Linear Programming and its Applications to Computer Science
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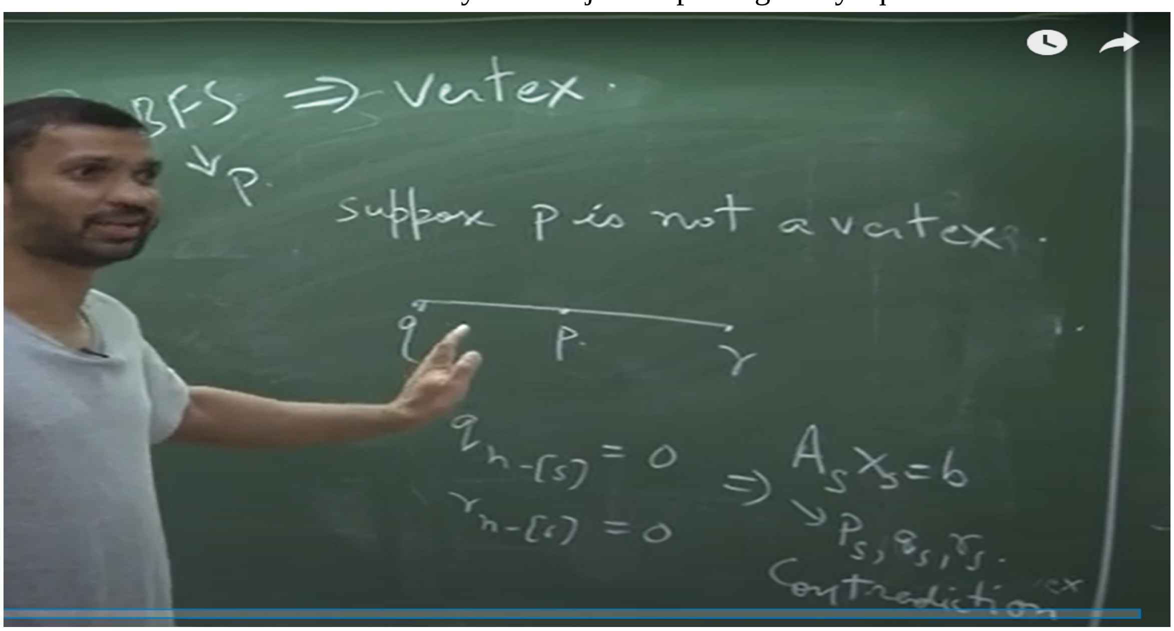
Lecture – 18
BFS and Vertices

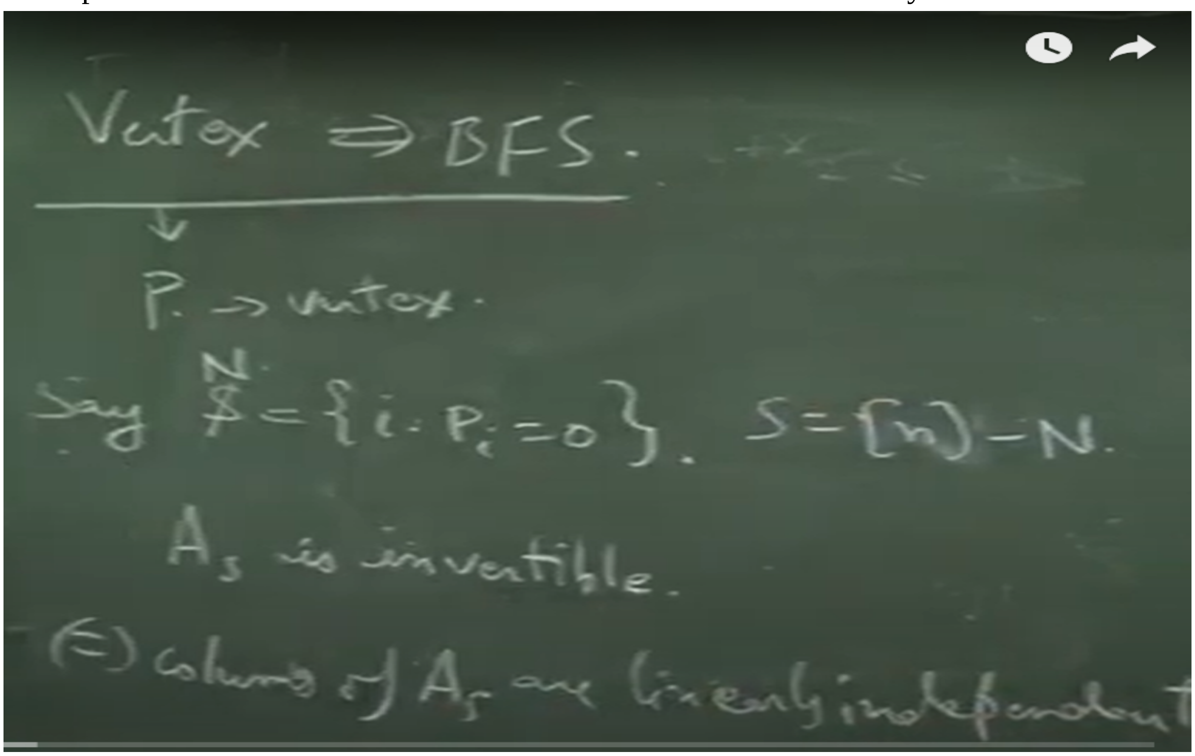
So, from this actually what we have shown is that a BFS is a vertex which I have already shown, but probably I did it in this term. So, let us do it once more ok. Why is BFS a vertex suppose it is not. That means, so let us say my BFS is sub point p by contradiction suppose p is not a vertex what does that mean? What does it mean? There is points q and r both of them are feasible such that p is a convex combination right. So, what do I know about q and r? Remember our intuition right if whatever coordinates of p are 0 all those coordinates in q and r should also be 0. Because I know that every entry of q is positive and when I take the convex combination it will remain positive questions right. So, that means, this implies now that As Xs is equal to B has 3 solutions Ps Qs Rs this is what we did here exactly I am just repeating my questions about this.



