Linear Programming and its Applications to Computer Science

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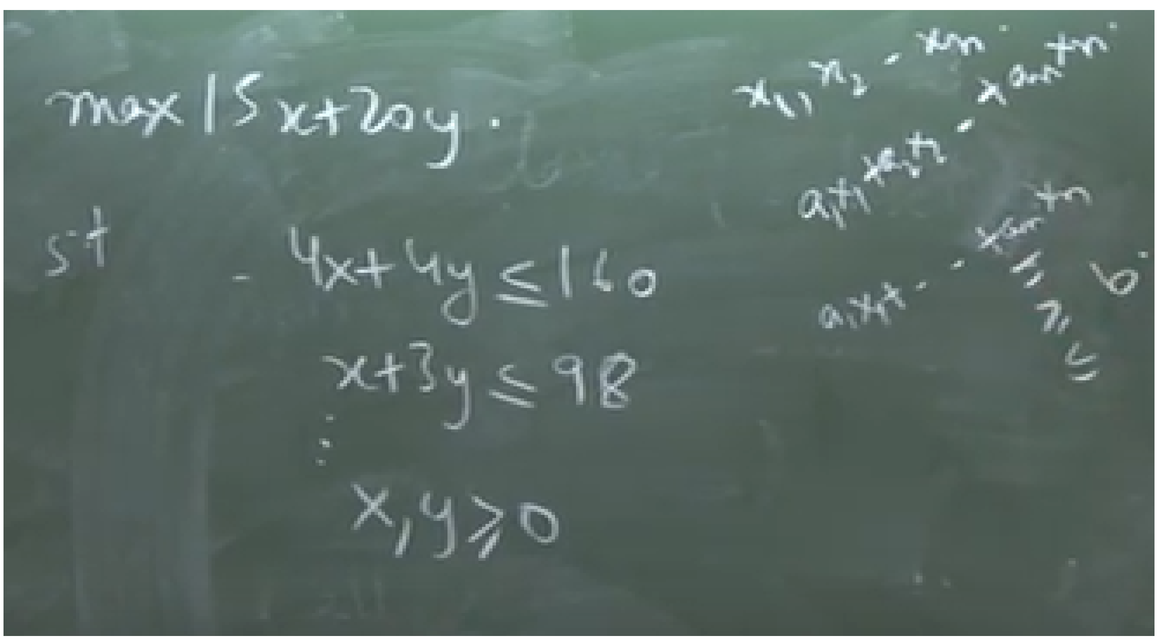
Lecture – 02
Introduction to Linear Programming

First formalize now what is a linear program? We have seen an example of it. So, we are almost there. Let us just formalize what a linear program is. To do that I have to define optimization in general. Optimization simply is maximization or minimization. When you would realize that most of the problems in this world are optimization problems.

You want to maximize the number of marks in this course. You want to maximize the amount of money you will get in your placement or in your jobs. You want to maximize the amount of peace you have in this world. So, many difficult problems like how to live your life, how to solve world peace, all of those are in general an optimization problem.

Mathematically if I want to write it, I have some variable, I have something which I can change. I want to figure out what is this variable. It does not need to be a single quantity. This could be a list of variables. I want to maximize or minimize a function of x.



In the case of beer problem, it was 15x plus 20y such that I have some constraints and so on. So, very general optimization problem is maximum minimize some function of our variables such that sub constraints are satisfied and As I told you, there are almost all problems can be written as optimization problems and that means solving such optimization problems is very hard. We have no hope of solving optimization problems in general. Great, we are stuck.

We have very important problems. We do not know how to solve it. What should we do? I think this is a golden rule of research. If you cannot solve a problem, simplify it. So, what is the next step? We ask are there classes or subset of optimization problems which can be solved? If we cannot solve optimization problems in general, are there specific kind of problems which we can solve? But then what specific kind should we be looking at? And this is more of a question about humanity rather than a question about mathematics.

