Linear Programming and its Applications to Computer Science
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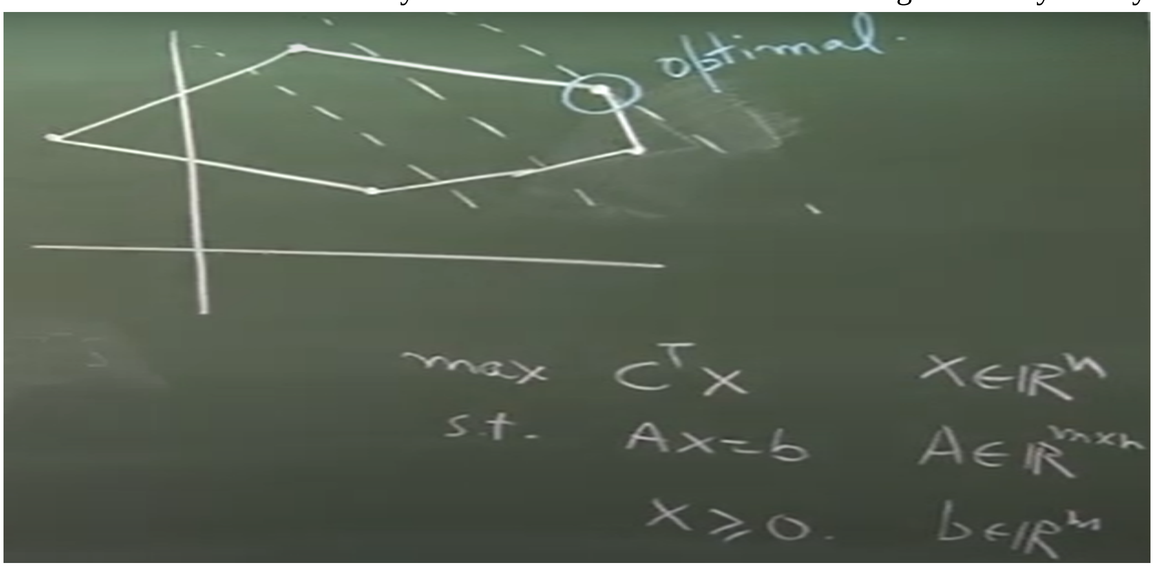
Lecture – 20
Details of Simplex Algorithm

Welcome, welcome to another lecture of linear programming. We are in the last stretch of technical details right, more of these math learning and doing simplex kind of thing. And after that we will go to duality theory as I told you in the email also it is one of the most beautiful mathematical construct I have seen and I am very excited to tell you all about it. And after that we will just be applications lot of applications in different areas. So, I think every week we learn basics of some new field and then see how linear programming has made an impact there. But for today let us finish our simplex method and I think even this has very nice clear Mathematical ideas using convexity theory.



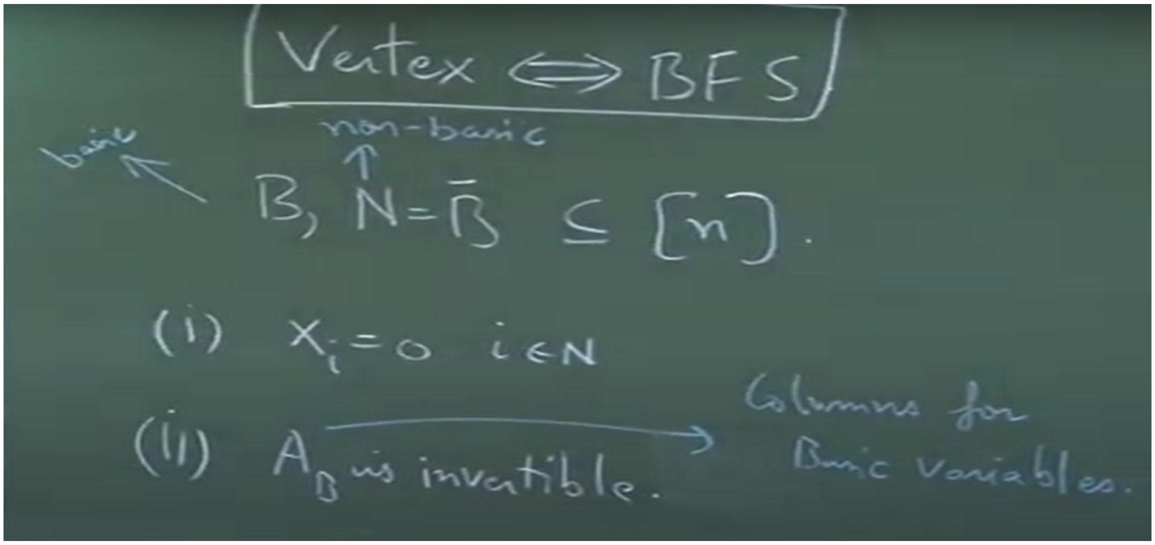
So, if you remember forget about this in general what we know is that our feasible region of linear program is a convex region right it is a polyhedra. Now that means, we know that since our objective function is linear the optimal will lie on one of the vertices right. And the other thing is that the global optimum local optimum is going to be the global optimum. This suggests this nice algorithm pick a vertex you might reach here and again I have not told you how to start with the vertex.

