Linear Programming and its Applications to Computer Science

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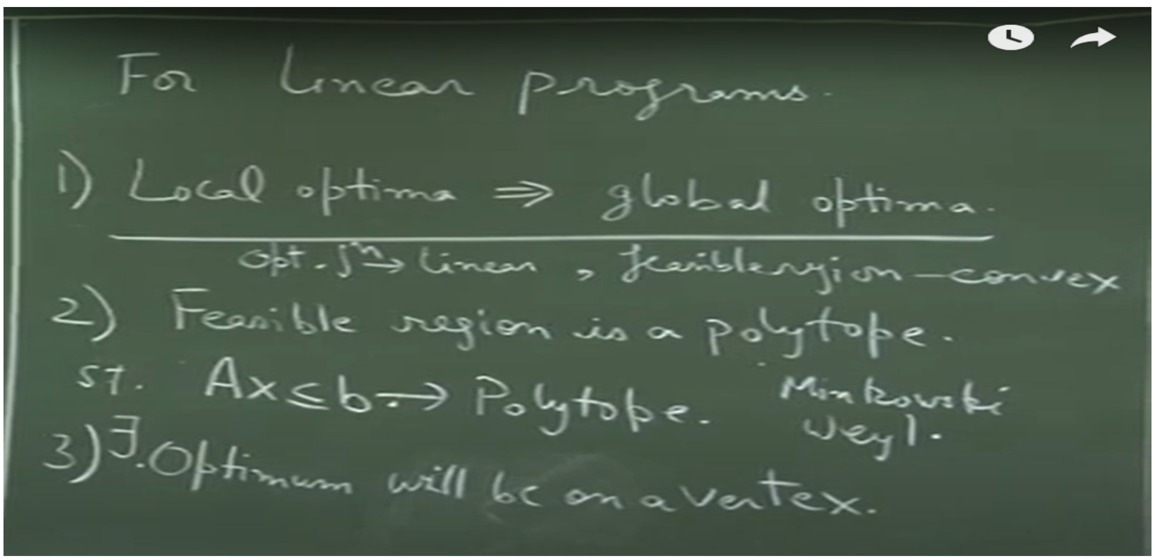
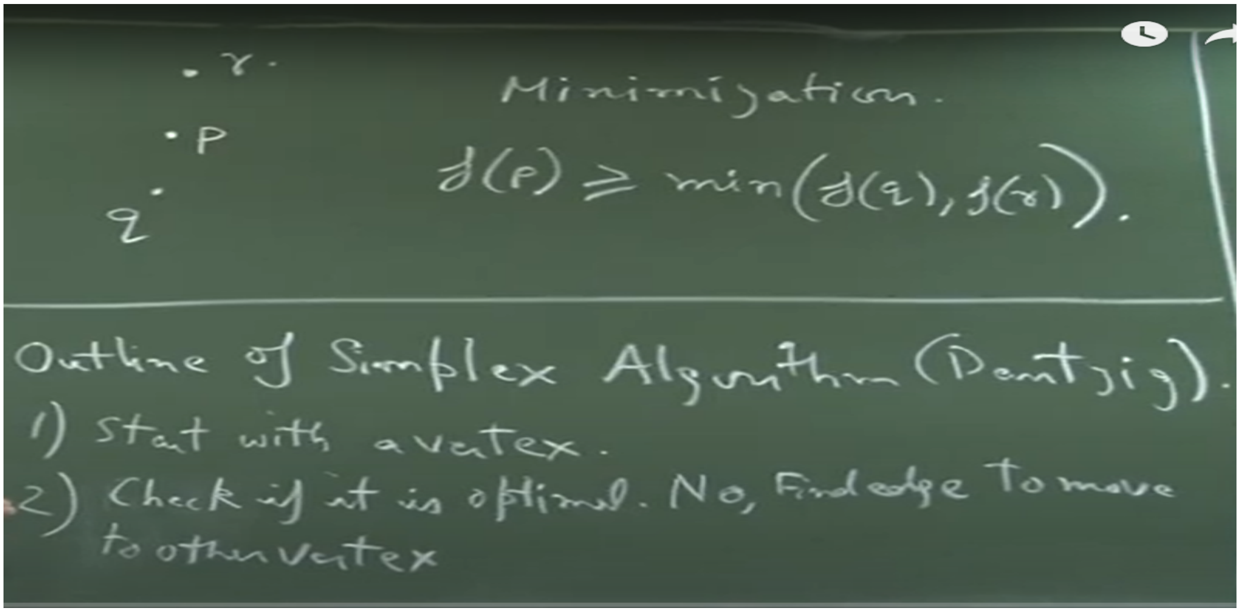
Lecture – 17
Basic Feasible Solution

