Linear Programming and its Applications to Computer Science
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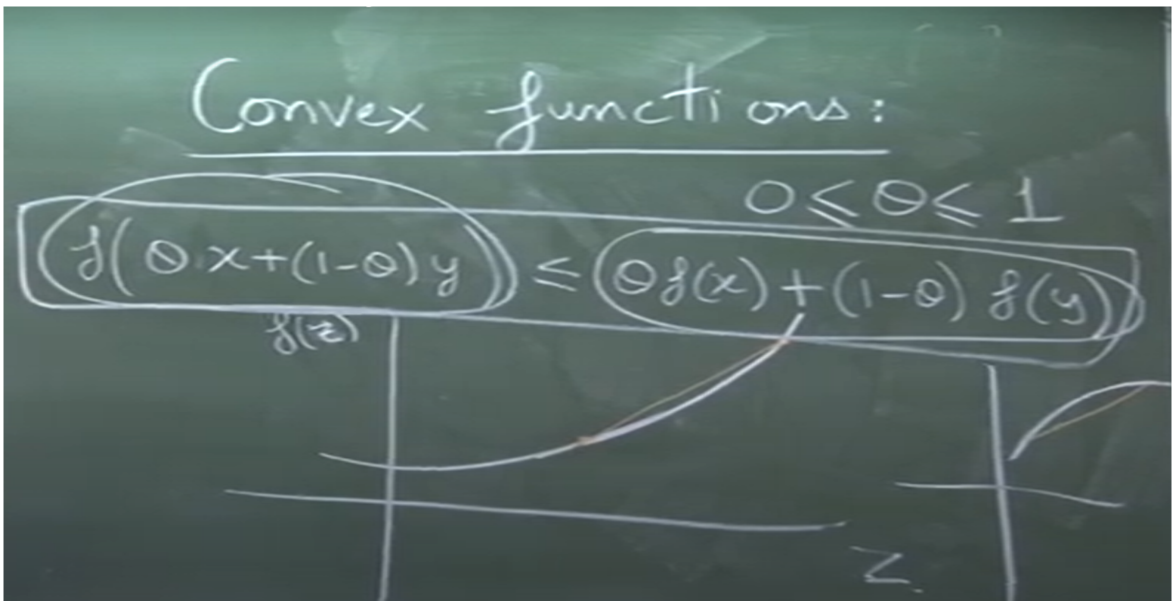
Lecture – 16
Properties of Convex Functions and Examples

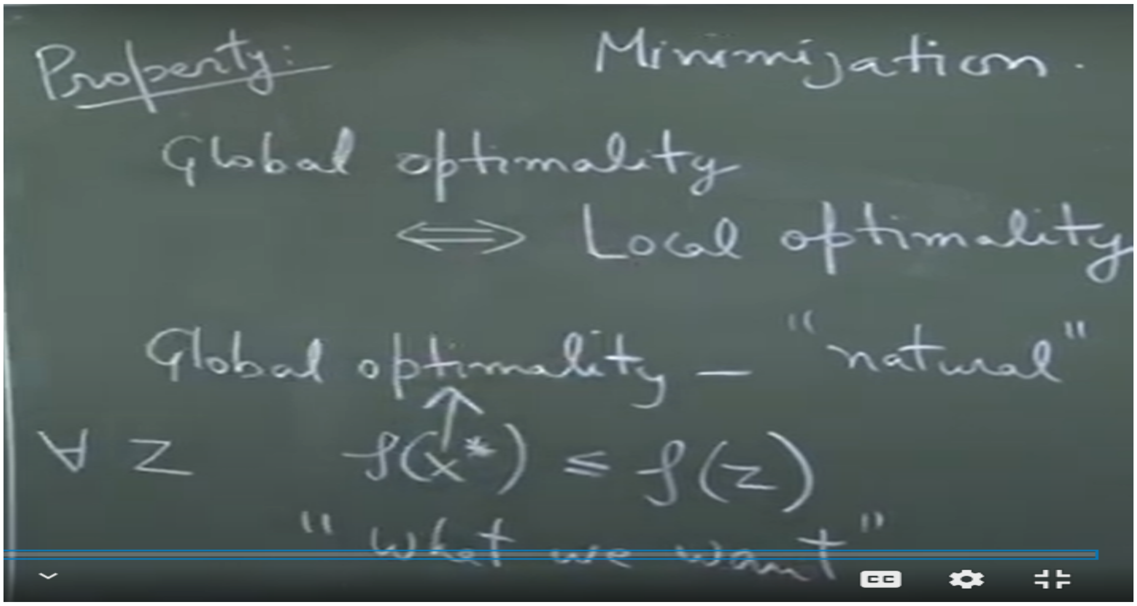
Hello, remember we were talking about convex functions. Convex functions are functions defined on real numbers or real number square or a convex region.



