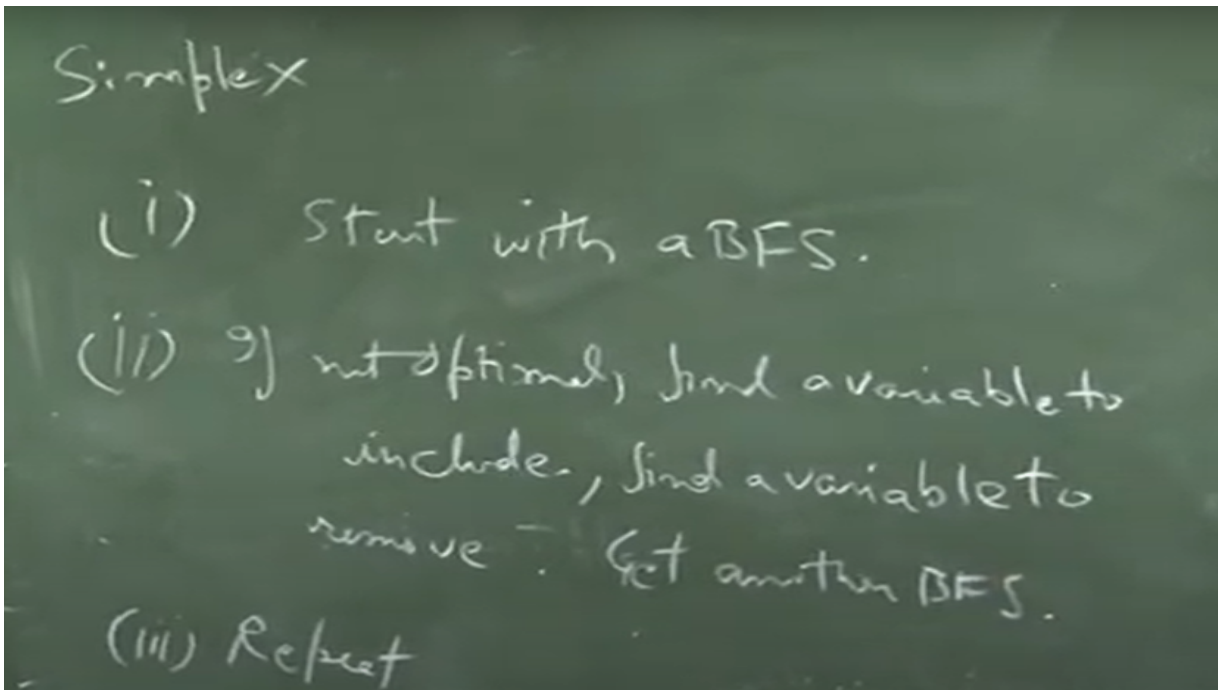
Linear Programming and its Applications to Computer Science
Prof. Rajat Mittal
Department of Computer Science and Engineering
Indian Institute Of Technology, Kanpur

Lecture – 19
Simplex Algorithm

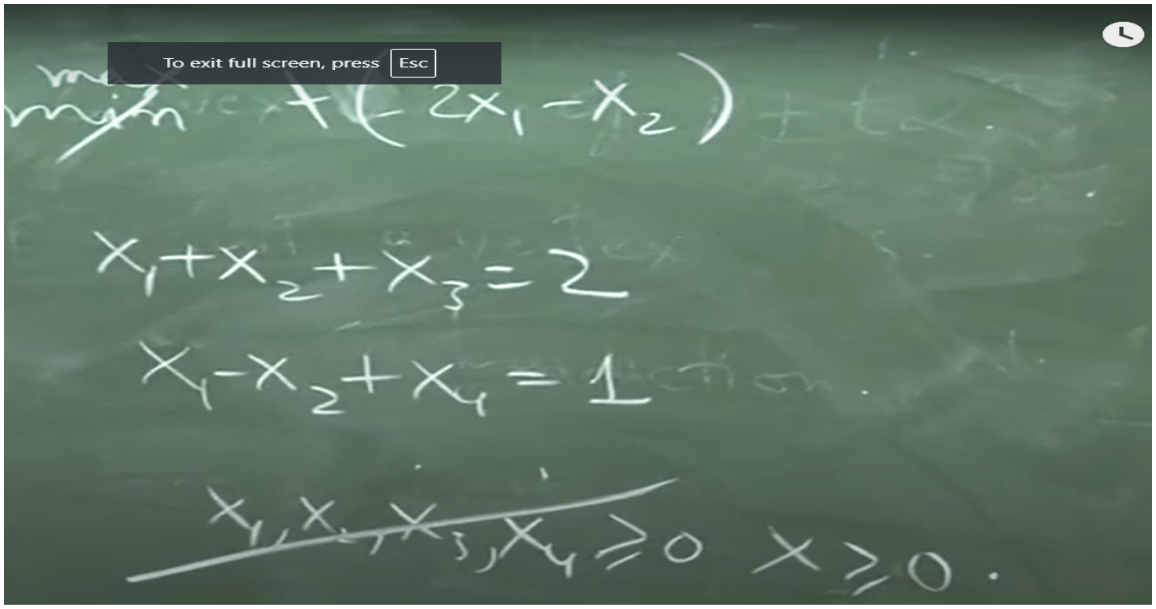
And now, what will be our simplex algorithm? Find the BFS, from the BFS find a direction and edge to move, what is going to be an edge? You kick out a variable include another variable right. There are two BFS's, you have to have an exchange of a vertex of a variable there. so now, your simplex algorithm is slightly more concrete. Again, how will I start with the BFS? I do not know yet, we will think we will worry about it, but then if not optimal, find a variable to include and I have not shown it, but you can show that if two vertices are connected by an edge, only one variable is going to be replaced, one will enter, one will leave. Ok So, that is not very difficult at most, so one variable. So, find a variable to include, find a variable to remove, we have another BFS right.