You have these big set of problems which you cannot solve. Now I am asking are there subsets which we can solve. What kind of subset should we be looking at? I would say simply put a class of optimization problems are interesting or we should consider that class which satisfies 3 properties. One, they should be solvable. If you cannot solve them, then studying them is not very helpful. But then we can take a very simple instance of it and solve it.

Obviously they are useful if many applications can be derived from this class of applications. Another way to put it is many situations I would say real life can be modeled by this class. So, this class is not just easy, it is also useful. There is a point of solving these equations because there are issues, there are problems in your life which need to be solved and which can be solved by this particular class of optimization problems. And the third reason is not really a humanity concern, but I would say as a Mathematician and as we go into deeper into this theory you will realize it is also a concern for us as a mathematician which is that this class of problems should have a beautiful structure.

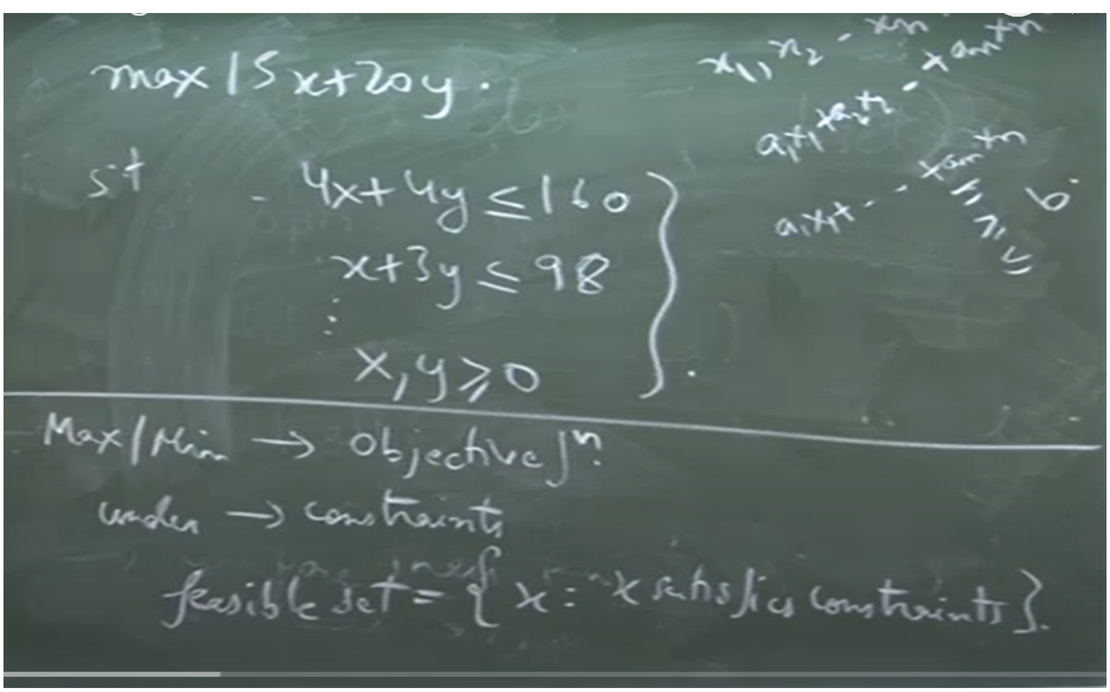
Now, this is pretty vague, but just to give you an example in the case of problems we talked about in that we had a nice geometry where the solutions were all continuous to align with each other and then we found that the solution was at a vertex. This was one example, but having this kind of a structure really tells us much more about the solution set rather than just giving us the value. I can't explain it more than this. If you are familiar with this word you will see this duality theory for linear programming which will tell you what I mean by having a beautiful structure inside this class of problems. It turns out we have a candidate and you shouldn't be surprised what will be the candidate? Linear programming.