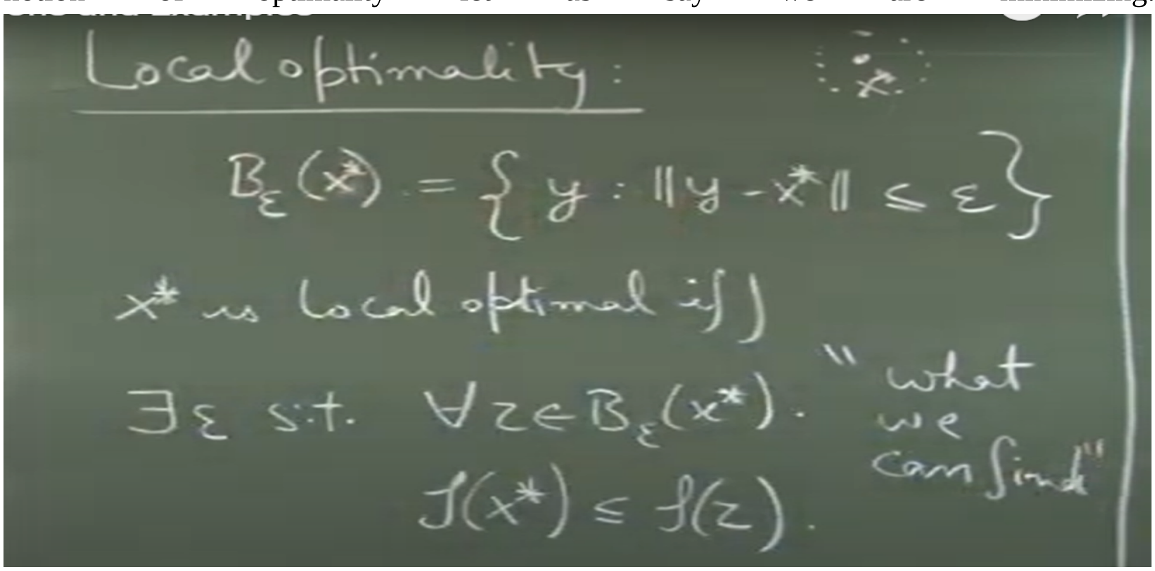
So, we will always assume that these are the functions defined on a convex region and the important property that satisfied was this particular property. It is said that if you take two points x and y and take a convex combination of them, then the function value is lower than the value you would ah you would anticipate if the function was linear. So, let me just say it in terms of the graph it makes much more sense. Let us say my variables are z my function is f of z.

A function which is depicted by this white line is convex because if I take any two points. So, I have taken these two points here these two orange points and if I join the line it is as though if my function was linear my actual function will lie below the orange line. So, the white line is lying below the orange line. This is the value if the function was linear this is the actual function value.

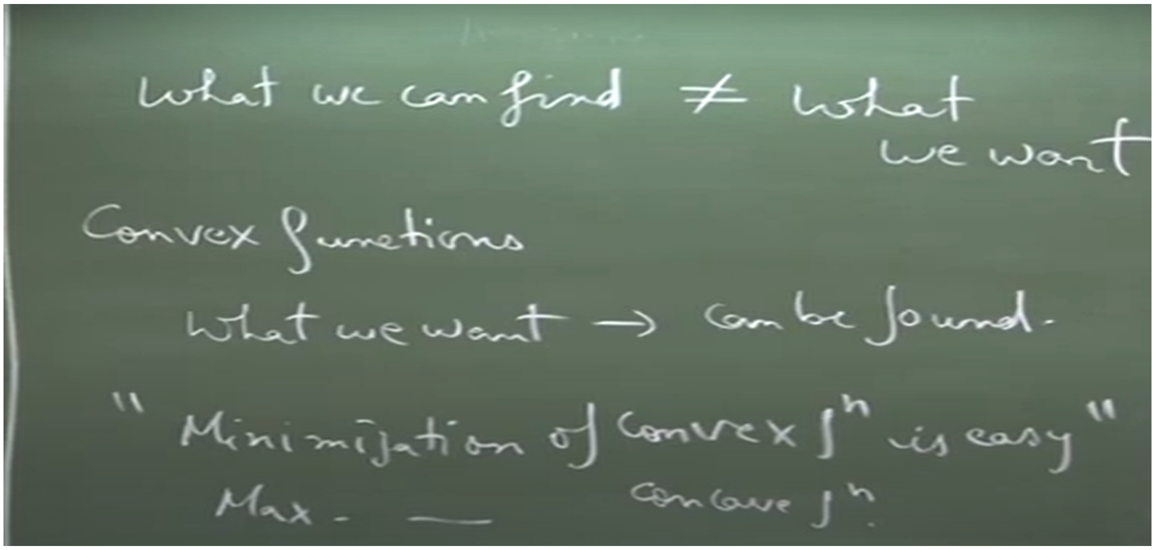
So, actual function value is is below the value anticipated if the function was linear. This is called a convex function the opposite would be a concave function where you know the anticipated value is below the function value, but we will be interested in convex functions



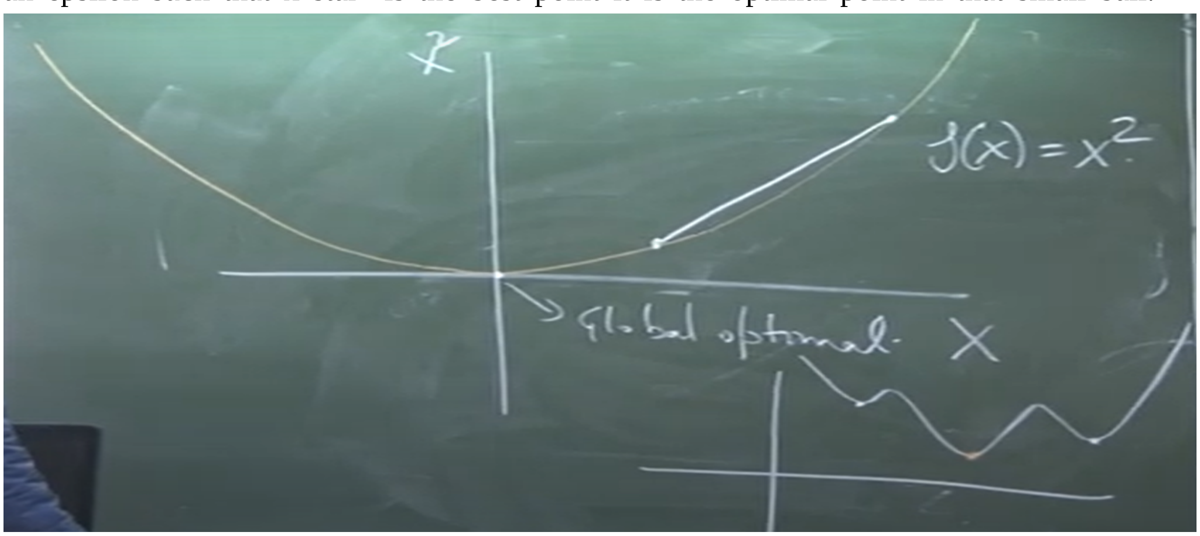
Ah. And ah we talked about the fact that there are very nice properties of convex functions more importantly the central property we want is that if you talk about global if you talk about convex functions you optimize it over a convex region then global optimality implies local optimality. I want to give you an idea of the proof here, but before I do that let us understand these terms ok. First global optimality this is the natural notion of optimality let us say we are minimizing.