So, good question, there is something which we call degenerate solutions. So, what I have shown to you that a BFS corresponds to a vertex, it can happen that multiple BFS's correspond to a vertex. So, the number of vertices need not be I have chosen and that leads to some details, I will shortly talk about what is a technical issue. So, it is not

exactly I have chosen number of vertices are not I have chosen.

Ok, so, great again remember we wanted to do simplex algorithm, we had an idea, we did a bit of research our algorithm became more concrete, but now we want to make this more concrete. So, we will do a bit of more research, more math which will help us make this more concrete. And the thing which I am going to do is very very simple, take an example and try to do simplex algorithm and this will give us an idea of how to formalize simplex algorithm. Let us take this example and please do not solve it using graphical approach. So, let us do simplex algorithm, this is the program I want to solve.



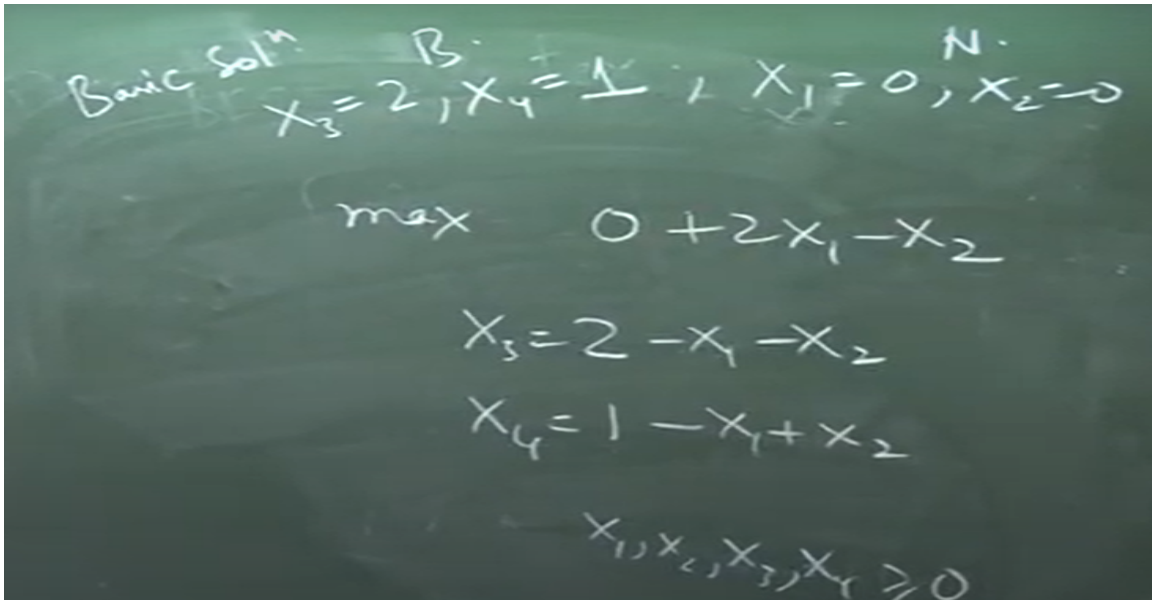
The first thing is to make it into the standard program, and how do we do that slack variables, we want to make these equality. So, if this will unnecessarily trouble, so let us keep it as max, but you can do with min then I have to put whole bunch of negative signs which I am not a fan of. So, I will just write it, in this case we are really lucky we have a BFS, can you see the BFS, if you see the BFS do not shout write in your copy. So, this is the BFS, this is BFS, what is m here 2 that means, I want just 2 variables which are non zero rest everything should be 0. So, in this case easy to find the starting BFS that will not always be the case this is going to be a big step, but at least in this case we can start.