But if we have a vertex let us say we start from here we know there is one direction where the objective function has to increase. In this case it turns out in both directions our objective function is increasing right. So, this hyper plane is giving me the increase in the objective function this is what I want to maximize. So, now, if this is the increasing direction in this way this way I can go this way and I am lazy. So, let me put the short way I reach this vertex.

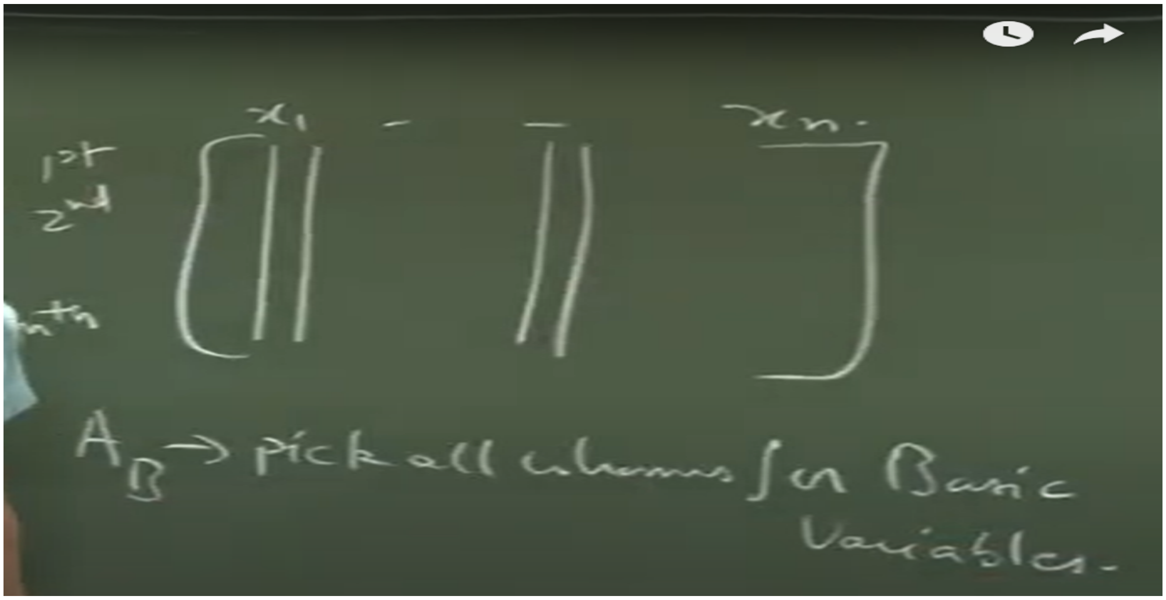
And then at this vertex again I always have an option to move towards optimal. And when I reach here I check in both the directions my objective function is decreasing great I have reached the optimum right. This was the idea and we saw a simple example right. That simple example kind of demonstrate this thing with this standard form and we will keep this standard form in mind. There are lot of simplex methods lot of implementations I am doing just one of it.

But the idea remains the same you travel from vertex to vertex find the local optimum. If you are at the local optimum you are done happy go home. This is what we are going to do, but we would not go home once you find the optimum. We are going to have another class today after that. So, this is the standard form you maximize a linear function such that you have this linear constraints and the positivity constraint.