Welcome to the lecture on linear programming. We saw in the previous classes that are feasible region is convex, our optimization function is linear. This allows us to have very nice properties for the optimization. In particular for linear program, we can write three important things which will allow us to given right. What are these three properties? Local optima implies global optima and this happens because of two things. One is our optimization function is linear convex as well as concave and our feasible region is convex.



That means, if I take two points the line joining them stays inside the feasible region. Because of this property, we can show that a local optima remains a global optima. If I find a point in my feasible region, where in every direction my value increases, I found the minimum point over the entire global domain make sense. Secondly, we realized feasible region is a polytope and this happened because one of the standard form the constraints look like this and we know that this is a polygon or a bounded or an unbounded polytope.

This is because of Minkowski Weyl. Thank you to both of these guys. We are very happy that they proved for us and then the third property is optimum will be on the boundary. Why is this happening? Again convexity right and again we will use both these things because if I take a point P and let us say it is a combination it is not on boundary. That means, it can be written as a convex combination of two points Q and R.



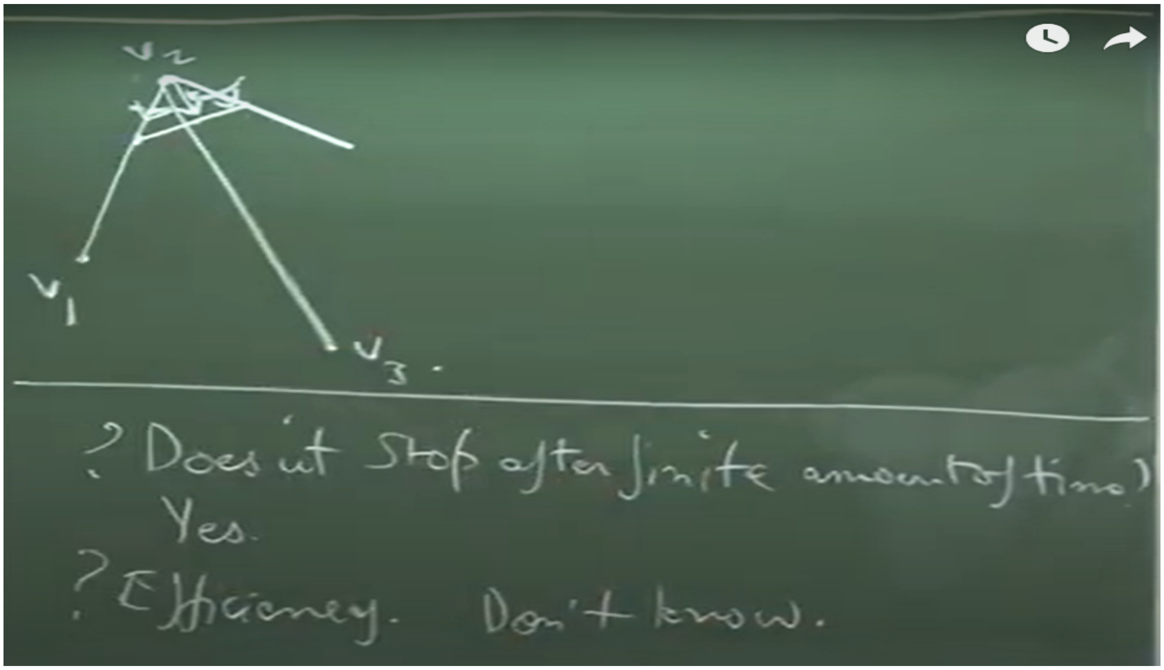
Then what I can show if I am interested let us say in a minimization problem F of P would be greater than minimum of F Q and F R. It is easy to prove how will you prove this? Take the contradiction suppose F Q and F R are bigger than this. Then you take the convex combination of that F P has to learn about this and this one. So, you can check the good thing is R objective function is linear it is convex as well as concave. For one minimization one thing will work for maximization other thing will work.

