Linear Programming and its Applications to Computer Science
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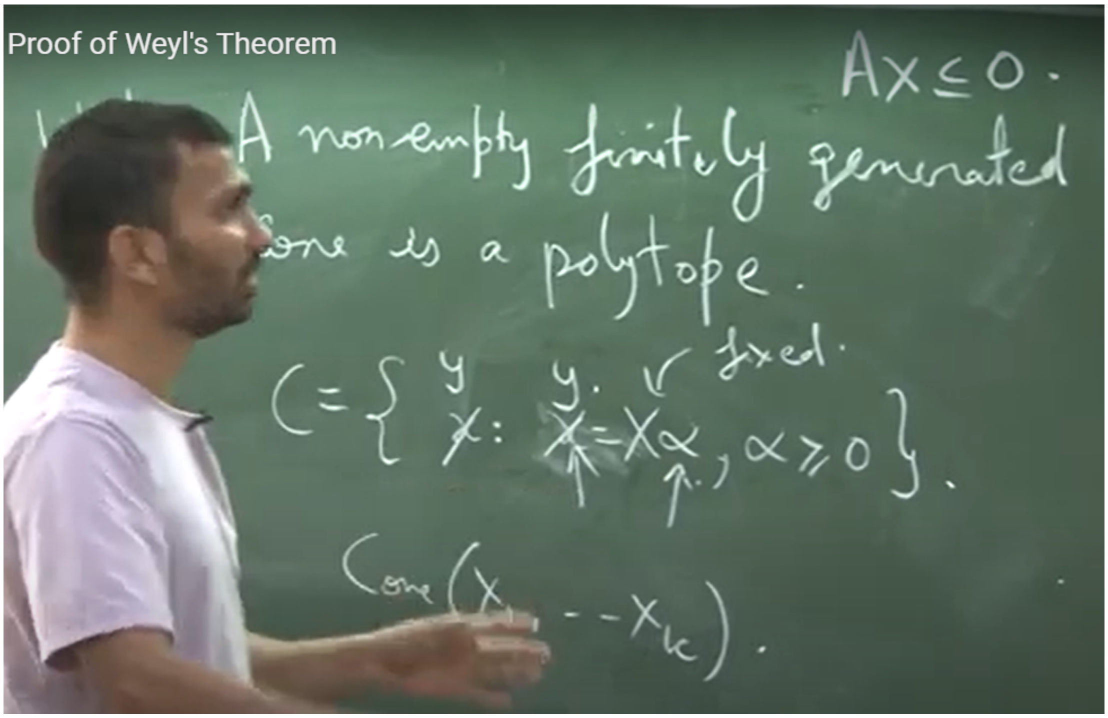
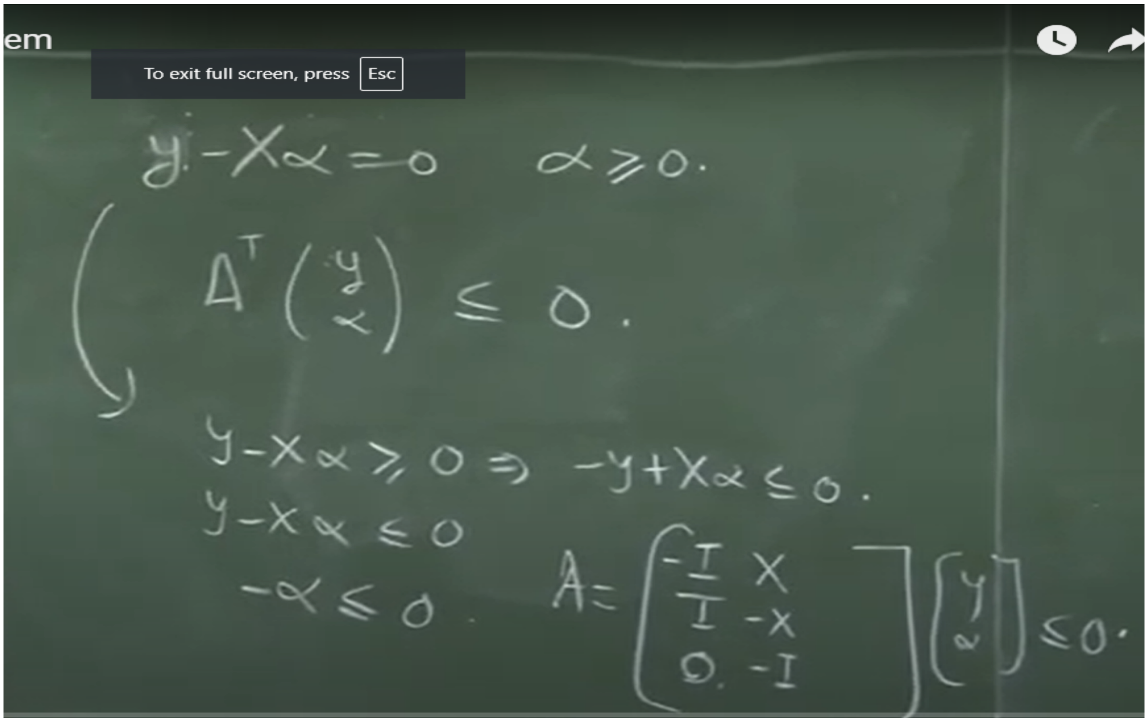
Lecture – 14
Proof of Weyl's Theorem