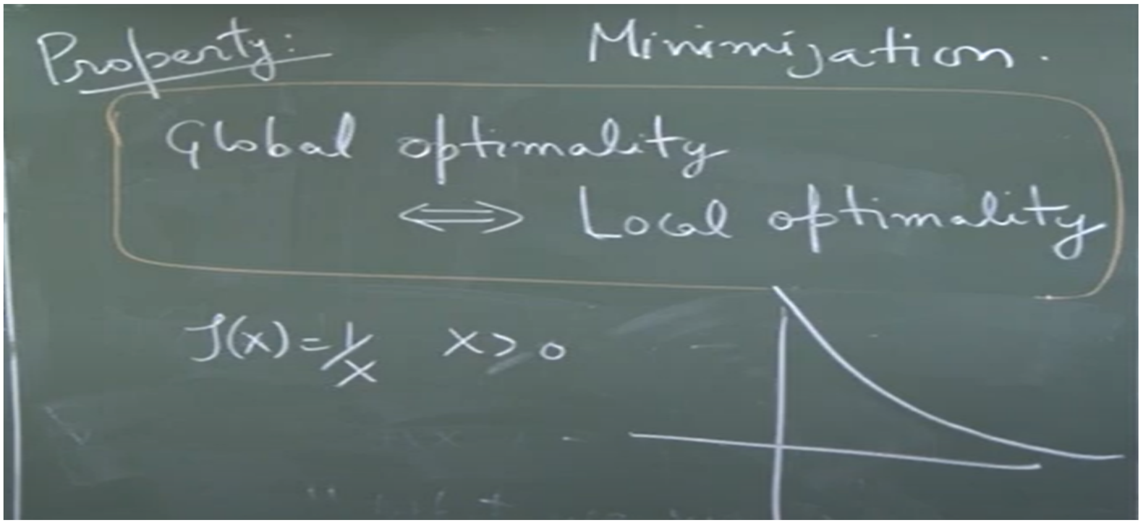


So, we are interested in minimization let us keep that as constant. A global optimal is the minimum point over the entire region ok. What it means is given for all z a point x star is optimal if f of x star is less than equal to f of z. That means, you take any point in the feasible region the function value is higher than the value at the global optimal. This is what we are searching for when we minimize a function over some region we are looking for this point.

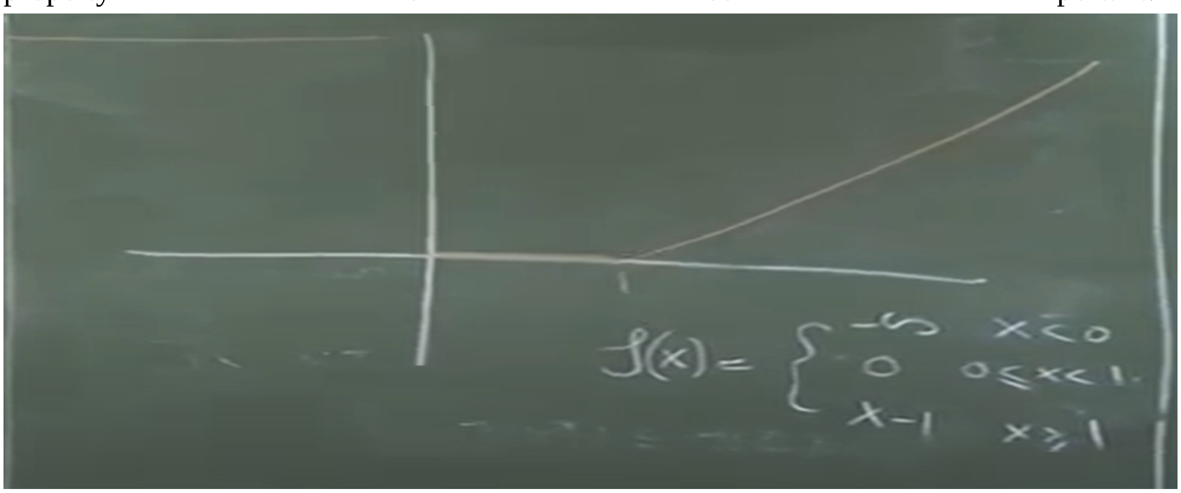


On the other hand local optimality you might guess is the condition which says that if you look at the point it is the minimum around it in the neighborhood. The way we would like to say it is let us look at a ball a very very small ball around I should say x star ok. This means these are all points y such that y minus x star the distance between y minus x star is less than equal to epsilon. Geometrically if this is my x star I am looking at a very very small neighborhood around x star. And then a point is locally optimal if there exist an epsilon such that x star is the best point it is the optimal point in that small ball.