So, it will turn out that in all the cases if P is the optimal then we know one of the Q and R should also definitely be optimal right. And I can this way I can keep going outside right and finally, reach to reach a matrix. That means, there will be there exist an optimum at a vertex. I am not saying all the optimal values are at vertex important there at least at least one optimum which will be at a vertex. And this gives us the outline of simplex algorithm.

I will start with the vertex check if it is an optimal vertex question is can I check it we will see, but again this is just an idea. And this algorithm actually will see in multiple stages give an idea formulize a part again give more idea then formulize and finally, we will have an algorithm. So, intuition is start with the vertex how we will check if it is optimal we will see, but if we can check if it is not then there has to be an edge. Notice I am not saying direction now I am saying an edge why can I say edge, but again this might be, but why should this be an increasing direction.

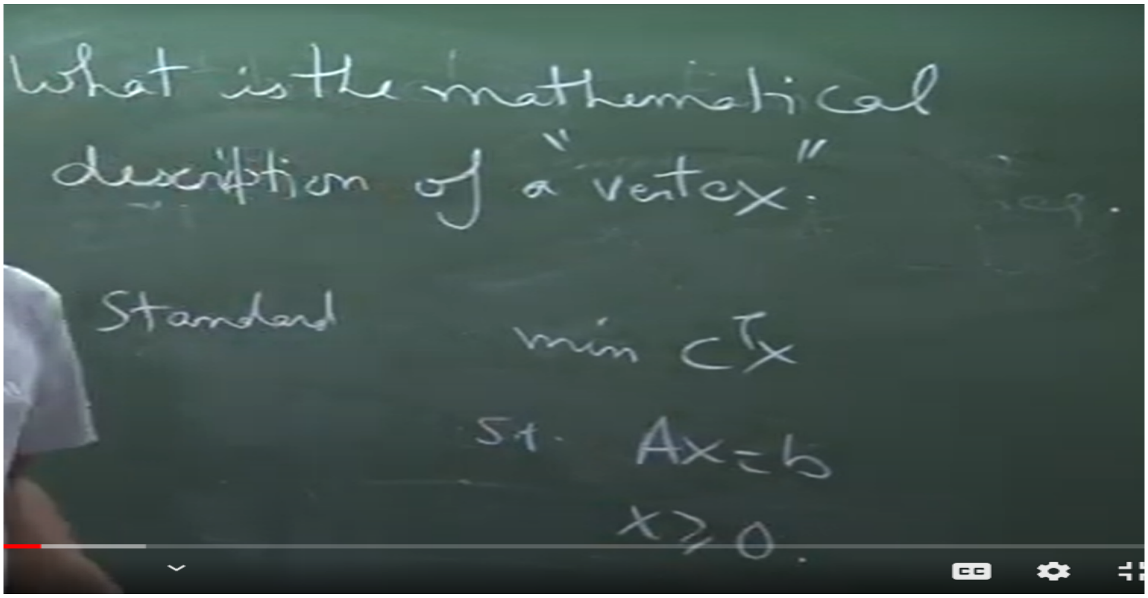
If both are increasing then it will be an optimal I am here this is not this is not a local optimum. That means, it says that there is some direction here where the value increases

I am saying there is one of these two are there where the value increases because this might



be optimal right. Exactly what you want to show is that suppose if I am increasing in this direction then because of convexity either in this direction I am increasing or in this direction I am increasing.