How many equals to the BFS, raise your hands if you see the BFS, yes good. So, the BFS is x 3 equal to 2, x 1 equal to 1 rest I do not care. So, this is the BFS, so this is the BFS, x 3 equal to x 4 equal to 1, these are my non basic, these are my basic. And now if you remember the last quiz question, can I express this thing, can I express this just in terms of basic variables and non basic variables. You remember how we did the last question, last question we had leading variables free variables.

And since every leading variable can be written in terms of free variables, just removed

it, it is the same thing that that intuition is not going to help us that solutions of A x equal to b. I can write x 3 as 2 minus x 1 minus x 2, I can write x 4 as 1 minus x 1 plus x 2. I am saying again what I am going to do is write my optimization function in terms of non basic variables, oh yeah great. But this is in this case this is super easy, but this is not, but we will have to do it, so as to find out what is the value there, what is the value of this solution, the value of this solution is 0 here. But then I can write 2 minus x 1 minus x 2 is equal to 0, so I can write x 1 minus x 2, can you help me out x 4 right so why I am written in terms of this in terms of non basic variables, because now it will help me find the variable such that my value can increase.

I know that x 1 and x 2 are positive, I can bring them down, I want to maximize, so now if I what can I do, I can decrease x 1, increase x 2 what should I do. So, now x 1 will be my variable which I want to input, but when I take x 1 when I increase x 1 from 0, I have to make sure these should not become less than 0. So, this will tell me how far I have to move on the edge when I encounter a vertex. So, x 1 and x 2 are 0, I cannot increase x 2, I want to increase x 1 this is the improvement direction, but if I increase x 1, I can increase x 1 only up to 1 because of this right after that this will become negative. So, I will keep this here, what will be my next BFS, now x 3 and what is the other variable of x 1, x 2 and x 4 are non basic correct.

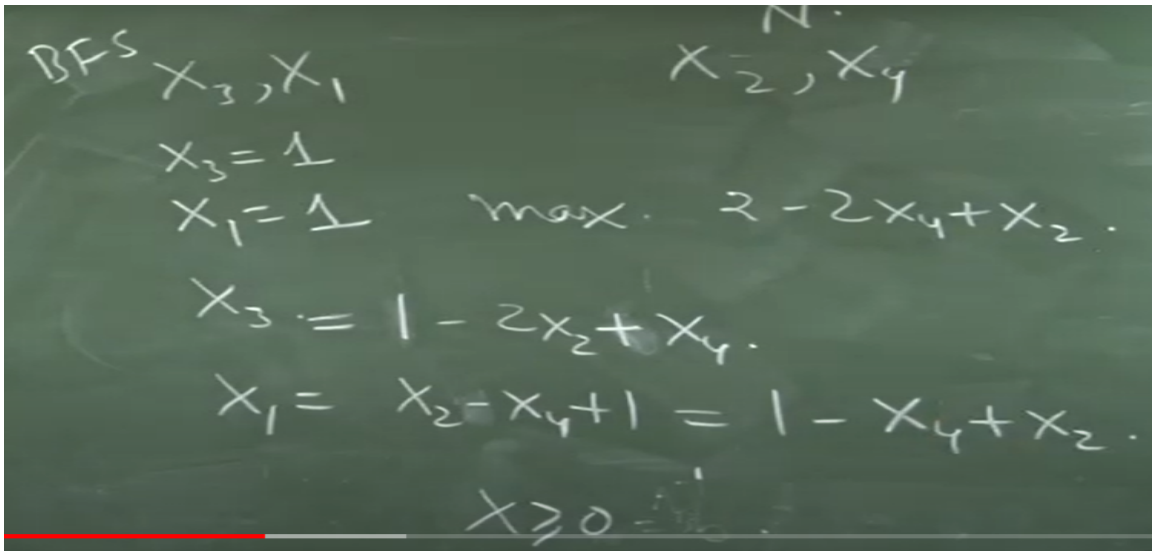


What is the value of new value of x 3, 1, what is the new value of x 3, 1, what is the new value of x 1, x 2 right. And this fact that I could now write everything, now I can write x 1 and x 3 in terms of non basic variables that happens because. So, again what am I going to do, I am going to write my objective function in terms of non basic variables. That means, I am going to express this right x 3 equal to this x 4 equal to this I want to change it. So, that I write it in terms of x 3 and x 1.