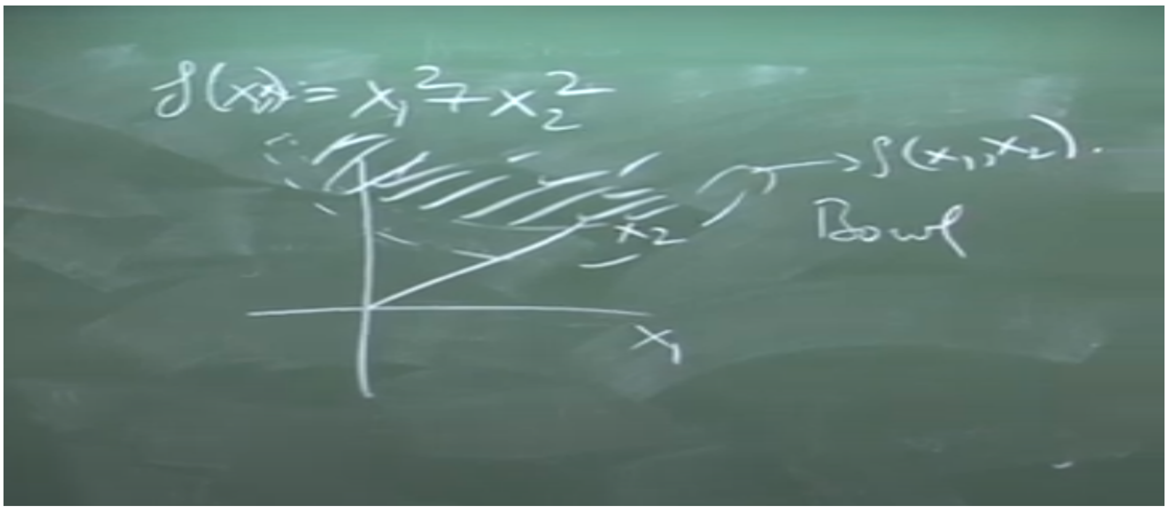


So, I would like to say a point x star is local optimal if there is a small ball of radius epsilon such that for all elements of this small ball not the entire region, but every point in this ball x star is the best possible, k. So, global optimal is when f x star is less than f z for all z this is what we want and if you think about it this is what is easy to find in the sense that we only have to check a local neighborhood to find out if the point is optimal. This means this is easy to find. And like in life it is what we can find is necessarily not equal to what we want right. This is the story of our life and in general if you want to optimize local optimal is what we can find, but we need global optimal that is why this property is so important.



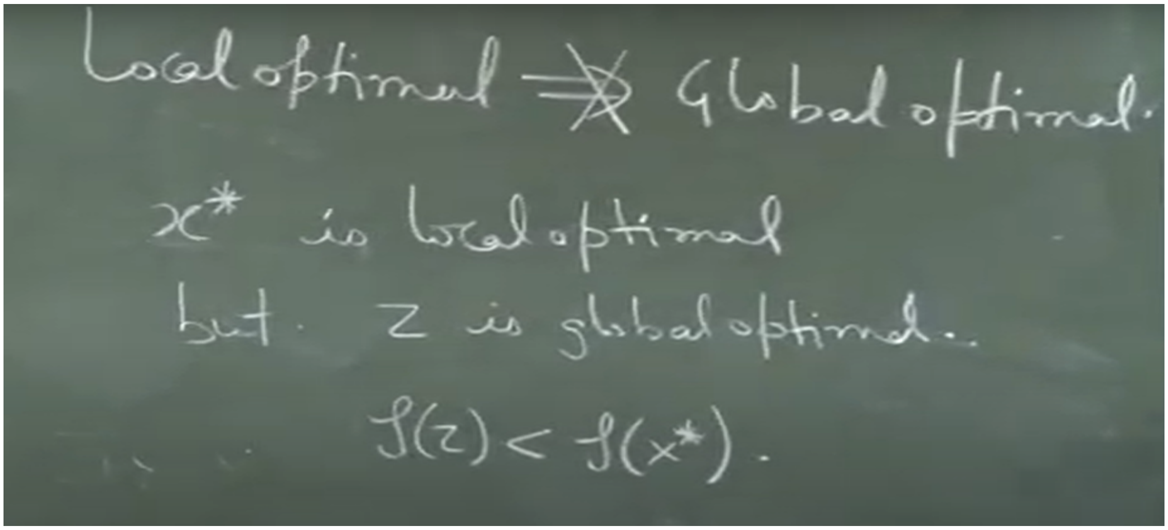
It is actually saying for convex functions these 2 are the same in the sense that what we want can be found also. Other way to say it is intuitively minimization of convex functions is easy and you can guess that if I write maximization here this will become concave function, but I think it is enough to look at one side of the story the other side is completely symmetric. So, this is the intuition and this is why we like this so much. In general again I want to emphasize in general global there can be many local optimals, but only one global optimal. Fortunately when it comes to convex functions any point which is local optimal is also global optimal.