This was the easy direction. So, if you are not happy with this then it is time to take a break not from you or me speaking, but I mean you would not move forward, but this remember we from the 2 dimensions we got this idea that oh you know look at the coordinates which are 0s. The remaining ones if they create an invertible thing we are happy with that idea we define the basic feasible solution and we are hoping that it characterizes vertex. And from this idea actually it was already clear that a B F S is a vertex and I have formulated on the proof here right.

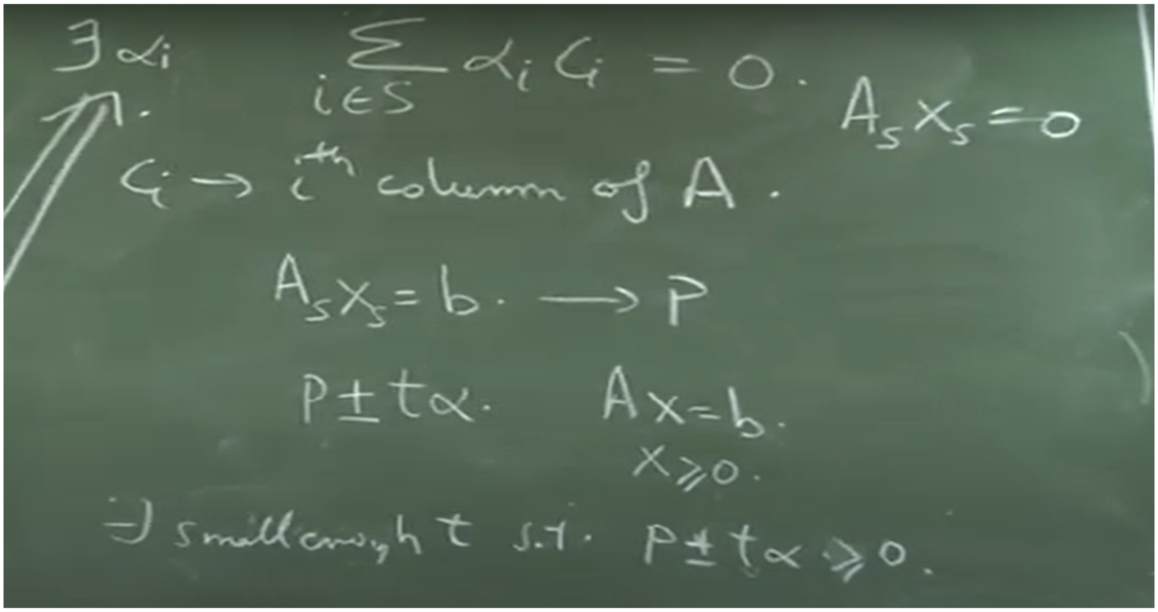
But the other important thing where we should keep our fingers crossed is if every vertex is a B F S because then we would be happy then we have a mathematical characterization of vertex. And now you see this description is way nicer than saying oh there are 2 points there are no 2 points whose convex combination is this point. This is much more concrete description of a vertex as compared to our previous general description and we could get this because our feasible region was of this special form right. This is not general true for any vertex of any kind of a shape or something right. This we are able to do because we are talking about A X equal to B X to the power of C good.