If direction is confusing you take a step of some t draw a line and then you can see that at least one of those will give me the increasing direction assume we are interested in minimization problem right. So, it is sufficient to look at edges again because of convexity and then if to other vertex and check if it is of or I should say repeat. I have not explained many things, but at least the idea is clear right. How do I start with the vertex? What do I mean by a vertex? That means, the first question we will answer, but if we know that how what does it mean by moving to other vertex? Even if you find a direction how far do we have to go right? Does it stop after finite amount of time? Does it stop after finite amount of time? Number of? Can there be a case when number of vertices are infinite? You have a polytope. If there is a cone and there is an increasing direction then the value is involved.



So, here if we find a direction to move and we are not reaching any vertex then the value is involved that is also fine. Circle. Exactly. So, then you should remember we cannot have a circle. We know that at feasible region is a polytope.

That means, it is a convex hull of finite number of points right. So, it cannot be a circle. So, the number of vertices are actually finite number of directions are finite and this since my value keeps decreasing it is a minimization problem my optimal is better and better it has to stop. So, this is fine. What about efficiency? Is it efficient? .

Exactly. So, do not know and this has been open at least for 70 years and this question depends on how you are doing these things. As I said yesterday there are multiple implementations for most or I think probably for all those standard implementations there are counter examples which cover exponentially many vertices. But still there is a possibility that there is some implementation which works in polynomial time. This is an exponential time algorithm and from whatever you have been taught in computer science you would say oh why care about an exponential time algorithm. But if you look at any algorithms literature and you know the marbles or gems of 1900s this will be one of them.

The reason it is simple exactly many industries still use it. So, in practice this is a great and this is very simple idea easy to implement easy to understand that is why this is very popular and mathematical. So, the first important question when you look at this kind of outline is how can I describe a vertex? Yes great it looks you know in a in a picture it is at the corner or something it is not the convex combination, but for my feasible region how do I interpret it in terms of my inequalities. That is the first important thing I want to figure out ok that that will be my handle to start this algorithm. Because if I look at it the I think multiple times I repeat the word vertex appears most number of times except probably a and with or something.

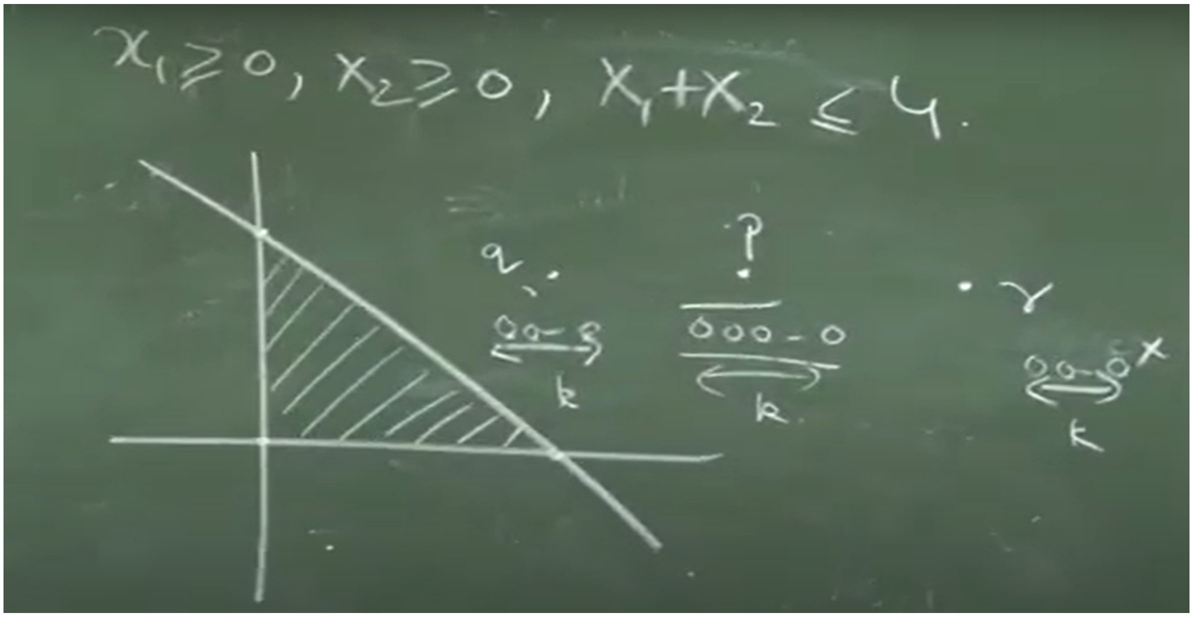
So, we really to figure out what vertex means for us mathematically right that will be our first goal. And again you will see this interplay where now suddenly my standard linear program will be this. When you want to understand the feasible region we said Ax less than equal to b fine, but that can also be written like this. And now we will play about look at vertices in this. But before I give you the mathematical description there is a mathematical description let us try to see if we can get some intuition of what a vertex should look like.

