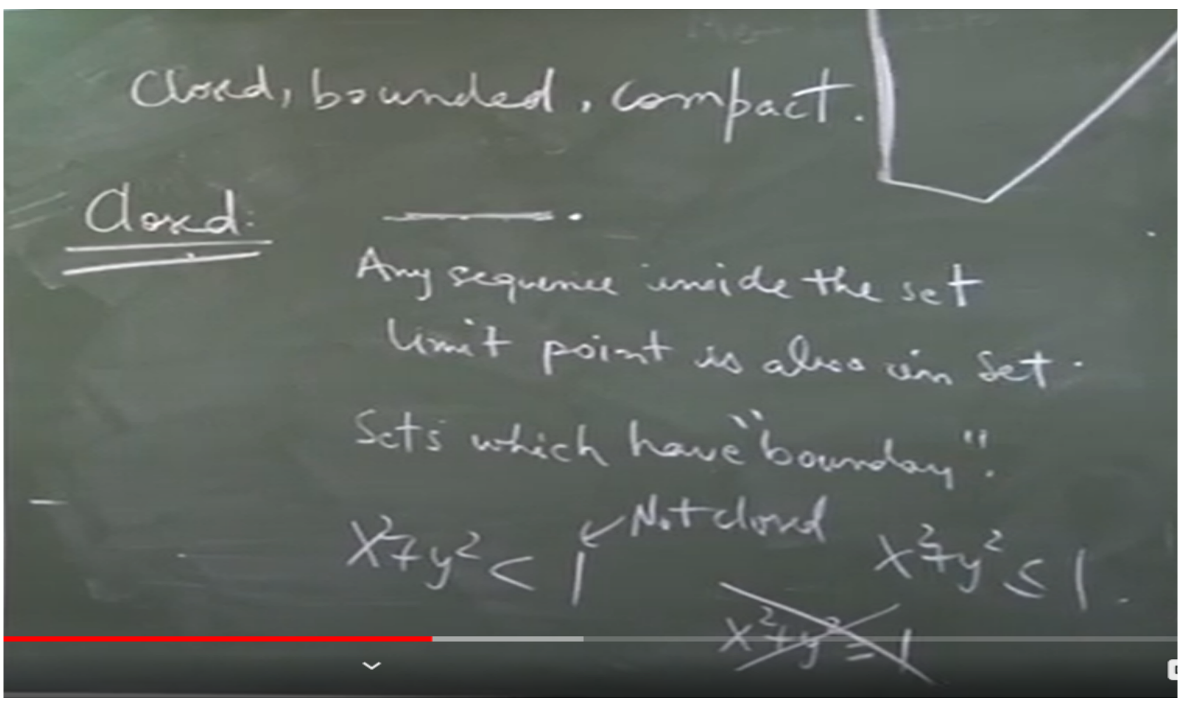


That means for every for C 1 as well as C 2 for everything a transpose x equal to b. So, that is not really any kind of separation at all. So, there are a series of hyper plane separations depending on the kind of convex structures I have different kind of hyper plane separation things can be given. And this is what we are going to see today you will take different cases you will tell me is it possible to give hyper plane separation it is not sounds good this is what we are doing. So, first is there a if this body is not convex then what is the problem you might not have a hyper plane separation at all.

For example, take a circle take a set which is all points outside the circle 0 is not inside that point, but there is no hyper plane separating from this point. So, definitely in the sets we need some structure and since we are interested in for convex bodies we can give very nice hyper plane separations and that is the reason we get duality theory. So, one convex the some properties of convexity theory give us simplex algorithms. Now, we see more properties of convex bodies and that gives us duality theory. The set here is all the points outside the circle origin is not in that set, but there is no hyper plane separating these things, but obviously this set is not convex.