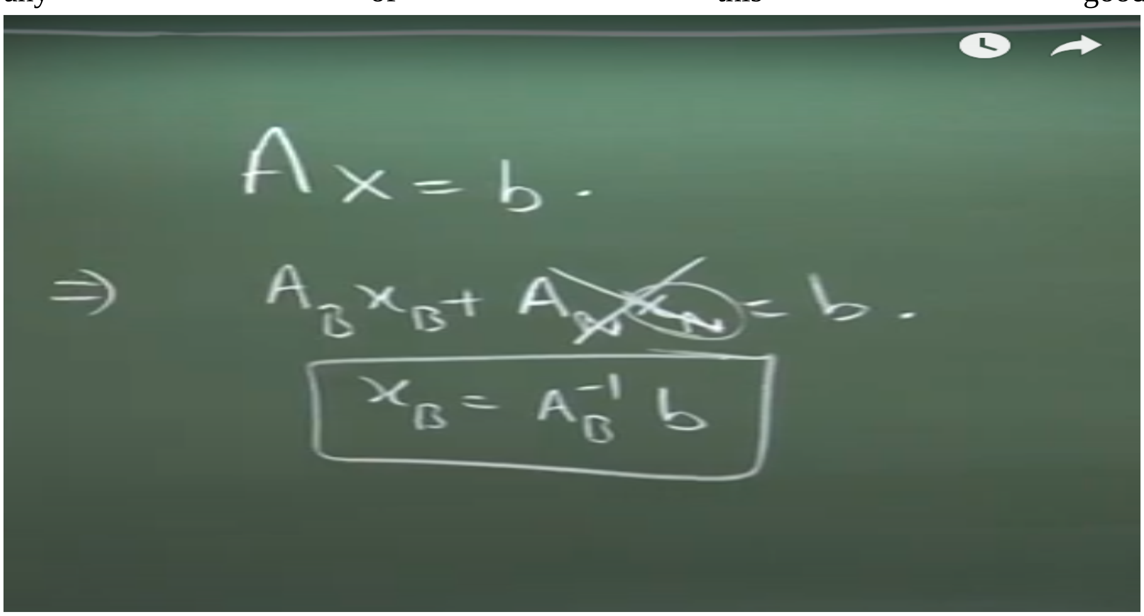
Here the number of variables is n right number of constraints are m. That means, b is the column vector of length m. And this in mind we saw that under this standard forms it is easy to describe a vertex. There was a one to one relationship between vertex and BFS we saw this proof last time right and just to remind you of what BFS is a basic feasible solution is actually characterized just by a subset of n. So, once you pick a subset of n once you pick a subset of n the remaining part B bar I am calling non basic.



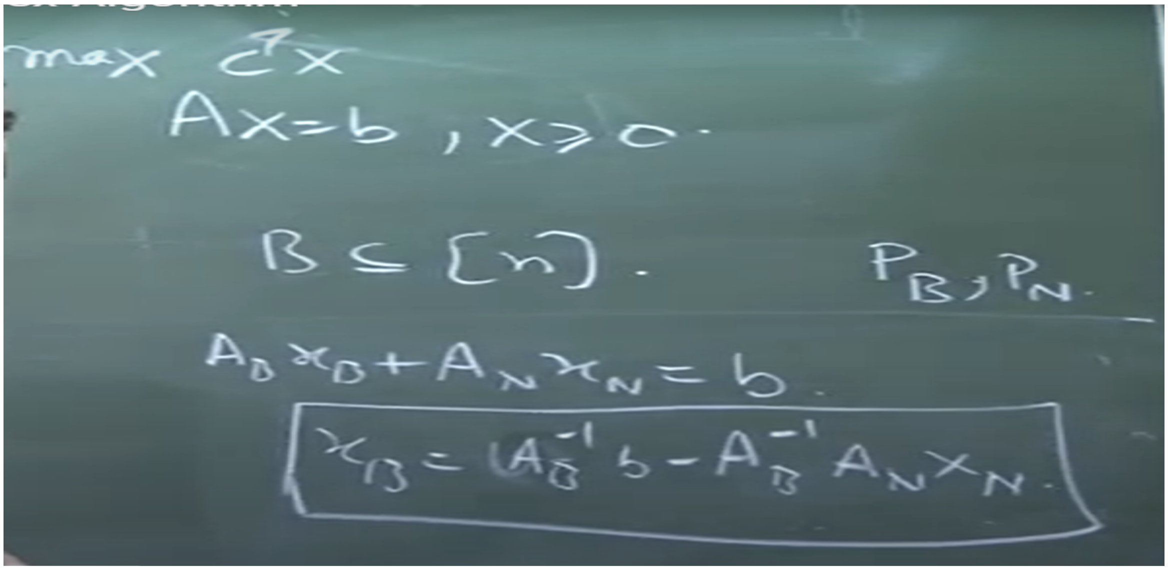
So, you have basic part non basic part what do you know everything in the non basic part is 0 right. Also AB is invertible what do I mean by AB once again A is an m cross n matrix. So, you have let us say n variables here and first constraint second constraint mth constraint. So, every column in some sense corresponds to a every column corresponds to a variable right it is giving you the coefficients in the constraints. So, then AB basically means pick all columns for basic variables.