And once again the idea would be to look at two dimensions right our favorite dimension. So, then let us say I am interested in this again optimization objective function none of my concern I am looking at the feasible region now that is what I am focusing on. By the way before I start does anyone have a nice description of a vertex then we do not need to seems like a complicated enough problem right mathematical. The only linear combination creating it is the vertex itself. Convex combination, but that is not a that is not a handle right that is not a good representation like Ax less than equal to b you see.

So, you will see what I mean is you are getting closer, but not really yet we are not there yet. So, I yes, but it is kind of a unique solution, but why n equations and n variables. If there are n minus 1 equation then we go like the solution would not be a. So, you want to have n equations from here. So, I can assure you if for equal to n or greater than equal to n equations this is a useless program.

Because this is equality I only need to put linearly independent number many of equations. If an equation can be written in terms of other equations if the constant follows the same behavior then that equation is useless. If the constant does not follow the same behavior infeasible is this simple point clear that I only need to worry about linearly independent equations right, right and if the number of independent equations are n and this has a unique solution this is nothing. And I just have one x, I can figure that out using or something put it and gets the answer.

So, the interesting case is. So, I would assume that if there are n equations m is less than n. Now, what do you think is a vertex? You are still though you are close. So, now, you are guessing that is ok. So, let us see let us try to guess, but in a more you know informative. So, let us try what happens right.