So, in some sense I am justifying why I am only talking about convex bodies. But before I talk about exact hyper plane separation theorems we need some concepts from analysis you know what are called closed or bounded or compact subsets. Have you heard of these? How many people know the definitions of these? Do not worry at all I will give you the intuitive meaning that is good enough. Do not get take the intuitive meaning from English take the intuitive meaning of what I tell that is going to work sounds good. So, first thing we start by is closed and if you do not remember that intuition then ask for him.

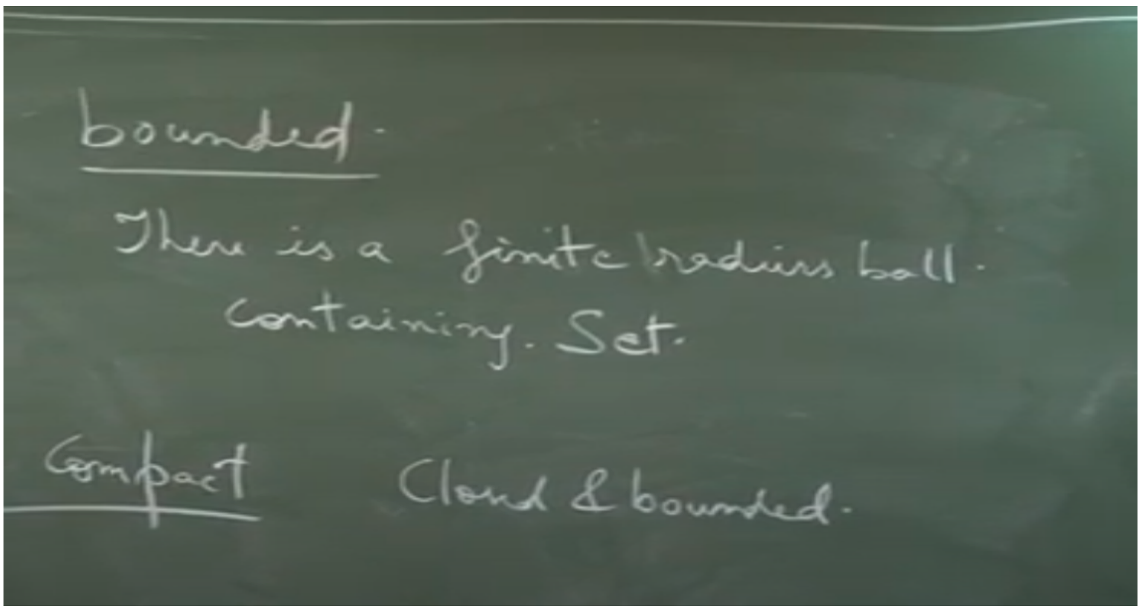
So, for him tell us what is a closed set? Very nice formally we want to say that if I look at the points a sequence of points I define a sequence you can form nice cost sequence whatever if that sequence if I take then the limit is also in the set. So, for any sequence inside the set limit point is also in the set sequence forget all about it sets which have boundary. If boundary is included it is a closed set if boundary is not included for us it is always going to be a convex nice body.