This defines my basic feasible solution you might ask oh there is no solution here there is no x here right I talked about basic feasible solution I am just giving you a subset of n, but we do have a solution here why right. Since A is AB is invertible I know AX is equal to B this is by the definition of matrix multiplication this right this is just how we multiply matrix and vector and then we know that this is 0. So, then XB is AB is B. So, yes a basic feasible solution is a feasible solution which have lots of 0s, but to specify it I need to just worry about the subset if I give you the subset I have everything good. So, now with this understanding and if you remember the last time we did an example of from vertex to vertex to vertex let us just formalize a simplex method questions about any of this good.

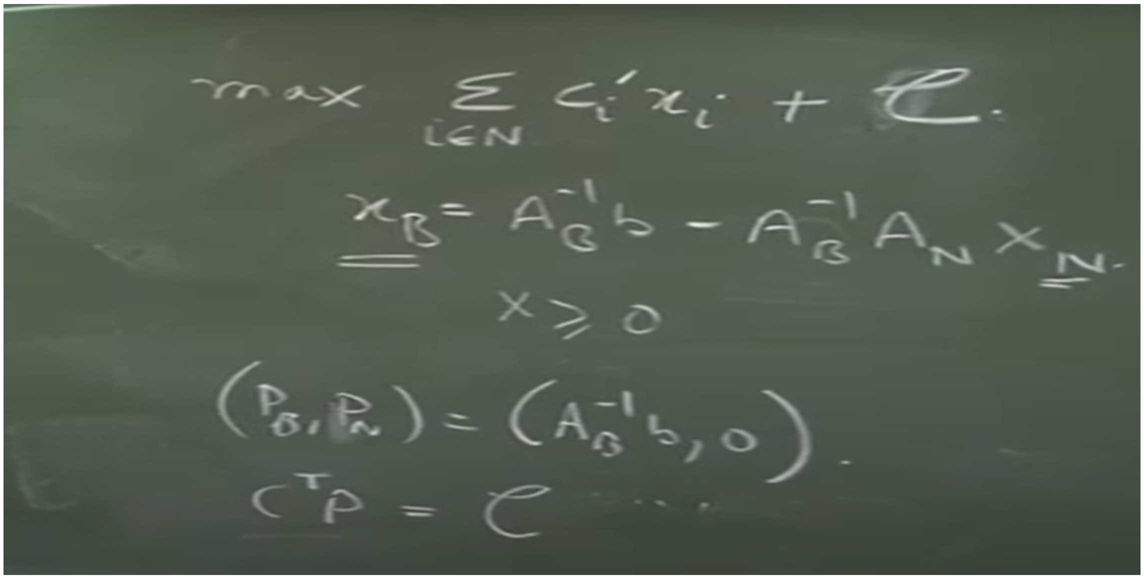


So, this equation is actually pretty nice I started with this, but then if I know this it is telling me that every basic variable sorry what did I do wait. So, let me redo it let us say I have a basic subset which is which corresponds to a BFS and let us say my solution here the actual solution is PBPN right, but forget about the actual solution at any point I can write my question like this right.



What is this telling you at this stage of the basic feasible solution everything every basic variable can be written in terms of the non basic variables right. And this is something which we did last time if you remember every time we move from one solution to other our basic variables are always represented in terms of non basic variables right. So, I have that and once I have that I can change my objective function also to be C prime psi let us say plus some big constant C.