This is what we have been talking about. There is a subset of optimization problems which satisfy all these three checks all three properties and this is called linear programming. That means we are going to have efficient solution. We will see that many applications the names of them I will give it to you today and many of them we will see it in this course specially in computer science. And It has beautiful structure as I said which is exemplified by what I call the duality theory.

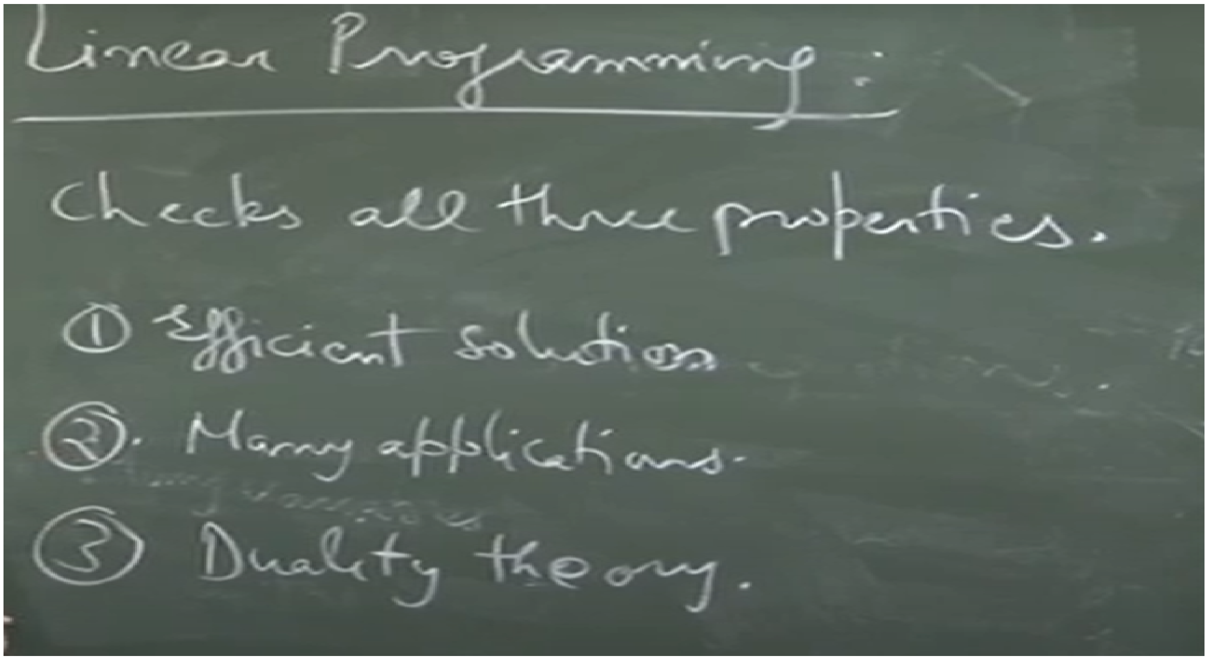
and Just to emphasize what is linear programming? It is a subclass of optimization problem where your optimization function is linear and your constraints are also linear. If you are confused about linearity remember our problem was that constraints like 160 equal to I think 98 so on. All these are linear constraints. In general if you have x1, x2, xn then a linear function is a1x1, a2x2 plus anxn. A linear constraint is anxn equal to less than equal to some b.

These are simplest kind of constraints, simplest kind of objective function. When we restrict our attention to these kinds of constraints and these kinds of objective functions we get this very nice class of linear programming which is going to be have efficient solution many applications and very nice structure. This is why we are learning linear programming. Just to set the notation remembers for an optimization problem we maximize, minimize some objective function. In this case the objective function was 15x plus 20y under some constraints here there were 5 constraints, 3 equations and x greater than equal to 0, y greater than equal to 0.