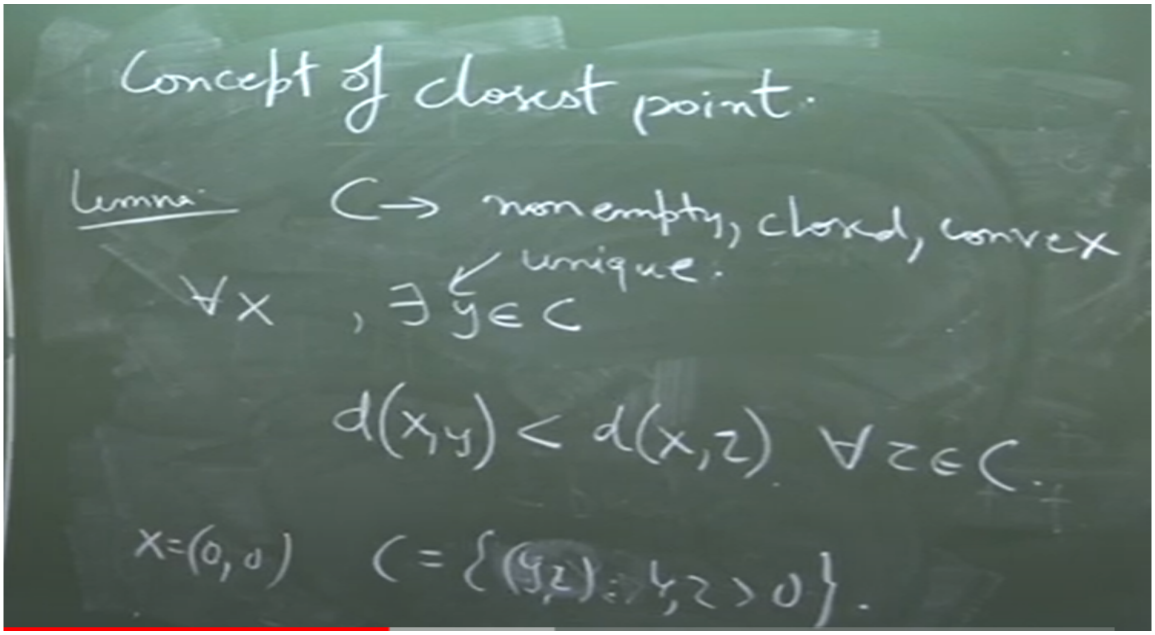
So, I will not worry about this definition what I worry about is in the convex case do I have boundary or not. This is not closed that is because it does not have the boundary this is closed clear because this has boundary closed not closed I do not care it is not a convex body.

I do not want to find out whether all sequences have a limit point inside this or not let us not worry about it I will only worry about nice convex bodies. The one example which you might want to say is something like this infinite body. So, it has a boundary at infinity in some sense this is also closed and if it is not closed then it is. And one reason why I hate this is because closed are the ones which have boundary then what is bounded? Bounded has nothing to do with boundary.

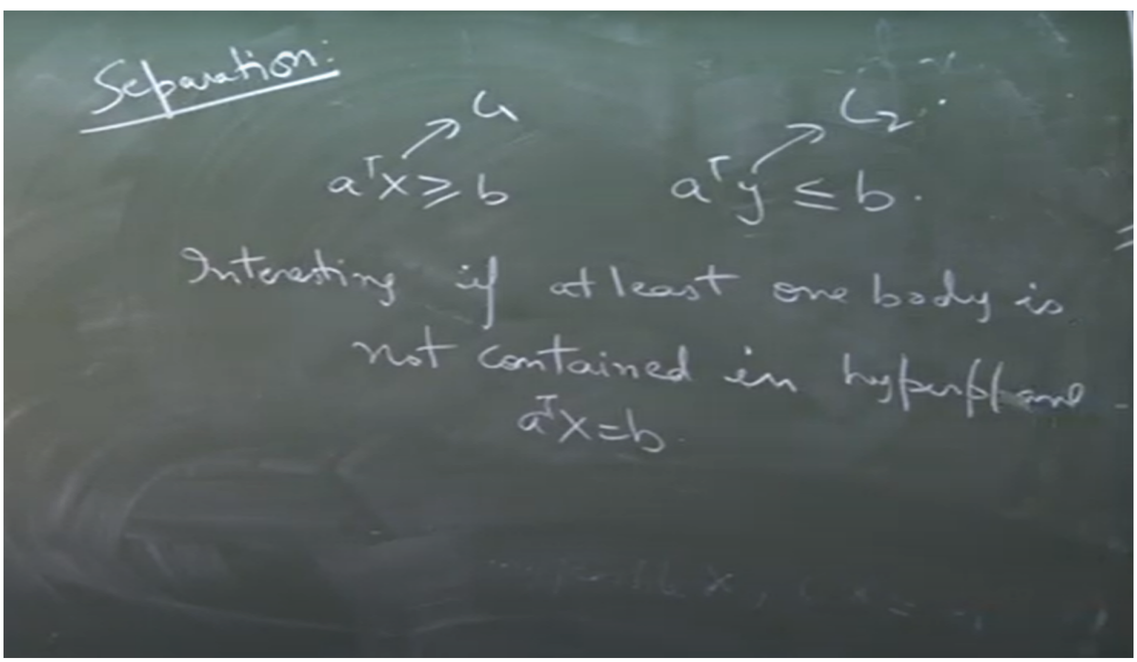
Exactly it does not look like this things are not going to infinity officially there is a finite radius ball containing set that is all we have to worry about again I am not going to ask you prove this thing is bounded this thing is closed it is just. So, that we understand our hyper plane separation theorems this intuitive definitions are. So, do not confuse bounded with boundary if you have boundary that is closed and now the easiest definition is compact. These are the nicest possible points a nice possible sets for us they are for example, a polytope is a polyhedron is a compact. A polytope would be bounded, but might not be sorry it will be closed it might not be bounded.