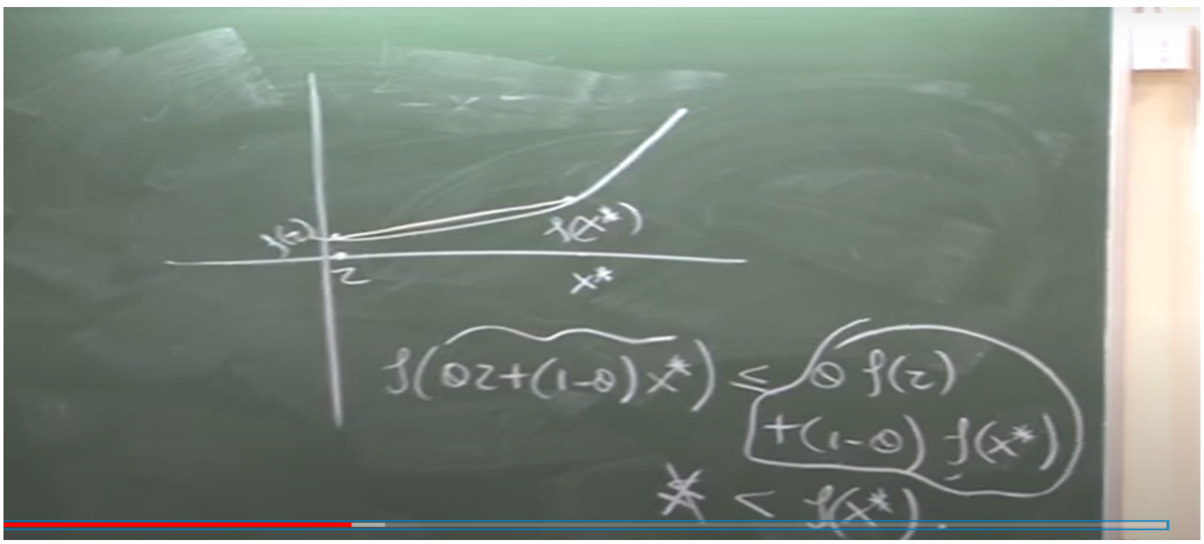
Again I am not saying that there is a single global optimal. What I am saying is if there is a point which is local optimal it is also global optimal. There can be infinite local optimal as well as global optimal points, but if you are searching for optimal if you find a local optimal point you can stop there you do not have to move ahead because you are because the function is convex you know it is global optimal also. This is why we really like optimizing over convex functions and if you think about this property this is not very difficult. Let me just show you this property you can visualize it by taking some examples.



The example of the easiest example of convex function is X square. I can plot X square with respect to X and the and it looks like a 1 dimensional part. Again you can verify that this is a convex function, but also it is easy to see from this graph. If I take any 2 points the line correcting those 2 points lies above the curve of the function. We know that this function is convex and in this if you look at this bowl shape there is only 1 global optimal or local optimal.

Notice that there can be functions where there are local optimals, but not global optimals. For example, look at this kind of a function. These 2 points are local optimal, but not global optimal. This point is global optimal. You can easily verify that global optimal is local optimal for f X is equal to X square.



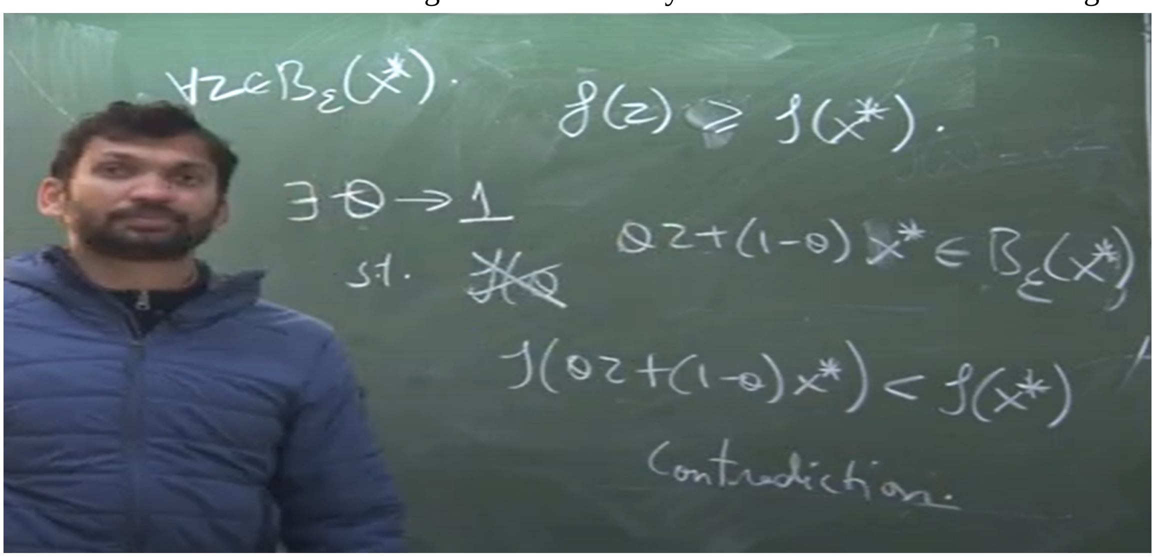


We can take other functions f X is equal to 1 by X for X greater than equal to 0 and I am sorry yes the thank you to fan for pointing it out f X is equal to 1 by X when X is strictly within 0 because it is not defined at X equal to 0. And then the function looks like this and again just by visual inspection you can see that the function is convex and also it is easy to prove that it is convex. And then you can see that there is no global optimal or local optimal. So, we are good. Convex functions can be of much different kind.