So, I will have x 3 equal to x 3 and x 1 in terms of x 2 and x 4 and that will just be done by replacing everywhere x 1 in terms of x 4 and x 2 right. But this fact that you can always write your basic variables in terms of non basic variables should be obvious to you, they should not come as a surface it is not this example where I am able to do it. Are the. Yes exactly I know this right I know. So, I am going to write this in terms of x 3 what did it do x b in terms of non basic variables sorry it should be.

So, implicitly I know that expression has to be possible I am just doing it in an easy way right. So, can you write now what is x 3 and what is x 1. x 2. Now, in x 3 minus you guys have pen and paper can you tell what to write here see if you help me out you go home early I do not know why you are resistant to helping me out.

Plus x 2. 2 minus 2 x 2 that is all there is no x 1 x 4 no x 4 and x 3 minus x 4. So, x 3 minus what am I maximizing plus minus I have to look at my calculations oh my god 1 minus 2 x 2 plus x 4. So, this is the first example I am going to write it in terms of x 3 minus 2 x 4. Now, I have this as my BFS the solution the value at this point is 2 I got an advantage right my optimal value increase I am maximizing I got to 2 from 0 right. This is my set of conditions.



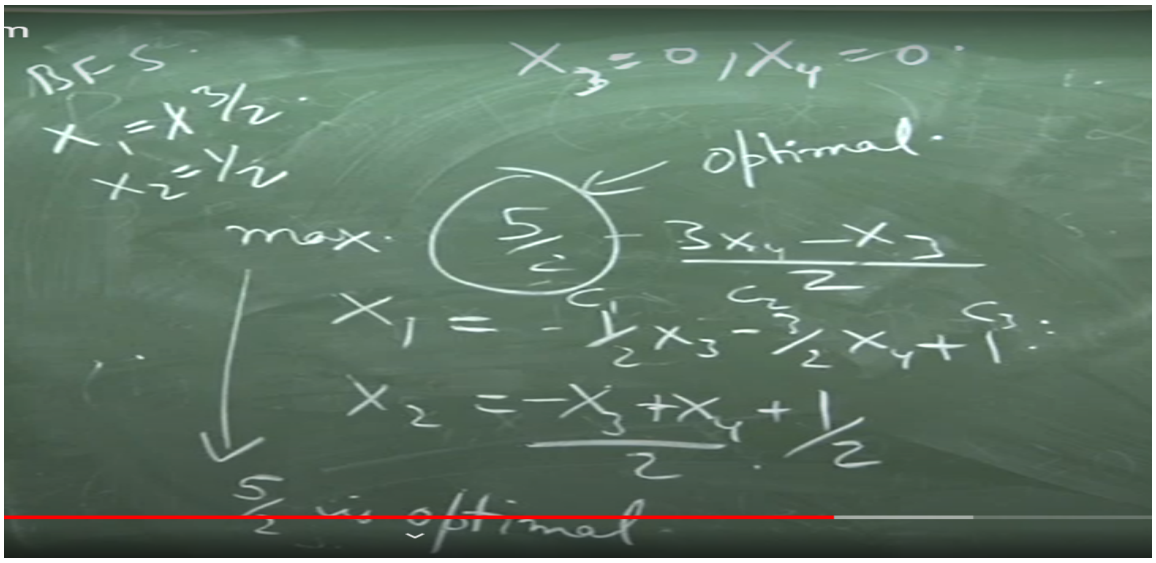
Now, I can put x 2 in my BFS how much is x 2 in my BFS. So, I have this as my BFS and I have this as my BFS. So, I have this as my BFS and I have this as my BFS and how much x 2 can I put in my BFS that will be determined by half good. So, now, my BFS would be yeah. So, x 2 is half x 1 is and x 3 is half.