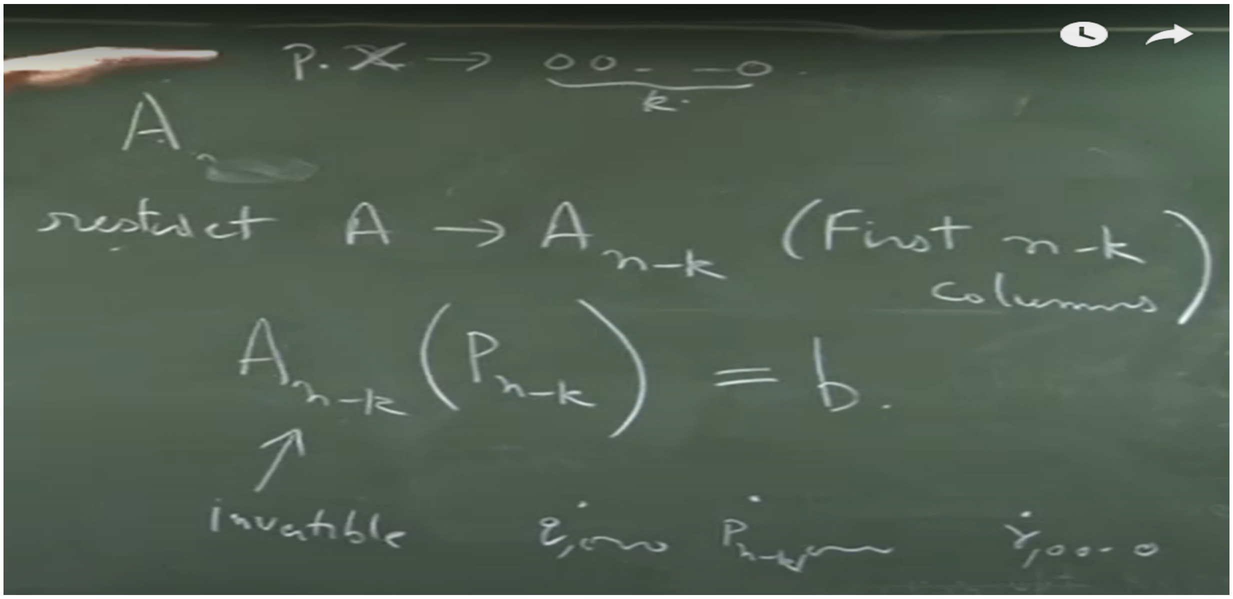


This is the graph what is my feasible region correct. Now, these are the vertices right and then what is happening at the vertex many of the coordinates are becoming 0. This is because we have changed this to equality and this is greater than equal to 0. So, basically this is a linear subspace which going to cut into orthons. So, then your vertices will have you know a certain number of 0s that is and you see it here, but how many 0s what are they that we have to still figure out right.

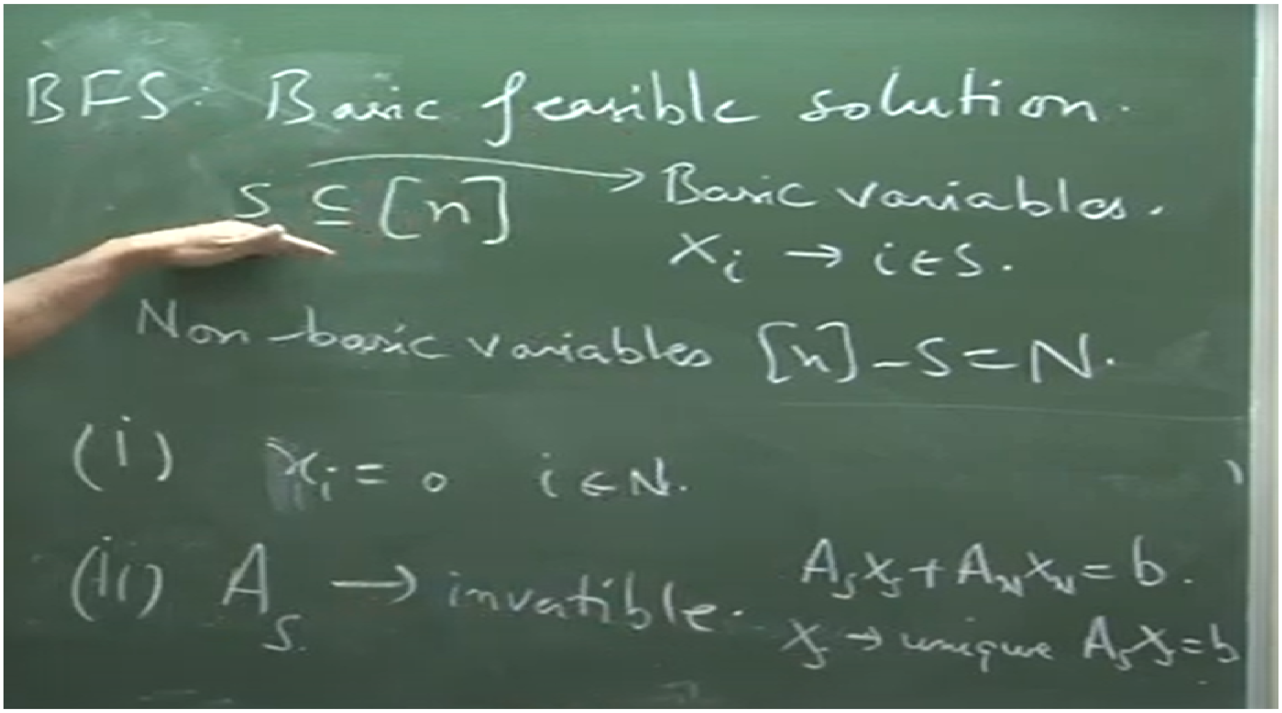
So, and the other point is if I have lot of 0s that is a nice way to get a vertex. Because if this is a linear combination of 2 points Q and R then what do I know? That these coordinates are 0 in both Q as well as once more. If P is a linear combination of Q and R and let us say P is 0 in last k coordinates. My claim is N Q and N R last k coordinates have to be 0. Because P is a convex combination of Q and R all these have to be positive I have this constraint.