What have I done? I had just written my linear program have not changed anything I have just written my linear linear program in a nicer basis you can think of it as doing row operations in your matrix. So, that my basic variables become diagonal right you remember Gaussian elimination right you what will happen if I am doing Gaussian elimination with basic variables basic variables are invertible. So, I can make it 1 1 1 and everything 0 otherwise right. So, I am to write all the basic variables in terms of this is not a new linear program this is the same linear program you can think of it in a different basis or after row operations only. But what is the advantage this gives us very clearly what is happening at this vertex right first thing is my PB PN comma 0 correct because x n has to be PN has to be 0.

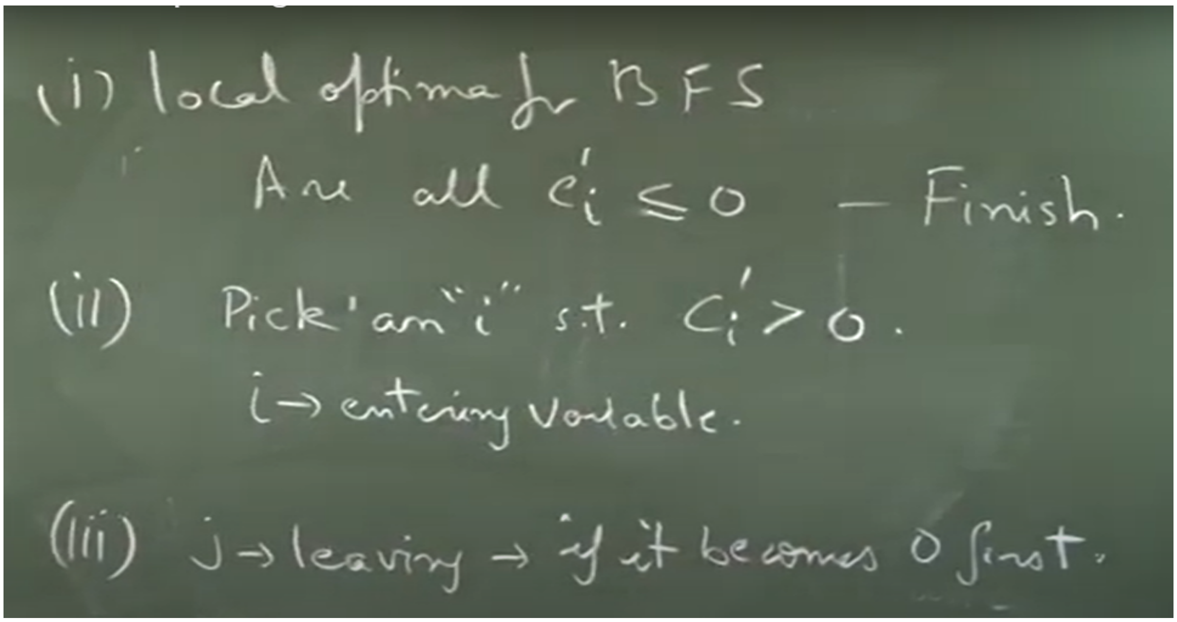


So, that means my P b has to be a b inverse b and my objective value is. So, I should say C transpose P where C is the original C this is equal to what do we know about non basic variables right. So, the objective value at this basic feasible solution is just this constant which is appearing here. So, this is the advantage of converting again when I say basic feasible solution it is the basic feasible solution here it is the basic feasible solution here, but this is much nicer and this is what we did last time too. PB and PN is the actual solution as x b and x n I want to keep it for the variables themselves PB and PN is the actual basic solution.