There were 5 constraints and then we have what we call the feasible set. The feasible set is set of all x such that x satisfies all constraints. So we want to find the best x from the feasible set. The objective function is maximize or minimizes in the feasible set that is the problem of optimization and now for the linear programming our objective function as well as constraints are linear. So, this question is answered this is optimization problems with linear objective as well as constraints.



This was the major part of the first lecture and now we want to answer the other 2 questions. Why do we want to learn linear programming and how are we going to learn linear programming?



Now why is already written. This is an interesting class of problems which have which we can solve which have many applications and has beautiful structure. Now one thing which I can give you more details about is the application. So let me tell you in brief how the amount of areas where linear programming has made an impact.

So if I want to just write out a list of things where linear programming is important. The first one we have seen which is resource allocation. This is used in management. Portfolio management can be solved by linear programming in many cases this occurs in finance production line optimization this occurs in manufacturing network design in telecommunications machine learning complexity theory computer science. So this is just a small list you can grab from the internet, but the point here is that there are so many diverse areas where linear programming has been helpful.

Another way to emphasize the importance of linear programming is by saying the number of departments in a university which can be offering this course. So if you are taking a linear programming course obviously you can be a computer scientist, you can be a mathematician, you can be a economist, you can be an electrical engineer, you can be a guy in finance so on and so forth. So there are at least six or seven departments in every university which offer this linear programming course. So this shows the breadth or the diversity of applications which this simple class of optimization problems span. So it is really really ubiquitous you can you can find linear programming almost in any place.

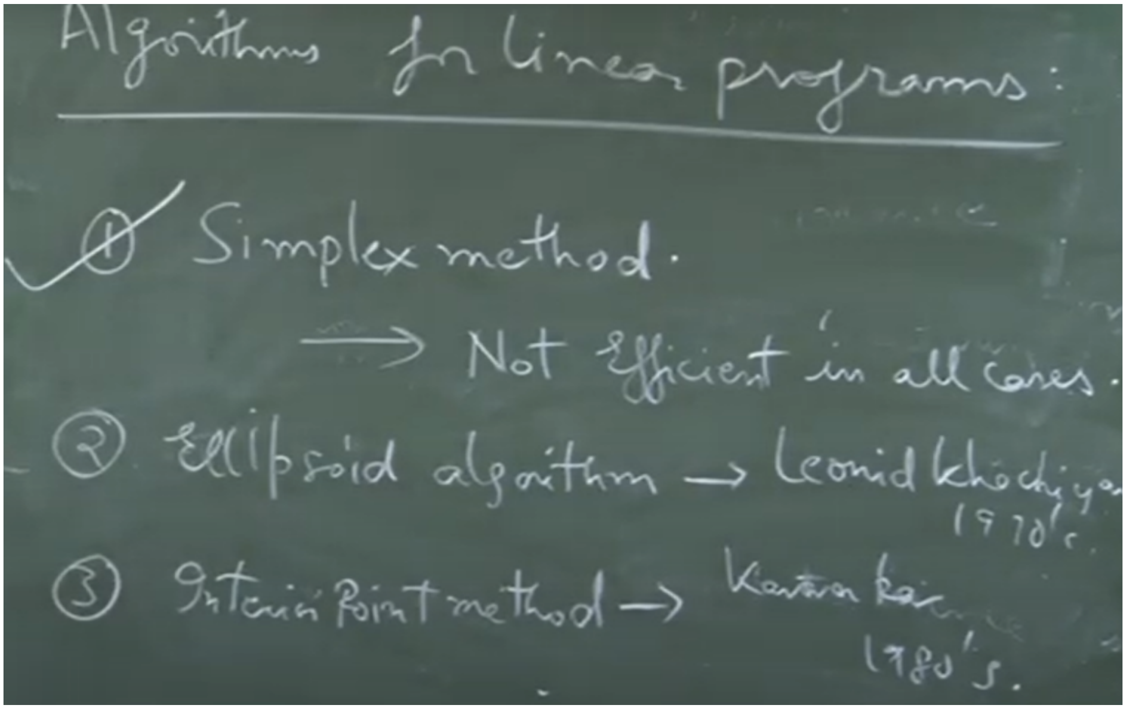
Many many problems can be framed as linear programming and solving them is important for multiple departments. But for us we will be concentrating on the aspect of computer science. It turns out there are many many areas in itself in computer science where we can apply linear programming techniques. Obviously algorithms and in general theory of computing, then machine learning, algorithmic game theory, network design. So even in just one area out of this there are multiple sub areas where linear programming makes an impact.