What we want to prove now is vertex is a B F S and why do not we give Dev the chance to prove this? It is you let him sleep. So, it is ok. So, what we want to prove is that a vertex is also a B F S we have proved that B F S is a vertex, but how do we prove that a vertex is a B F S? This is not clear from what we did right ok good, but what we know is ok. So, let us look at point P let us say it is a vertex we know that P is a vertex and we want to look at columns which correspond to entities of P which are non zero. Remember the remember the old ideas here what we looked at was all the columns which corresponded to the non zero entities of P and here they were inverted.



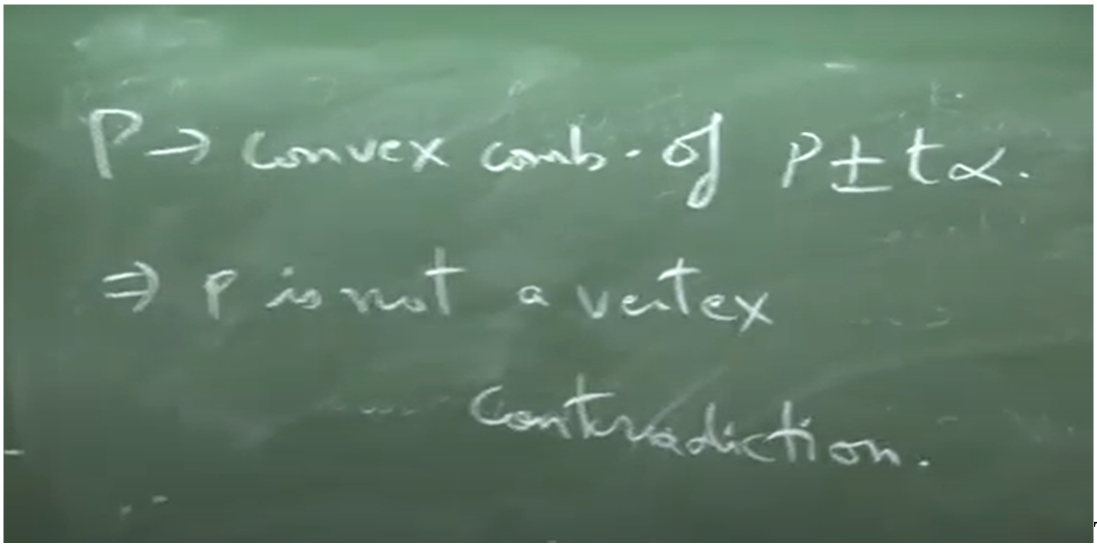
So, say S is the set of I such that P I equal to 0 or let us keep our notation like this define S. So, to prove that P is a B F S what do we need to show? You have the definition of basic feasible solution B F S written there tell me what do I need to show? So, that I need to show that P is a B F S very nice what I need to show is that on other words right. Notice that if there are very few columns in A S which are linearly independent they are not m I can just join arbitrary columns of columns and make them m right. So, that I get the complete. So, the only description only difference between this or actually between this equivalence is that it might happen that A S is not full.

It is not the size of S is less than I know size of S cannot be more than because of the rhyme constraint right. No A S in A can have rank more than m because the rank of A is m itself right, but this equivalence like this is almost equivalence because it might happen that A S has less number of columns. So, it is not like a m cross m matrix, but that is then I can pick up arbitrary columns of A which are remaining and make it a complete rank the corresponding entities of that will be 0 it does not matter. Here I am not saying that the x i for i element of S has to be non 0 is it. The only constraint here is that the non basic variables have to be 0 basic variables could be 0 might not be non 0 that is that will be the unique solution which comes out from this it might have some more 0.