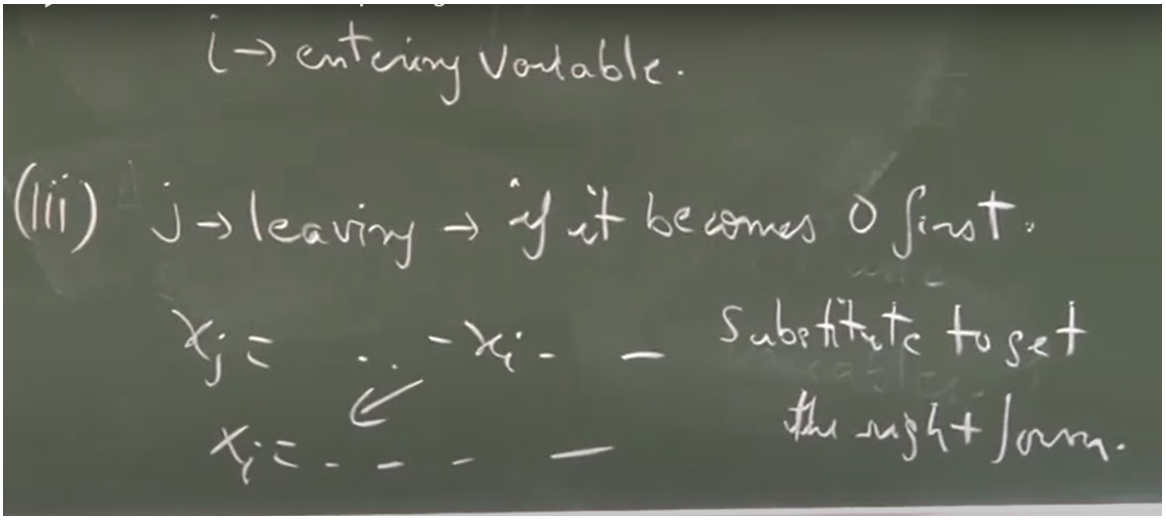
So, actual solution at BSF right and once I write it this way it is very easy for me now to see is there a direction which is increasing the objective function right. So, let me just write down the 2 conditions there is no sketch pen here. So, let me just write down here what do we know about BFS the value of non basic variables is 0 and A B very good right and what we did by example now we can see that it almost works in every case right. Now is C optimal how do we know that right if all these coefficients are negative right then I know that good. So, how to check local optima for BFS are all C i primes negative or let us say even less than equal to 0.

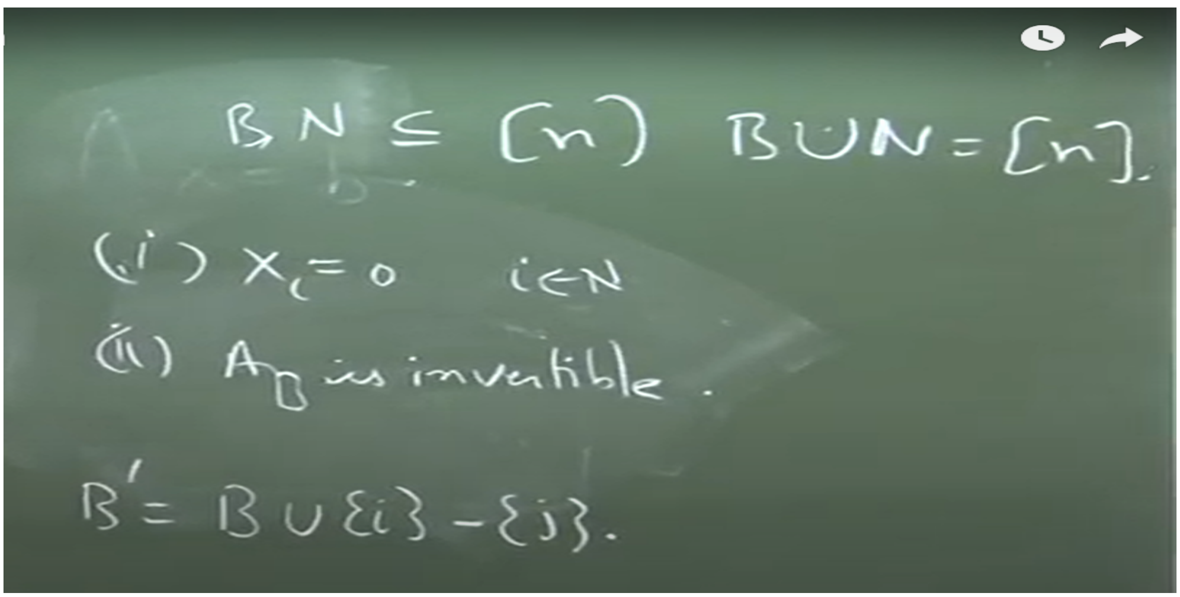
So, if they are all less than equal to 0 finish I have local optima is a global optima we are finished question. Sir, what will be my better C i is negative and negative because that should be better again you are getting confused between x i s and C i primes these are coefficients. So, if x i s are all 0 even though this is 0 you can make x i to be anything any delta this is not going to increase the value right. So, do not confuse between C i prime and x i right. So, now this is definitely our criteria for finishing right if we do not finish what do we need from our intuition of simplex algorithm direction of progress right.