So, now, I have x 3 equal to 0 x 4 equal to 0 sorry x 1 is 3 by 2 and then see because you guys are not doing this in exam I will give you a simplex algorithm to simulate that is the only way because I it is clear that you cannot do this simple equation substitution one will say one will say 3 by 2 only way to ensure you can do it is to give it as a question in

exam and then you will simulate for half an hour this entire thing ok. So, now, let me do this thing and then I tell me what will come here maximals first equation. So, this should be x 1 or x 2 what should come here x 1 x 2 or x 3 x 4 in what terms should I write x 1 x 2 good. So, x 1 equation you will tell me x 2 equation cover will tell me x 1 equation yes I told you tell me what will be the maximum what which one x 2 is minus 1 by 2 I have the correct thing written there. So, you cannot. x 4 minus x 3 plus 1 x 4 minus x 3 plus 1 x 2 minus x 3 plus 1 x 1 minus x 3 plus 1 x 4 plus 1 x 3 plus 1 x 4 plus 1 x 3 plus 1 x 4 plus 1 x 5 plus 1 x 6 plus 1 x 7 plus you can tell me x 1 and you can tell me the objective function 2 of you got same because you have to write it. So, here you just put x 2 right what do you want to write x 1 right. So, you just put. x 3 plus x 2 minus 3. You are not supposed to answer.

So, you have x 1 you have x 2 in terms of x 3 and x 4 just substitute it here right. Similarly, here in place of x 1 and x 2 you have x 3 and x 2 you will just substitute this equation to give me the objective function. So, what is the objective function and what is x 1? So, here do you have the objective function.
Yes. I have x 1 and x 2. So, I want objective function in terms of x 3 and x 2 that means, I have to replace x 2 this was the hardest part which you already did not hard part, but using this can substitute here and give me x 1 and x. 5 minus 3 x 2 minus x 3 by 2. Sorry 5 by 2. Minus 3 x 4 minus x 3. I do not agree this is correct I do not know about x 1. 2 minus 2 x 4 plus. Are you telling me x 1? 1 of max. You are changing your answer here. It is correct. It is correct.

It is correct. What about x 1? No you do not tell x 1. Minus half x 3 minus 2 x 2 x 4. And now what do you know for. That is not correct. Which one x 1? Minus x 3 minus x 2 minus x 3 minus x 2. So, the important lesson here is that for this solution this basic feasible solution the value is optimal. Optimal first, but what is the value? 5 by 2. 5 by 2 because x 1 is 0 and x 3 is 0.



So, your actable is written here. Now, question is that the best. Yes. And that is easy to see because x 4 and x 3 are 0 as soon as you increase it is going to go down. So, now, it is in the form where it is easy to see at 5 by 2 is right.

Let us look at what was the output right. Start with the BFS check if it is optimal if not find a variable to find a variable to leave get on the BFS right. Even though we did this simple example some of the steps now we know can be done. Can you tell me do we know how to start with the BFS? No. In this case you are actually very lucky that we had a BFS solution.

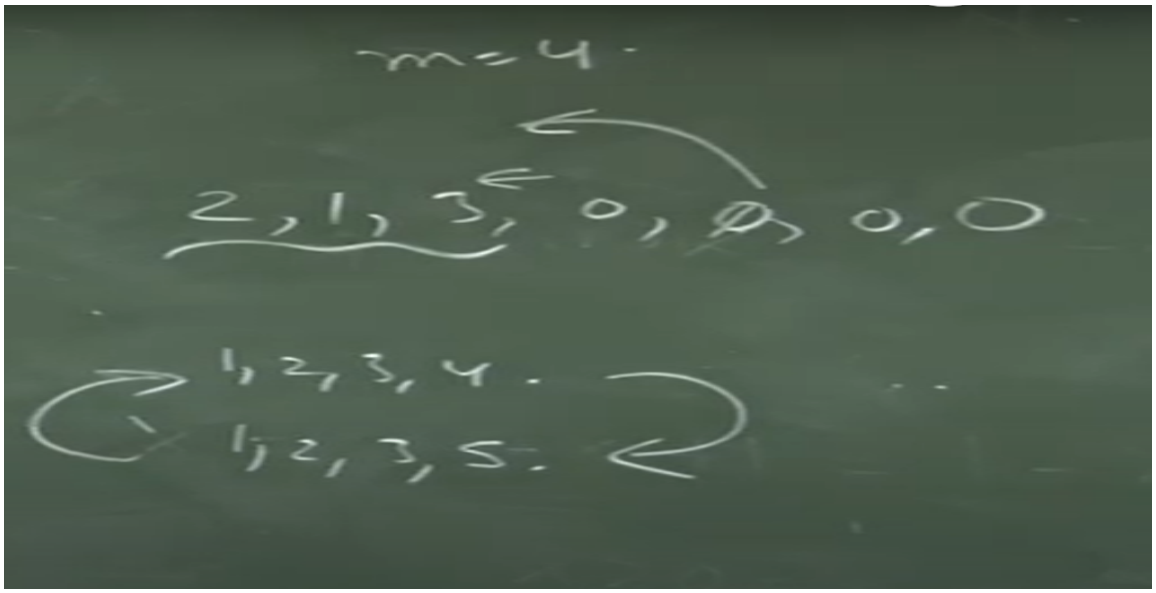
So, what is the solution? BFS is the optimal solution. So, what is the optimal solution? BFS is the optimal solution. X 3 and X 4 right, for example, if we had a greater than equal to thing then we would have a minus sign and then we would not right. So, we were lucky to start with the BFS that is not going to happen all.