In this course once we have learnt what is linear programming we will see its many applications in such areas. So we will briefly talk about some of the basics in this area and then show how certain problems in these areas can be modeled as a linear program. We have answered what? Why because it is a small set of or it is a simple set of problems which arise many many times in different different areas that is why we should know how to solve these problems that is why and in terms of how you will see that we have these linear programs we will write these standard forms. The first thing to learn would be linear algebra and I will talk about it very soon. Once we understand the basics we will talk about convexity theory and all these are small steps which arise very naturally when you want to understand linear programs.

This will give us some of the solution methods and then the rest of the course will be focused on or half of the course will be focused on applications in CS. This is the plan to understand linear programming. Before I finish this lecture I just want to tell you why we want to understand linear algebra. But before that let me just clarify one of the things in terms of algorithms for linear programs. This is something I already said that we can solve them efficiently.

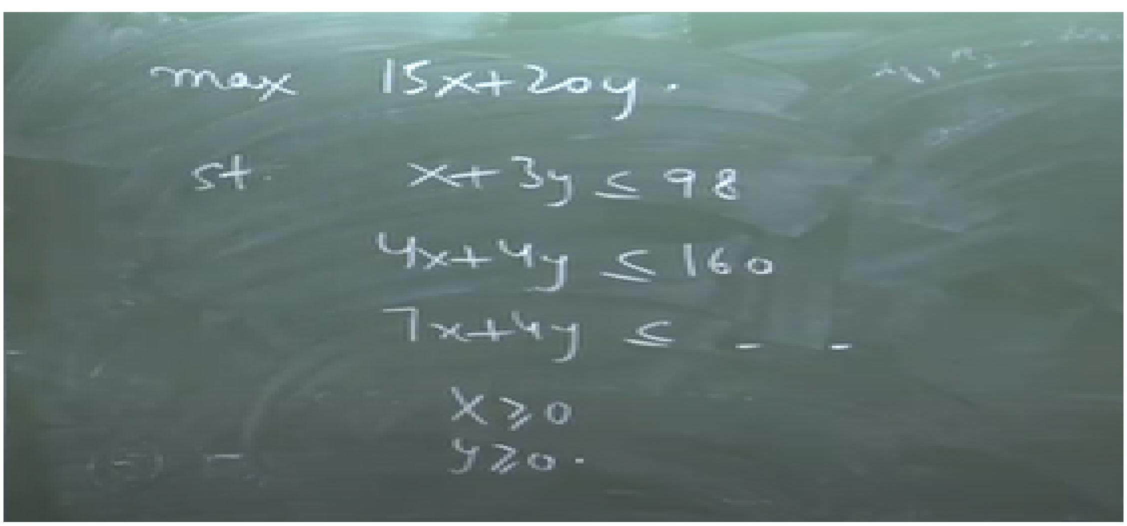
Let me just mention that the first or the most prominent method by which people solve linear programming is called simplex method. We will see it in this course. This is not efficient and efficient means polynomial time if you are familiar with time complexity and stuff it is not efficient in all cases. There is lot of research in finding modifications of simplex method which is efficient for all cases but still we do not have such modification. The first efficient method was called ellipsoid algorithm.

This was given by Leonid Khachikyan. This was around 1970s I think. And the method of choice now if you want to really solve it efficiently if you do not have weird cases is called Interior point method. This was long series of work but one of them was given by Karmarkar.