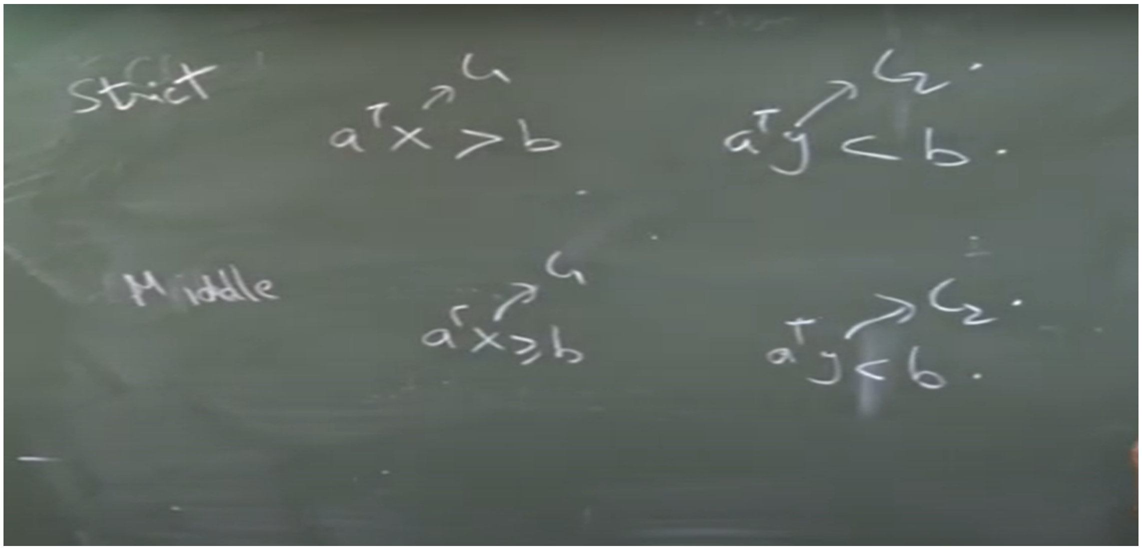


And one already nice thing which you can see with the close set is the concept of closest point. I am not going to prove this lemma, but this is going to be useful for us what does this say if C is non empty closed convex. Then for all x in our space there exist a point y in the C such that the distance between x comma y is less than distance between x comma z. And actually this point is unique f I am not going to give the proof of this, but I can actually still justify few of the things one thing is what if it is not closed clearly this is not true right. I take x to be 0 0 my C to be y such that yeah exactly that is what I want to do right.