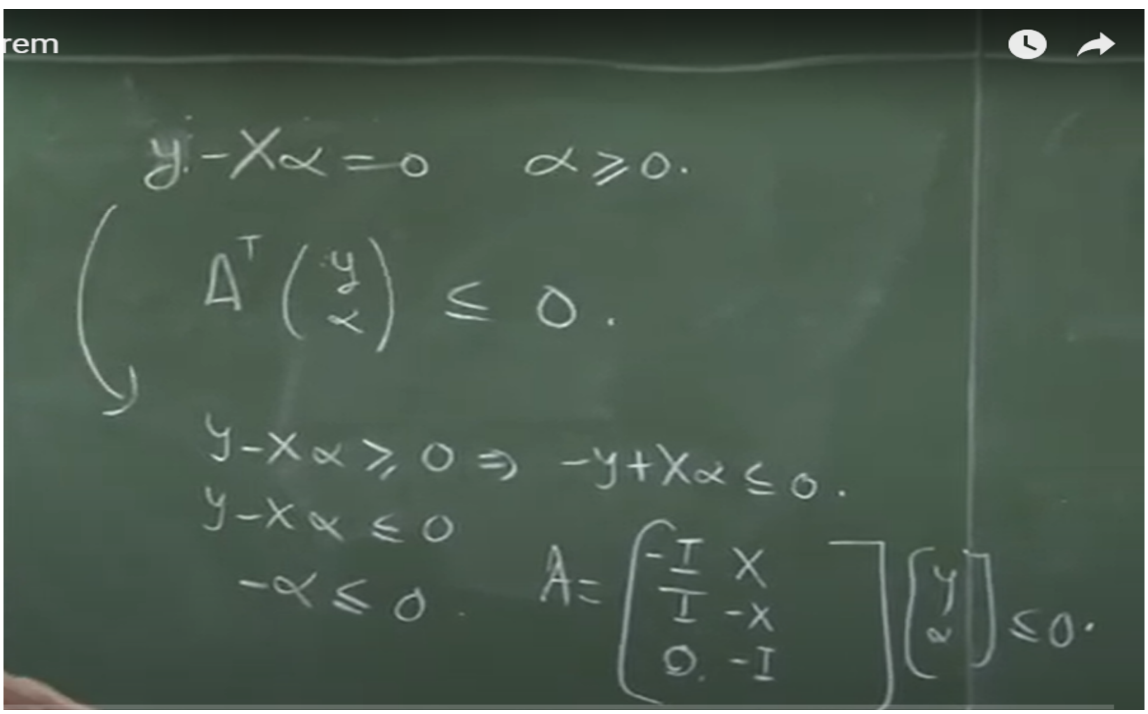


These are my equations right. And my question is can you come up with a matrix A such that this is equivalent to saying some you know Y alpha less than equal to 0 right. So, this is equivalent to Y minus X alpha correct. Now, with this description you can come up with A is that ok with everyone.

What is A minus identity So, this multiplied by right first equation will be minus Y plus X alpha less than equal to 0. Second equation will be Y Y times identity minus X alpha less than equal to 0. And sounds good great. So, we are almost there.