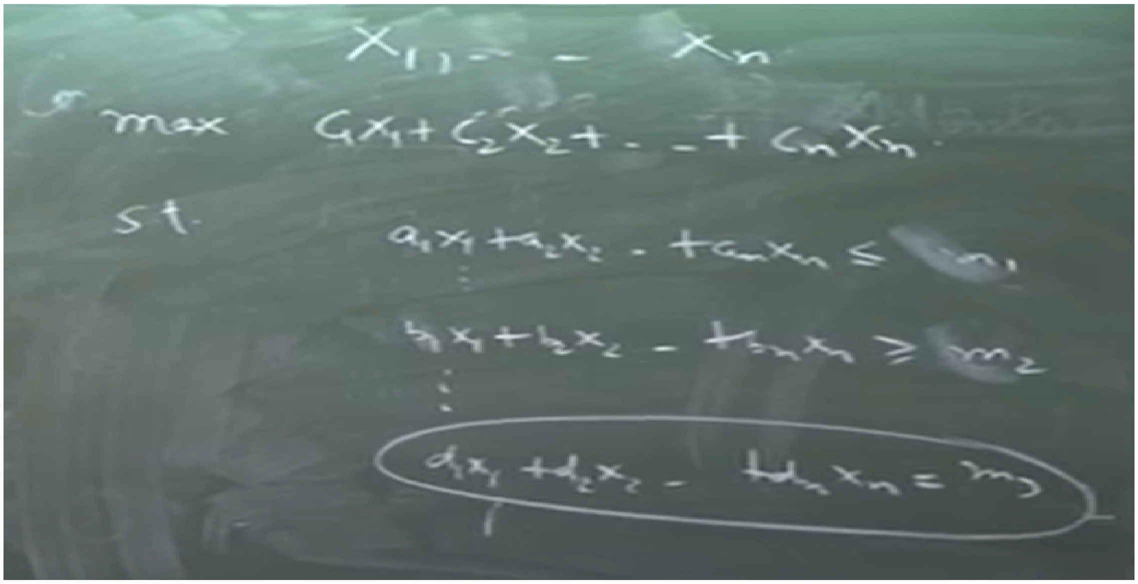


This was I think in 1980s. So this is the method in practice most of the time. This is theoretically efficient but it is very hard to implement. People do not use it. This is useful in practice as well as it is provably efficient. We will look, at this and if time permits some idea will be given here.