If I take convex combination of this coordinate and this coordinate it is still going to remain positive. So, this has to be 0 this has to be 0 these have to be 0 agreed right. So, to get a convex combination of P get P as a convex combination and there are lot of 0s this is pretty hard right. So, you can see it in multiple ways linear subspace cutting these axis in this way you should have your vertices should have lot of 0s that is the idea right. So, let us look at a and let us say x is 0 at some coordinates right. Ok.

So, let us say last k coordinates some point let us call it x is a variable let us take a point P which is 0 at last k coordinates. Now, I look at the first n minus k coordinates and I can restrict a to first n minus k coordinates. What is a again? a is your constraint matrix this is the linear program this is the feasible region we want to target we want to understand the vertex of this point right is this clear? Now, you know that a times P equal to b, but last k coordinates are 0. That means, a n minus k times P let us say restricted to first N minus k coordinates right. And suppose this is invertible your idea and probably you might have had this intuition at some point, but now if this a n minus k was invertible that means, this is unique and that means, this is a vertex think about it you have P n minus k you have 0s you have q and r what did we realize this has to be 0 and this has to be 0, but then q and r are also feasible solutions correct.



That means, A n minus k q should be equal to b, A n minus k r should be equal to, but that is a problem I cannot have three different solutions for this invertible a equation it has a unique solution right. Again this is just and I am going to give you the mathematical description, but the idea which is going on behind this is oh there has to be lot of 0s and once I look at 0s I look at the remaining columns of a if they are invertible job done. With this intuition. Now, let us define a vertex actually I will not define a vertex I will define what is called a BFS and then we will show that a basic feasible solution is a vertex a vertex is a basic feasible solution. So, what is a basic feasible solution it is described by a subset remember I had N variables and this subset is going to be n minus k . It I had written it as k k coordinates, but these 0s could lie anywhere. So, this set will be called the set of basic variables x i's such that I element of S and I will have non basic variables which is N minus S let us note them by N.



Then what do I need 1. So, x i equal to 0 if i is right I this is how I started with this is the set of 0s the non basic are the set of 0s remaining part for the set of basic variables. And part 2 if I restrict a to S this is N part 2. So, what is happening here As Xs plus An Xn is equal to b right this is just Ax equal to b this is matrix multiplication no this is this is a column right. So, this is a linear combination of these columns.

So, this is S this is N right this. So, this dimension matches this, but the length right. So, think of it as this is a linear combination of columns of a from basic set same dimension right. So, this we are still working in the column space of it this is linear combination of non basic columns basic columns non basic columns will you be happy if I change S to b or keep it as S right. And now what do we know x N is 0 right that means, x S is the unique solution for As Xs right. What it means is that if I fix my S correct then I get Xs So, my basic feasible solution is described by S in tell me what is the size of S and we have assumed that every row in a is linearly independent right. Exactly because row rank equal to exactly yes. So, cardinality of S is number of rows in a or m yes you are right what is the problem with that you seem confused this is what you want to show yes. So, you see right all my rows are linearly independent that means, the rank of the matrix is m.

So, if I want to make it invertible the size of S should be equal to m. Other words a basic feasible solution is described by a subset of N of size N you fix a subset create everything else outside 0 then remaining things they should be invertible if they are invertible then only it will be a basic feasible solution that as you found it. If it is invertible you solve this equation this gives you the value of the remaining variables you have everything. So, you see a good reason to define this kind of a nice solution from here right.