So, how do we find direction of progress here whichever C i prime is positive if there are many C i primes take your pick start with take your pick I will have something to say about it later. So, pick an i such that C i prime is strictly greater than 0 this is my direction of progress right. Now, if C i prime is greater than 0 this i will become my entering variable what do I mean by entering variable entering in the basic feasible solution. So, this now in this subset in the B prime B prime the new solution is going to be B union i, but then I need to remove something I need to have a leaving variable also how do I decide the leaving variable I want to make sure that the x remains positive right. So, now what is going to happen all these things were positive you had some constant and then you had these x axis as you start increasing x's right you know that each of these are positive why x B is a feasible solution p B is a feasible solution right.



So, this is all positive we know that. So, now this is all positive here everything is 0 as I start increasing i one of the variable will become 0 first that is my leaving variable. Again I am writing this in informal way, but I am sure you can write down this mathematical equation in terms of the coefficient here and the coefficient here the ratio of this coefficient to ratio of x i right also the ratio the coefficient of x i should be positive if it is negative then we are happy then we can just increase x i to whatever we want if the coefficient of x i is positive in those equations this divide by that constant you want to take the minimum of that that is how much I can increase x i right. So, you can write that formula right this is not ok. So, important point is and we did this last time we had 4 equations we saw oh if I am increasing x 1 does this become 0 first does this become 0 first does this become 0 first whichever becomes 0 first that is my leaving variable.

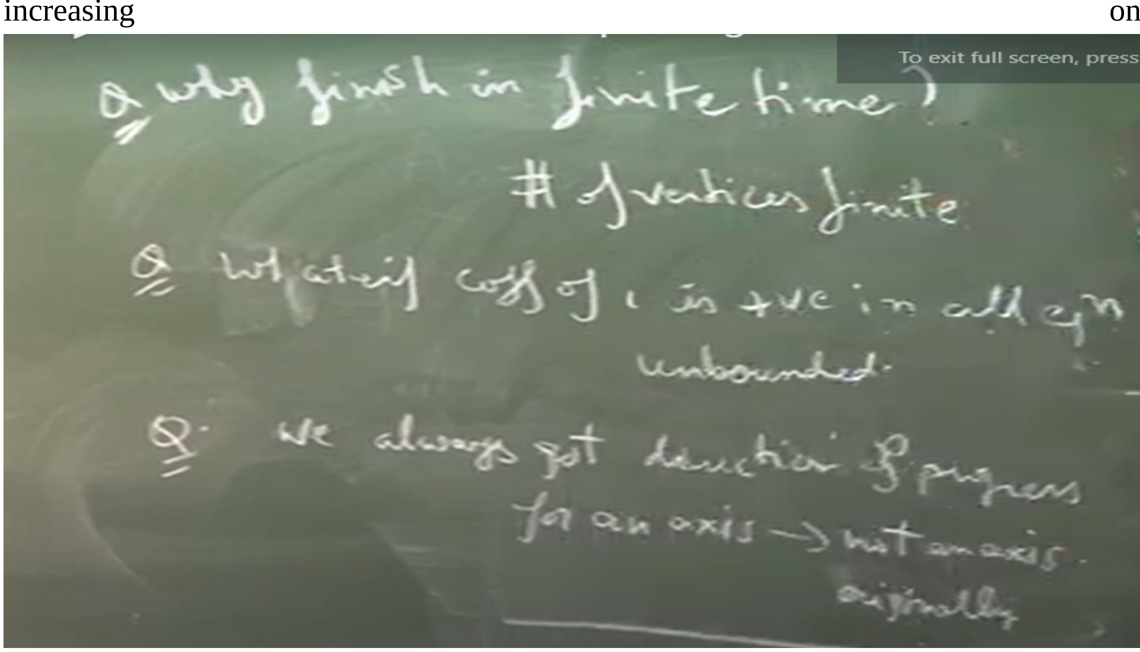




Now j I have gotten my new BFS pick one if the two ratios come out to be same pick one and put that right. Now once I do this I want to make sure that I get the similar form for B prime now, but that is not very hard why there is a single equation here where it is x j is in terms of non basic variables right. So, there is a equation yeah good there is some equation minus x i something just convert this into x i equal to and now this is a substitution which you will make in every other equation as well as the objective function. So, substitute to get the right form right this will keep going at some point since why is it going to finish why does it have to finish in finite time? Right number of vertices right. So, let us see how good you understand this ok.