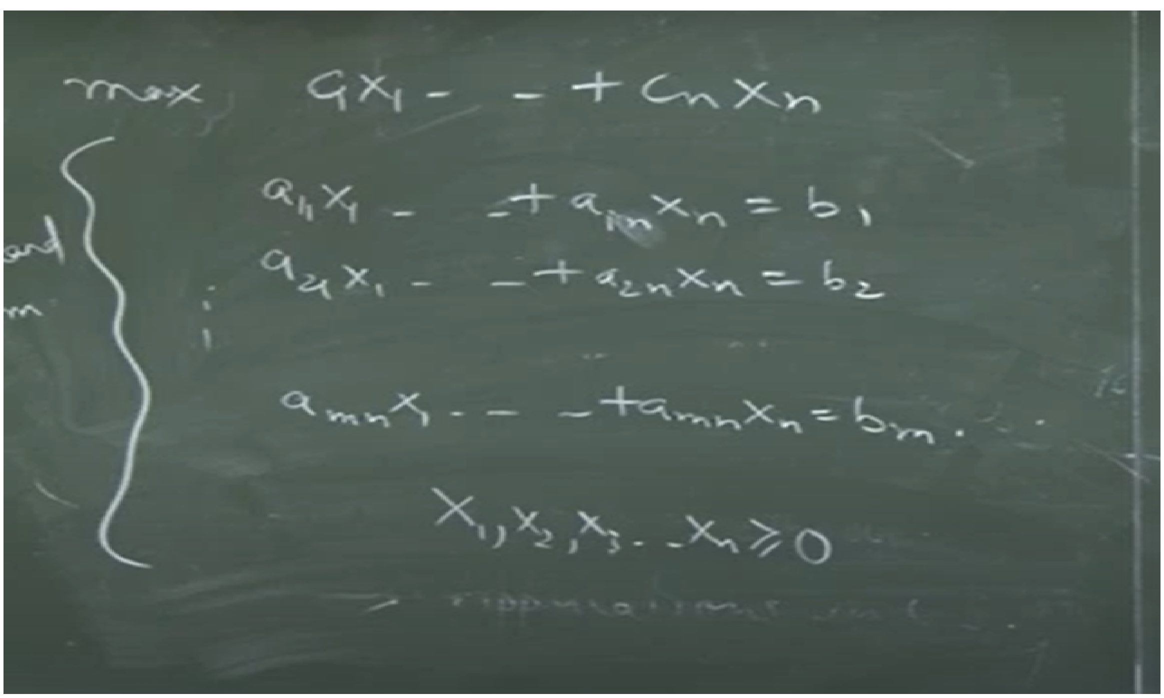
This is just to tell you that linear programs are efficiently solvable. Ok, we are almost done. Let us just see what will be the next lecture about. As I said to understand linear programming we have to understand linear algebra. So why linear algebra? Remember Our problem to start with the example was this such that I might be slightly off with the coefficients but it looked like this.



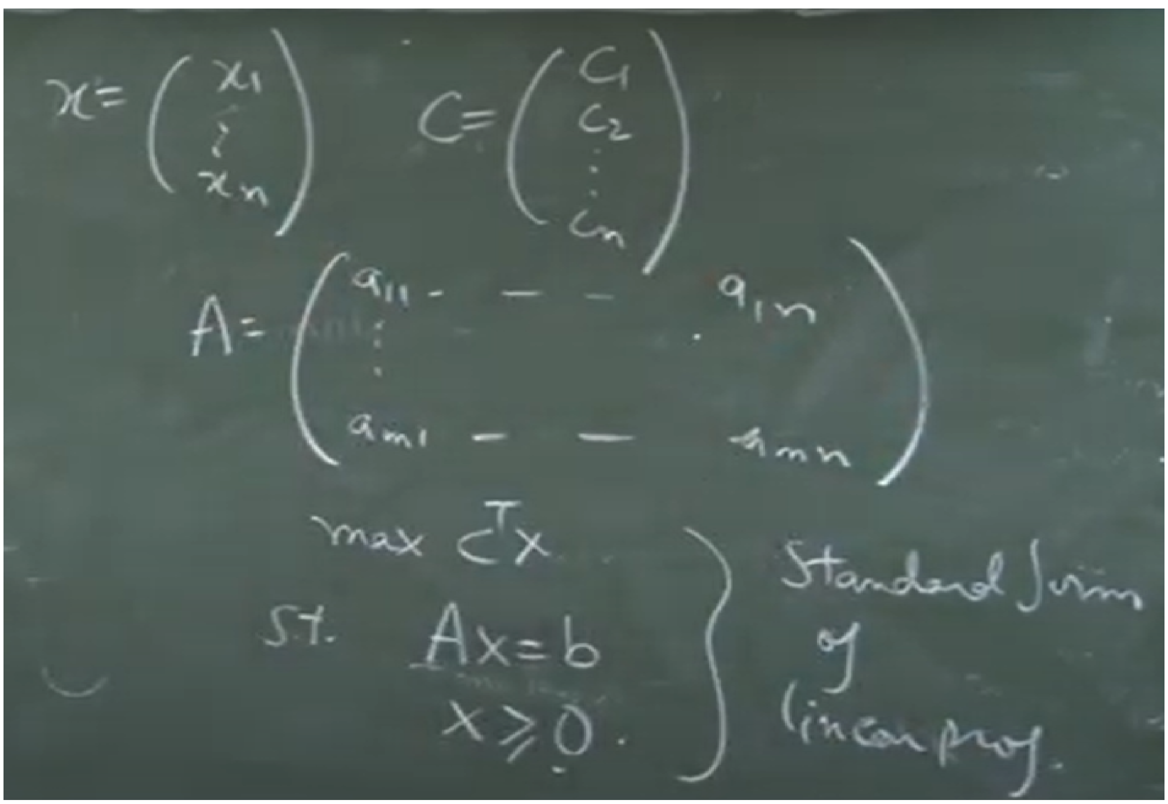
y greater than equal to 0. The other way since I said that linear programming is a set of problems where you have linear objective function and linear constraints.