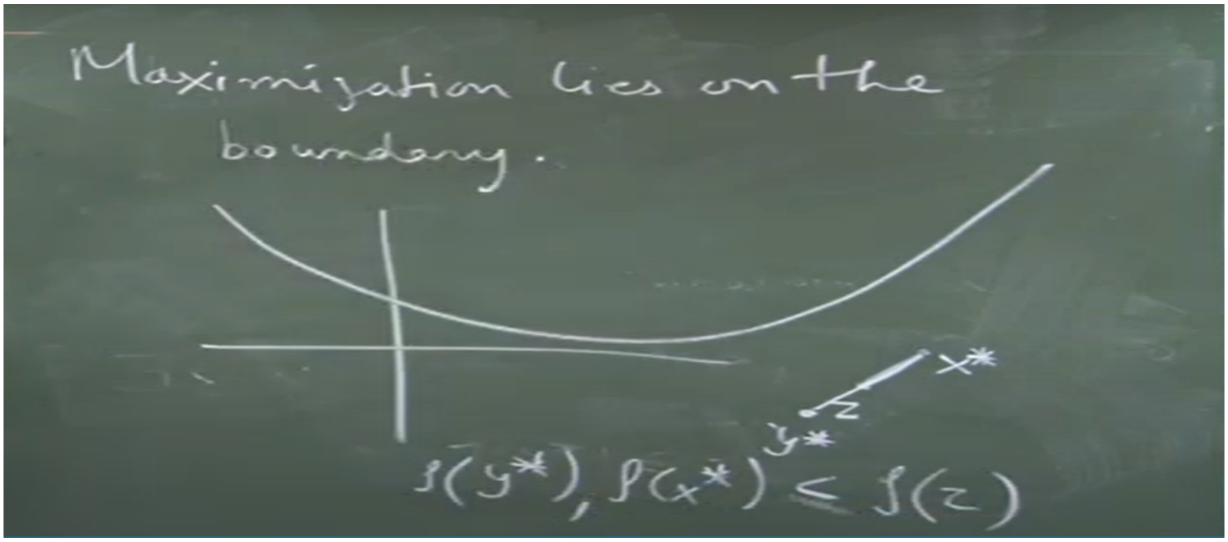
One function could be such that it is a very big number when X is less than 0 some constant function then up to 1 it is 0 and then it is X minus 1. So, if I want to define it this is you will think of it as minus infinity 0 this is 0 X greater than equal to 1. And again you can see that the area above the curve is a convex region or if you take any 2 points the line lies the curve lies below. So, then this is fine and even here local optimas and global optimas are the same. In this case there are many local optimas, but all of them are global optima too.

In general the idea in the convex function is that if you look at the shape of the curve it is like a bowl which can hold water that is the intuition. So, even if it will be probably be very hard for me to draw this, but this will be a bowl shaped picture where the region above the curve is going to be convex. All these are convex functions and you can see that the global optimal is local optimal. So, I think at least by picture I have convinced you that this is the case, but as mathematicians we are not happy till we see proof this is in 3 dimensions. So, I should say yes I should say X 1 comma X 2 and this graph is f of X 1 comma X 2 and then come down.

So, X 1 X 2 are in the you can think of it as a plane and then this is the axis. So, it will look like this, this kind of a thing symmetric amount both the axis is coming down very nice. So, as to fan said if you want to visualize this as a picture you can think of it as a circle. So, my X 1 and X 2 lie in a plane my f of X 1 X 2 I am drawing on the line going up and then you have co centric circles whose radius is decreasing as you go down. This is the shape which will be there and you can see that this is kind of a nice symmetric bowl which is a convex region that is why it is a convex function great.



Without further ado let us see the proof of this. We want to prove that global optimal is local optimal. How can we do this? Let us assume the contradiction, assume there is a local optimal point which is not global optimal. So, let us say X star is local optimal, but Z is global optimal. What this means is around the neighborhood of X star f X star is the best value, but actually there is a point probably far away where f Z is less than f X star.

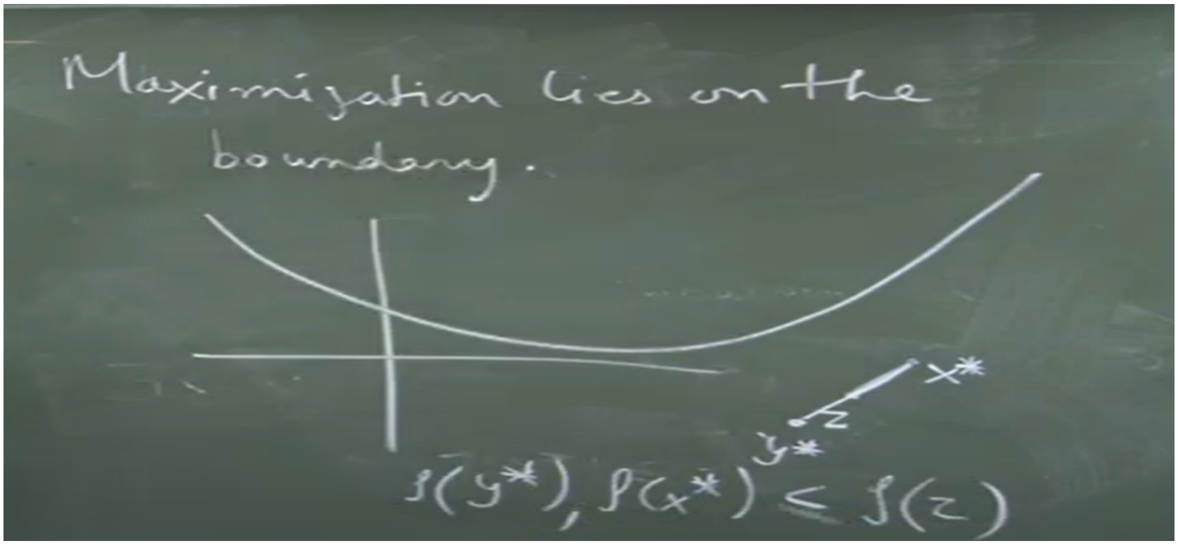


And you want to say that such possibility cannot arise why because f is convex and the region on which the function is defined is also convex this is the region. So, let us look at it I think 2 dimensions are good enough to give you the intuition. Let us say this is X star f X star and somewhere there is a point Z. So, let us say this is f of Z also. So, my curve lies somewhere like this the intuition is if I look at the line which joins Z and X star what is the function value.