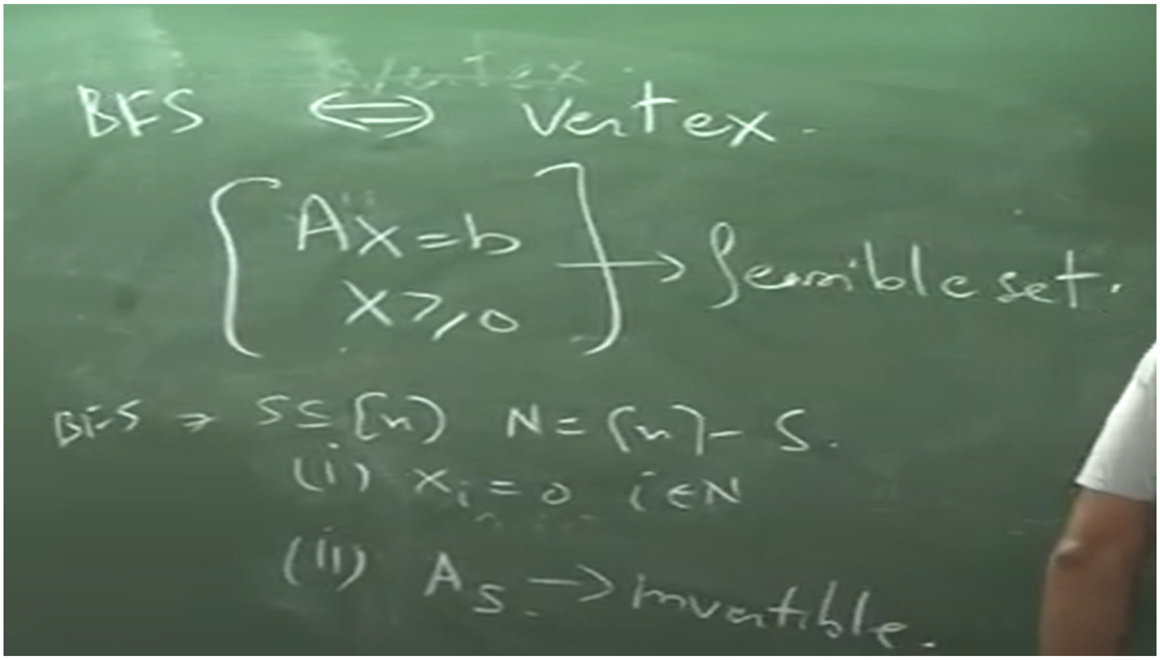
So, this again will correspond to that case it might turned out that I only need m minus 1 columns those are linearly dependent they give me a solution the rest the next column it does not matter because that will anyway be multiplied by 0 that is my B is a combination of m minus 1 columns that can happen completely fine. So, this is what we want to show if I find out the columns of A S are linearly independent I can convert it into a BFS if size of S is equal to m great if it is not just add arbitrary columns and this you can convince yourself ok this is a small trivial technicality, but if you are stuck with it you can ask me after the class this is sounds good this is what we want to check now. Probably would not have cared if I had not pointed out this technicality this equality, pointed out but it is not a big deal. So, this is this is what. So, then assume there are they are linearly dependent there exist alpha i's such that i element of, this is what it means C i is the i th column of should I denote by a i is that help or that is right.

So, this how is that useful can you find more solutions of can you find more solutions remember what were the solutions of A X equal to B you take a solution and add A X equal to 0 solutions what is this this is the solution of no you want to prove this is to prove.



That means I want to prove that these columns are linearly independent this is my task now assume for contradiction they are linearly dependent and I will come up with a contradiction. Assume that these columns because if they are linearly independent I am good I have the BFS if they are linearly dependent I want to show there is a contradiction. So, if they are linearly independent, but linearly dependent by definition there has to exist alpha such that their linear combination is 0 my claim is this will show that your point is not a vertex .can you give me more solutions of. So, we already have one solution for this right which is p agreed. this is what is means p is a vertex s was the set where it was non 0.

So, As P s was B right because P N was 0 again I will write this P N is 0 this means I already have a solution of this now what does alpha allow me to do add not just alpha right this is a solution of A X P, but how is it helpful we wanted feasible points in this what is the guarantee that this is these are positive. Exactly all coordinates and p are positive now we are going to use that since all coordinates are positive there exist small n of t such that p plus minus t alpha is and why is that is. So, what we were able to prove is a very nice characterization of what a vertex is and this nice characterization was possible because this was the feasible solution.



So, if my feasible set is of this kind I can define my BFS what were the two conditions X i should be 0 for non basic variables and if I look at and if I look at the columns corresponding to the basic variables.