Then I can think of my constraint as suppose I had n variables x1 to xn in my equations for n variables my maximization might will look, like c1 x1 plus c2 x2 plus cn xn. I am writing down a general problem in a particular instance we would know the value of c1 c2 cn and would like to figure out the value of x1 x2 xn. In this case this is x1 this is x2 c1 is 15 c2 is 20. Such that I can have conditions like a1 x1 plus a2 x2 plus an xn less than equal to b1 some conditions like this then I can also have conditions like b1 x1 b2 x2 sorry I should say m1 let us say m2 I can also have conditions like d1 x1 d2 x2 dn xn equal to m3 so on. Right,



Generally the problems would look, like this, but again by some small trickery we can change all our constraints to be of this kind. And this we will learn very soon in the third or fourth lecture even though at this point I allow my constraints to be less than equal greater than equal equality there is a way to convert between these constraints. So actually I can say that any general problem will look, like this is my first constraint and this is my mth constraint, but this equivalence possible if you have this constraint in addition. Greater than equals to zero. So yes from what we have discussed till now a linear program will look, like this, but you will very soon see that I can change it into these kind of equation I can only worry about equality constraints if this greater than equal to 0 constraint is there and we will prove it in next few lectures. So great so then any linear program actually look, s like this and if I want to write it succinctly I can think of x as my vector of variables I can think of c as my cost vector which keeps all the constants here and my constraints I can write them as a matrix and if I write it like this then I can write this what I call the standard form of linear program simply.



The standard form is actually maximize $C^T x$ if I take the dot product between these two vectors I will get c1 x1 plus c2 x2 plus cn xn and this equations will become such that Ax = b where x is positive that means all x1 x2 xn are positive this is called the standard form of linear program. What it means is whatever linear program you take it can be viewed like this equivalently. Let me warn you this is not the only standard form we will look, at many other standard forms the point is if there is a standard form then any linear program can be transported to this standard form. Now coming back to why do we need linear algebra if you look, at this in terms of words you want to look, at all solutions here which optimize $C^T x$. So, all x such that Ax = b, x is equal to 0 and then find max of c transpose x all these x's. Right.

Remember this is what we call the feasible region or feasible set. So if these are the two steps that means I should first concentrate on finding the feasible set that means all x's such that Ax equal to b and x greater than equal to 0. We want to find all such x's such that Ax equal to b and x greater than equal to 0 as a student of mathematics you should automatically light up your eyes should light up and say oh this I have seen before this is just a set of linear equations which we have solved for centuries we already know how to deal with this. This is a central problem in linear algebra. So, our technique should be what do we know about solving Ax equal to b and then extend it to find the feasible set of all x's for Ax equal to b and x greater than equal to 0.

That is the plan now for next two lectures we will be looking at linear algebra that means how to describe the solutions of Ax = b. Once we know that we will go back to our linear programs learn more about it and then apply these tools to the feasible set Ax equal to b x greater than equal to 0. This will allow us to have a an efficient solution for linear programs. So, in next lectures we will be covering linear algebra in other words how to solve Ax = b.