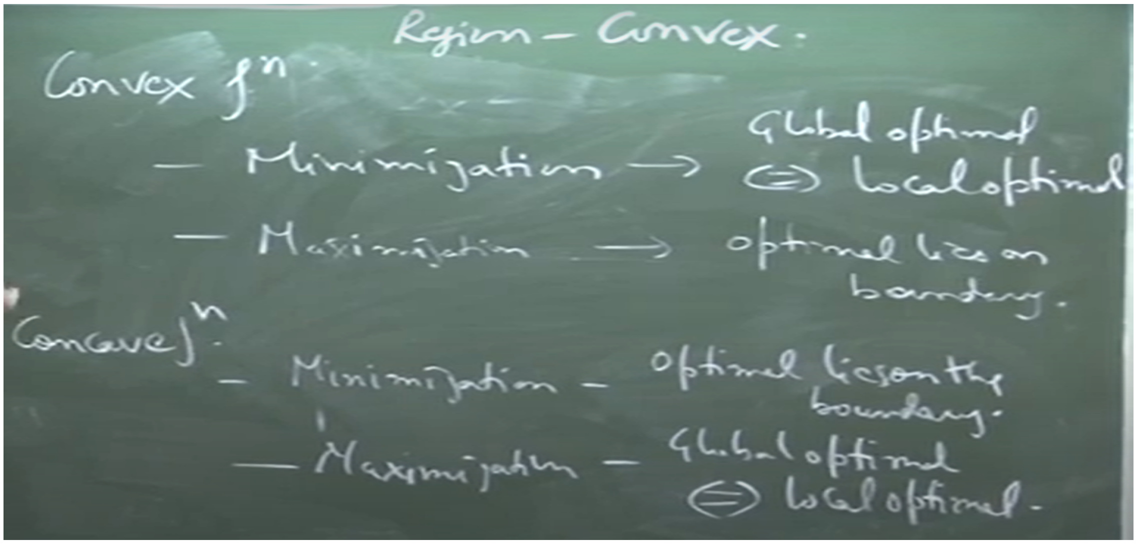
And I know from the property of the convex function f X star this is just because f is convex notice that all these points lie in the feasible region because my region is convex the region on which function is defined is convex. So, that means, all these points lie in my region. And now the claim is that this quantity is I should say strictly less than f X star this is an easy exercise notice that f Z is less than f X star. If f Z was equal to f X star this quantity would be exactly be f X star, but since f Z is strictly less than f X star these two coefficients add up to 1 because this quantity is less than f X star. So, that means, everywhere on this line my value is less than f X star, but then some part of this line has to lie in the neighborhood of f X star.

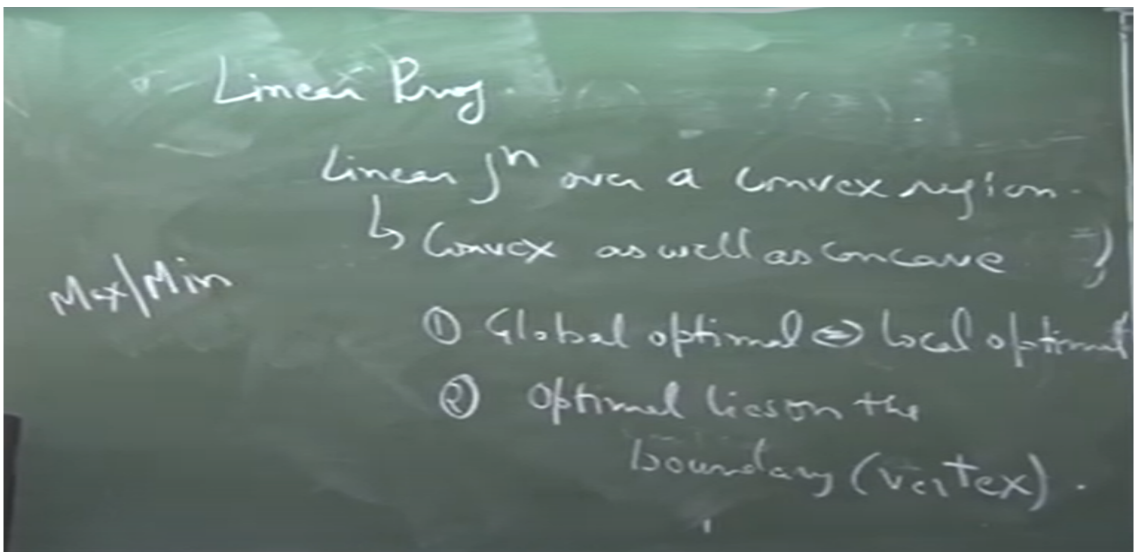
However, small my neighborhood is. This finishes the proof if I want to write it in the formal terms I will say assume this is the ball around X star such that for all Z element of f Z has to be more than f X star this is by the local optimality property of f X star of f and X star, but there exist a very small there is theta I have taken it as 1 minus theta. So, I should say there exist a theta which is approaching 1 such that notice this is the line joining Z and X star. So, as I increase theta as I approach theta close to 1 the point gets closer and closer to X star. So, if there is a ball around X star there has to be some part of line inside it.

That means, there exist a theta such that this point lies inside the ball. So, then there exist a



theta such this is part of B epsilon X star, but from that equation I know this is again because of convexity property here and the fact that f of Z is less than strictly less than f X star, but this is contradiction this point was supposed to be bigger, but this is smaller. So, this small proof shows a very nice property that global optimal is same as local optimal and again I would like to emphasize that this shows that it is I would not say easy, but that difficult to check optimality and again the intuition is if I am at the point and I want to check it is optimal or not it be very hard to check it is globally optimal, but if I only want to check it is locally optimal I just need to search in the neighborhood probably just want to see in all directions that the value decreases and that is enough to show that my point is locally optimal and this intuition will help us build algorithm for linear programming. But before I give you the idea of the algorithm for linear programming I also want to say one more property of convex functions which is that maximization lies on the right boundary. If you minimization if you want to minimize global optimal is same as local optimal, but if you looked at all the shapes for the convex functions the maximum always lies at the boundary.