So, they were equalities. In this case you always had slack variable, which you could put at BFS. So, this we still need to work, but once we do this we can always express our objective express our objective function in terms of non basic variables and then it is easy to check if it is optimal or not. So, this now we have to check whether it is optimal or not. If there is some coefficient in the objective function this positive then we are not at optimal otherwise we are again this one maximization you can do it for right. Now, find the variable to include pick any variable whose coefficient is positive fine.

Find a variable to leave this I kind of overlooked. So, what happened is now here x 2 goes inside right. Now, x 2 went inside and I could increase x 2 because this was 1 what that this was 0. This is actually the case when multiple BFS is give rise to the same vertex and since it is the same vertex the optimal value is the same at all these points. So,

there is a chance that you keep rotating in under those BFS you are at the same vertex, but your algorithm is rotating in those BFS.

So, this is the case when you have your size of s is actually less than n. So, there are your in your BFS the things which are 0. Let me write it what will happen in that case is. So, you have 2 1 3 and let us say m is equal to 4 correct. Now, these are linearly independent now it might happen that I can add this column also or this column also and still keep it as linearly independent.



Then that means, I have 2 BFS's 1 is 1 2 3 4 1 2 3 4 other BFS's 1 2 3 4 both the BFS's correspond to the same solution x. And then potentially it could happen in your algorithm you go from here to here and what is the guarantee that we do not get into this. This is a small trivial technically not small, but technically this has to be taken care of. So, we do not know we know how to remove, but not exactly. It might be that the coefficient of this is positive like here there was only 1 coefficient which was positive.

It can be there can be 10 coefficients which are positive or there is just 1 coefficient which is positive, but that positive takes you there right. So, that would happen if this constant comes out to be 0. But they finally, will give the same vertex right. Yes, but our algorithm will say that oh I have removed x 4 and put an x 1. So, we do not check the optimal variable to remove.

So yes, so this good. So, some idea what happens if there are multiple coefficients which are positive here. At this point I have said pick any agreed, but there is a way to say that oh I will if there are 5 coefficients which are positive I will take a particular one and in that case we will not get stuck in such and such and such. But the way I have described it is look at the objective function if the coefficient is positive that is the other one to get included right. So, my point is that there are 2 directions there are multiple

directions how do I know this also does not decrease the value this also does not decrease the value.

But it might become expensive slowly. The number of. Yes. So, there is a way to select. So, that you do not get into the same vertex that is the problem sorry I probably did not answer your question. So, if we check all the edges from this node and we go to the one which has the most increase which has the most optimal time.

What if you are stuck here this is the point where the increase happens 1, 2, 3 this is the cycle these are same BFS's, but the increase you have to go from here not from here. So, from at this point every direction you are not increasing you are staying the same. You just Google degeneracy a linear program they will give an example where this happens yes, but what I am saying is that now you are visualizing in terms of vertices do not visualize in terms of vertices in terms of BFS this is the structure you have these 4 BFS's they are connected to each other you can only move between them like at least from this vertex you can go to this BFS, but from this BFS there is a way to improve the solution. So, you have to be moving from here you cannot stay here, but from here in every direction you do not increase it is the same.

So, which direction to increase. So, there is a way in which you do not come back to this and that depends on time whether I will do it or not, but yeah. This is partially we have seen partially we have not seen and then this is also fine. So, next time I will formalize this in terms of the matrices and then we will see these two sorry this degeneracy and this term that will complete our simplex algorithm distribution. Once again more intuition now we have more content then still some points we can again see make it complete.