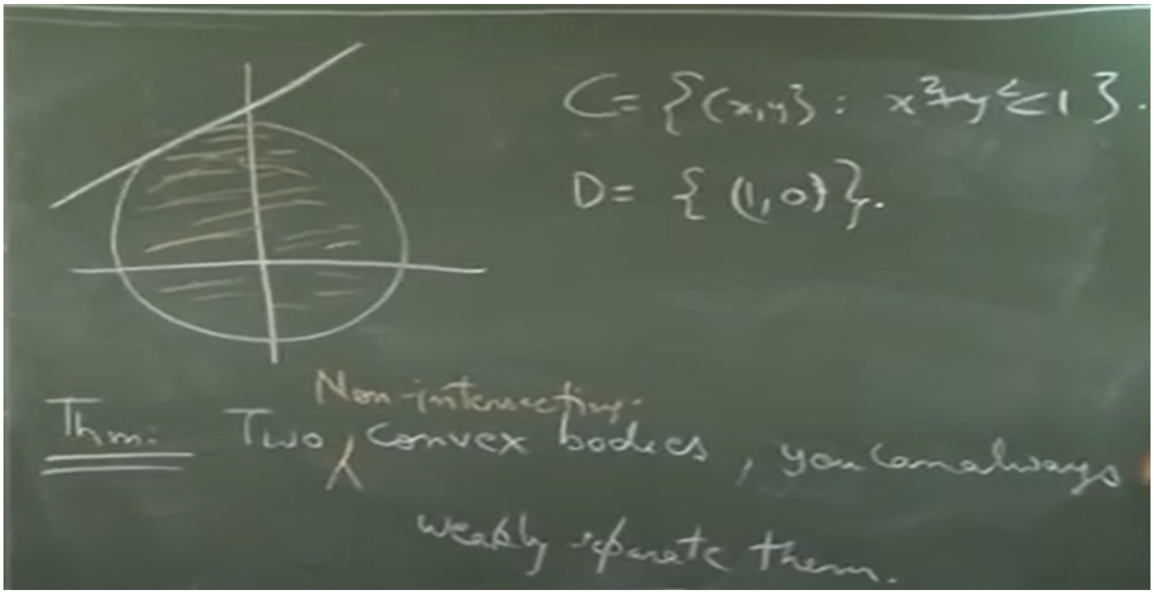
So, you have an open set you take anything on the boundary there is no closest point I have to include the boundary right. So, that means closed is required what about uniqueness again convexity is required you have a circle you have a point the closest points could be much more. So, there are cases when the closest points are not unique this allows us to give a unique closest point and that is important. C is a set C is this set non empty closed convex it can be it is a closest point not the farthest point .yes it is a it is an it is r to the n. So, now, once we have these definitions we have this lemma we talk about separation and as I told you there can be multiple kind of separation the weakest separation is when we say that you know this is coming from C 1 this is coming from C 2.