So, let me ask a few questions what can I erase and yeah we play this game where someone answers then they are out of this game then they cannot answer the next question then someone else answers yes. Sorry it has to be negative if it is positive then it is not a problem it is some negative some coefficient here if it is positive then I can increase x i and x j will never be a variable right. So, question why finish in finite time? So, there is out of this game number of vertices finite what if coefficient of i is negative in all equations I do not I would not find j right it could happen that if you look at all these equations the coefficient of x i is negative everywhere I never said that this quantity is going to give you positive coefficients right sorry it is all positive it is positive in all equations then what does that mean? All of you are out from the game good. So, this was right good always we are able to find entering variable we are always able to find leaving variable ok. Now, this is a nice question right if you look at this what I talked about was once we found out that this was not a local optima we had a direction of progress right and this direction of progress comes out to be in the direction of a variable right.

So, we always got direction of progress for a for an axis right it is aligning with the with the axis right first see that there is a problem here there can obviously, be shapes where the direction of progress is not aligned with any axis right for example, if my feasible region was like this right I cannot have x 1 or x 2 as my direction of progress, but here my direction of progress always turns out to be a particular axis that seems wrong how can I. So, yeah Prathush wait let us do you have an answer it seems to me no your question first you answer my question then I answer your question ok. So, but is this dilemma is clear right that it seems like we are doing a great thing here every time we are able to move in the direction of an axis, but you are out of the game. This means like either x 1 I am increasing or just x 2. So, x is means x axis y axis right I am I am just increasing one



particular variable right this is this is a direction of the axis.

So, y is my direction of progress always an axis does not seem to be the case in many many places tricky question I understand ok. So, your final answer is correct, but I am not sure whether you have connected all the dots you need to say more. So, that I am sure you have connected all the dots anyone who are part of the game. So, not part of the game also that that was the I think that is what Kumar was saying no I will keep asking after 40 years you allowed to ask this question multiple times. So yeah anyone know I give you the answer is the axis here an axis in the original solution no we have changed the basis.

So, many times this is an axis now right. So, do not get confused by this notion that the equations written here and the equation written here are in the same basis you have changed the basis lot of times. So, what the x I which seems to you to be the direction of basis now it is some direction of basis depending upon you know 3 times a b 1 inverse multiplied by a b 2 inverse multiplied by a b 3 inverse or something. So, no problem right. So, this is not an axis originally we have or other way to say it is we have rotated our axis in such a way.

So, that are edges are aligned with the axis that is another way to better way to think about it in the original way we are changing everything, but in this 1 we are actually just changing 1 x I. So, what if coefficient of I is positive in all then you can just make x I to be infinite everything will remain positive all variables will remain positive and your coefficient in the objective function was positive. So, your maximum is infinite. Those were the only things those were the only allowed leaving variables right leaving variable was 1 of the variable whose coefficient was positive everything is negative then we have a local optimum we are done yes because we have to finish in finite time. If I have to increase infinitely at some point I have to find a direction where I increase right.

No, no feasible set is convex that is all we know right and now we come to co-ordinate question right. So, what have I not described what case have I put under the rug and you did not bet an eyelid though that will help later, but that is not the problem. No great I am already at the optimum you are I am let me write it. What was it c plus summation correct and then what was the equation here x b.

Sorry here. So, again you are roaming around the right solution, but I said this is it positive or right. So, is it positive or is it non-negative right that is the only thing I know it is non-negative. So, what if my equation looks like x g equal to 0 I have not told out that possibility right. This corresponding constant could be 0 and there is a negative non-zero coefficient in front of x i. So, now I know that I cannot increase x i.

So, I can switch x j to x i that is x j, but what if this becomes a cycle right. So, in some sense you are at the same solution you are not changing the solution you are just denoting them by different p. There could be 10 variables. So, probably you can easily argue that it cannot be just 2 variables, but what about a cycle of 10 variables.

It can be 2 that is what I am saying yes. So, then your objective function does not increase you are not saying that you are at a local optimal. What if I stay in that vertex I am just roaming around the basic feasible solutions at the same vertex. This is what we call degeneracy with the whatever is the right spanning. Except this have I covered everything let us jog your memory what have I not told you. In the implementation of everything happy selecting the first.

These are the 2 things which are remaining everything else I have either given you an idea or explain to you how we will do this. This is a complete description of simplex algorithm except these 2 parts and you realize what is degeneracy cycle of basic feasible solutions.