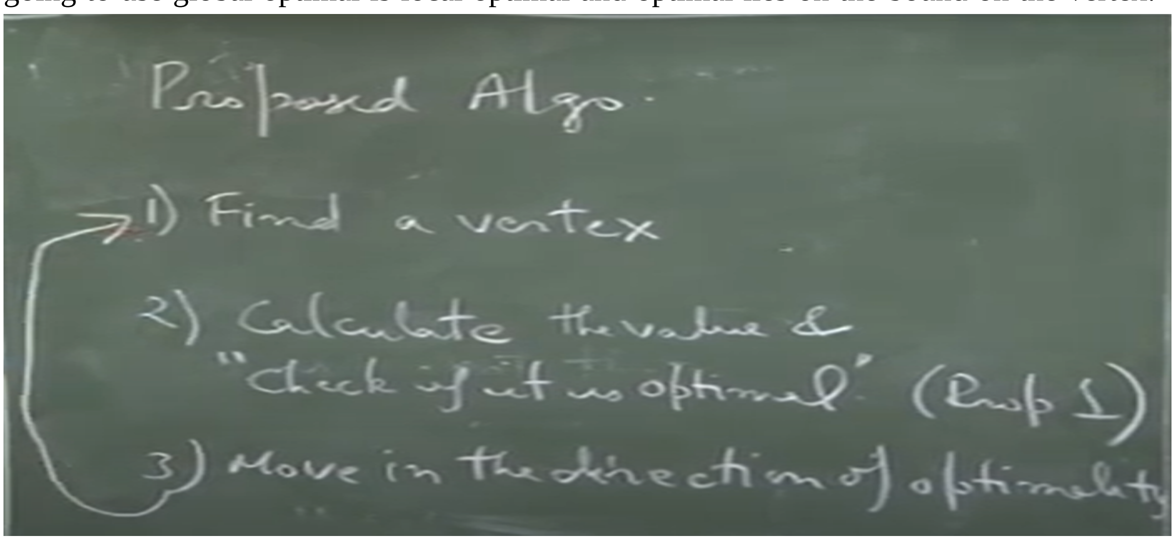




And this is clear from the picture if you have a convex function the function keeps increasing and the maximum lies on the boundary I will not prove it I will give it to you as an exercise, but the simple idea is that if the maximum let us say z does not lie on the boundary look at the point at the boundary and the line connecting these points and you will be able to see that if. So, z should be in the z should be written as a combination of 2 points at the boundary right every point which is not on the boundary we should be able to write it in terms of the 2 points. And then if f y star and f x star are less than f z or yes if suppose f z is the global optimal then this is higher than f y star f x star then you can actually show that any point on x star and y star will be below the maximum of these 2, but again I would not go into the detail I will leave it to you as an exercise. Great we know nice things about convex function notice we will always assume that the region where the function is defined or are feasible region is convex. Now, if you have a convex function minimization global optimal is same as local optimal maximization.

So, optimal lies on the boundary these are the tip 2 properties we have seen we have seen the proof of this I have given you some idea of why optimal lies on the boundary you can fill in the details. Now, similar thing if I write it for concave functions I would say that if it is minimization optimal lies on the boundary or the vertex if your feasible region is convex and for maximization global optimal is same as local optimal. This is what we have seen today and just approaching it in the same way a symmetric proof will give it for the concave function. Why is this nice? It is nice because when we talk about linear programming we are going to optimize a linear function over a convex region and now you should be happy what does it mean that a function is linear a function is linear that means, it is convex as well as concave. That means, we have this properties as well as these properties for linear programming that is great.

So, then for linear programming for whatever maximization we want to do for whatever optimization we want to do whether you want to maximize or minimize it does not matter we always have global optimal equal to local optimal. And the second property is optimal lies on the boundary or for us because we have a convex region it lies on the vertex. These two properties are central to defining the simplex algorithm. The idea of the algorithm is not very difficult I will give you the idea and then we will see the formal algorithm in the future lectures. Remember these are the two main properties we are going to use global optimal is local optimal and optimal lies on the bound on the vertex.



The proposed algorithm is first find a vertex obviously we are interested in comparing value of our function on all the vertices. And yeah actually let us let us let us talk about this we know that optimal lies on the boundary I give you a very simple algorithm look at all possible vertices look at all possible values on those vertices find the optimal that is going to give you the answer why is that not a good solution why is that not a good algorithm what do you think? You can take all possible vertices of the feasible region and ask what is what is the optimal among those vertices we know it has to lie on the vertex good Tufan has very nice idea he says there might be many points many vertices in the feasible region. And you can actually show that there can be convex regions which have exponential number of points in terms of the dimension. So, this algorithm is really not efficient we need smarter way to search for a vertex which is optimal we cannot just list out all the vertices that will take lot of time. So, that is not a good idea, but again using this we can refine it.

The idea is find a vertex I am not telling you how to these things will become clearer from the next lecture the proposed algorithm is find a vertex calculate the value and check if it is optimal. Now, this is easy because of property 1 what it means is for r convex region if this is my convex region I am at this point I just need to check whether in both these directions my value increases or not and that is good enough to tell us whether this is a local optimal or not, but local optimal implies global optimality. So, we are just going to check the local optimality here. So, checking if it is optimal is easy because we can just check all the directions from that vertex all the edge from that vertex and that is enough. If it is not optimal move in the direction of optimality that means suppose we were at this vertex we wanted to find a maximum the function value if it does not increase here does not increase here we stop because this is optimal, but let us say the value increases in this direction if the value increases in this direction we can go and find another vertex in this.

So, then we in the move in the direction of optimality and find a vertex again this is the idea of the simplex algorithm. Again there are many many things which are not clear here how do we find a vertex how do we actually check if it is optimal and then what does it mean to move in the direction of optimality. So, describing these will finish our exact description of simplex algorithm and that is what we will see in the next lecture how to make these this idea concrete.