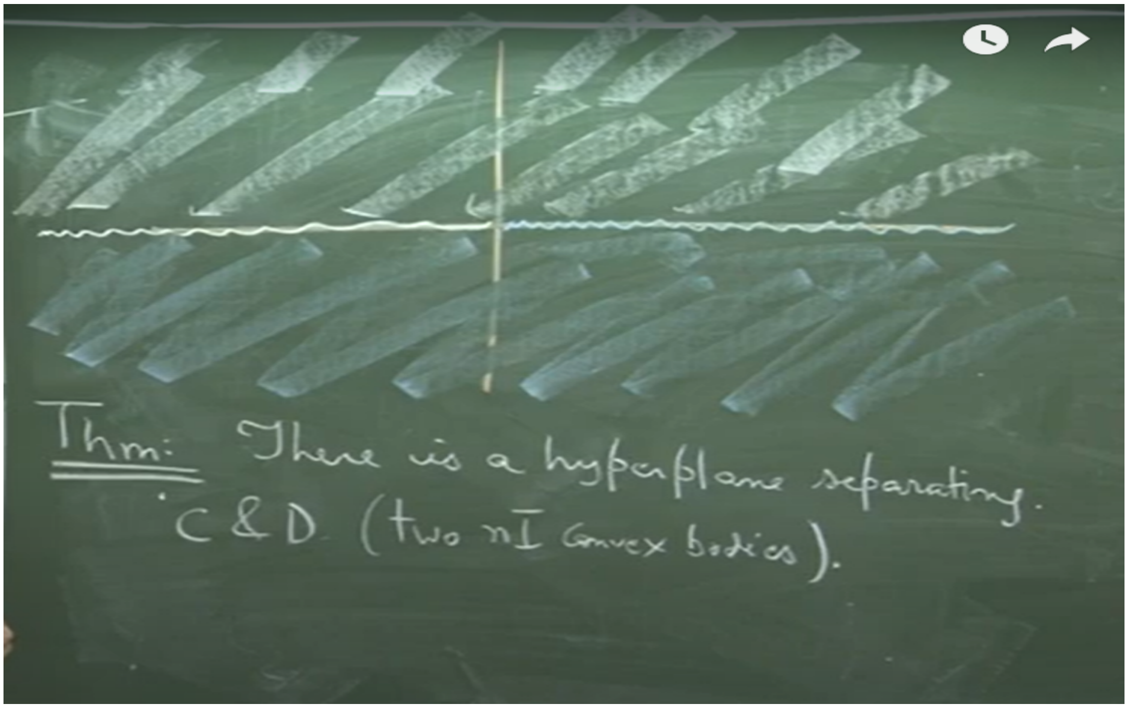
And this is interesting if at least 1 body is not contained in hyper plane right that would be trivial you just take the entire space entire space as the hyper plane then everything is contained. So, this is this is not very interesting in most of the cases we will not be doing this weak kind of a separation we will be interested in stricter separations where right. And there is a middle one also where right. So, all these kind of separations mathematically are possible and can you find a point and a convex body or 2 convex bodies such that there is no hyper plane separating them non intersecting yes geometry draw the graph right draw the picture sorry. I am saying can you think of 2 convex bodies which are non intersecting and they have no hyper plane separating them.

So, the example is you have the circle it is a circle, but the points are only inside the open set it is convex. So, now I know for this point I cannot give a strict separation, but obviously this gives a this kind of separation. No, they are not no how are they intersecting this is a single point on the boundary this is x square plus y square less than 1. So, d is just this point. So, they are non intersecting, but I cannot get a strict kind of separation it turns out that you cannot think of 2 convex sets and it is true for 2 convex bodies you can always have a weak kind of separation does not seem very difficult, but like believable right I should not say difficult in sense of proof, but this seems believable.



So, yes I will leave it at that there is a theorem which says that given 2 convex bodies you can always non intersect. Now the fun part why did I say weak separation why did not I define my separation to be this draw you see the point I said 2 non intersecting convex bodies you can always weakly separate this is what I am saying as a theorem, but from whatever you did right it is the it seems like I should be able to give this kind of a separation and even for forums example I am able to give this kind of a separation. So, can I construct 2 convex bodies non intersecting such that even this kind of separation is not possible or it might be true that you can always give a or probably for every convex body I can have this kind of a separation where do you put your money for all convex bodies you can only weakly if I want to separate all convex bodies is it only weak separation is possible or even this separation is fine this is the weakest separation. So, you have to pick a side right. Do not go on what I am writing I might be trying to confuse you.

This non intersecting convex bodies always assumed even if I forget to write at least when I am talking about hyper plane separation right. You only have to look at 2 dimension do not go in 3 dimension let me. So, let me take the vote then how many people think that middle separation is possible ok. You are raising your hand or not do not raise it like this that is not allowed ok. How many people think middle separation is also not possible only weak separation is possible 1 2 3 4.



Arpreet did you raise your hand twice no ok 4. So, as we always know majority is always. . They are convex right 0 comma 0 I give it to you can keep it for to yourself if you are very generous you can give it to one of the sets it does not matter still it turns out the blue set is convex the white set is convex they are non intersecting, but you even cannot have a middle separation right. So, this tells us the need to have these convex bodies nice convex bodies and then we can have interesting hyper plane separation.