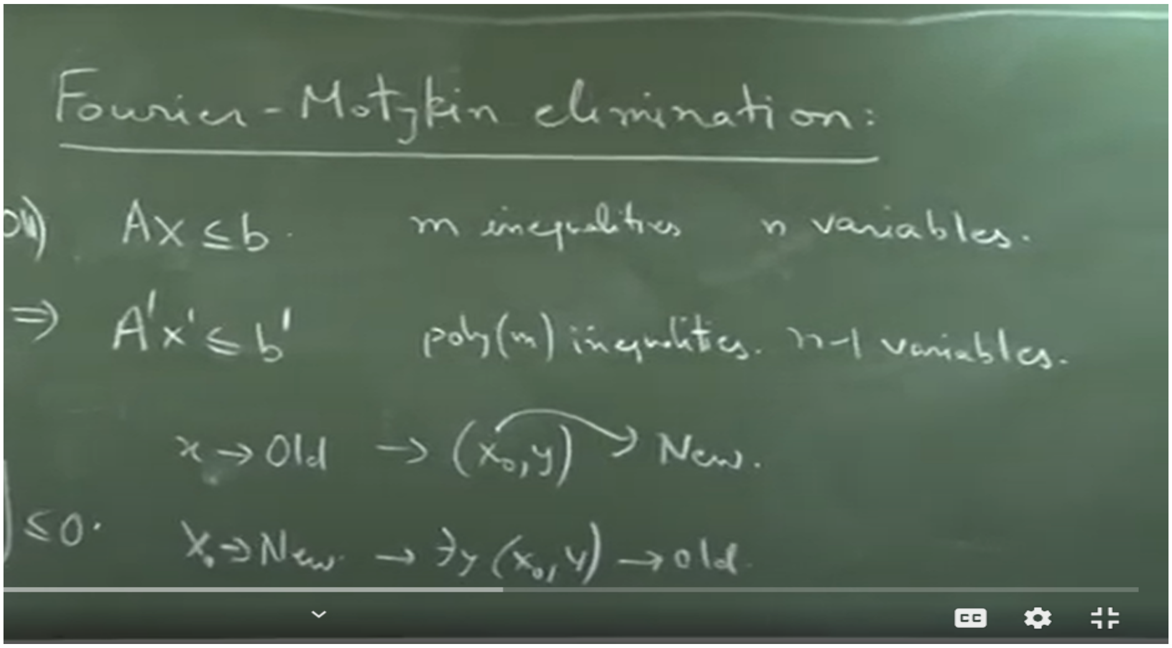


We have these inequalities with 0 on the RHS, but with more variables than we want right. So, how should we need to remove alphas?



How can we remove alphas? No but then you will lose like you want is remove doesn't mean just erase, remove means come up with equivalent system so that solution there is solution here, solution here is solution there right , You have show that they are equivalent.. So, this is done by something called Fourier Moleskin elimination. So, this is done by something called Fourier Moleskin elimination.

This is a nice trick works in generality and you can probably come up with it if I had not



told you already right. So, what does it do? So, let us say we have the system A X less than equal to b where you have m inequalities and n variables. Fourier Moleskin elimination changes it to b prime where now you have n minus 1 variables and X prime is you can say X 1 X 2 up to X n minus 1. So, it is the first n minus 1 variables of X and you incur only small overhead in terms of number of inequalities. And then the point is so, this is let us say the old system this is the new system then if you have X which is a solution of old system you break it down into X 0 comma Y, X 0 will be a solution of new system.

And if you have a solution for the new system there exist Y such that X 0 comma Y is a solution of this is what we are going to. So, if you believe in Fourier Moleskin elimination do you agree that a finitely generated cone will be defined like this a Y less than equal to 0 that is easy to show. No like I mean a finitely it is a convex cone that you can show A X less than equal to 0 set is which is A X less than equal to 0 is a convex cone that is not hard to show right. Now the question is given a convex cone can you convert here right. So, now we are here my question is assuming Fourier Moleskin elimination are we ok done we can get to a Y less than equal to 0 no you are asleep.

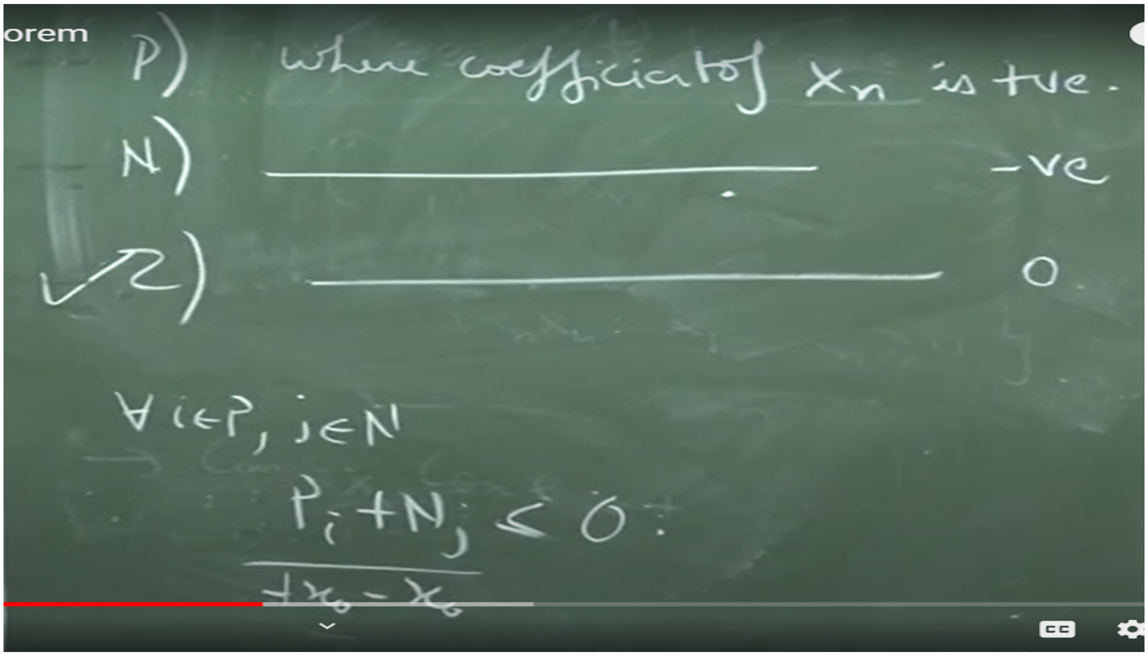
Sorry. So, but yes you are missing something. So, wake up tell me what you are missing. Exactly in this case we have to make sure that if we start with 0 we end up with 0 right this is not clear from this definition of Fourier Moleskin elimination. So, I am going to tell you how Fourier Moleskin elimination work and you have to make sure that it keeps this right hand side being 0 property true.

And then we are sure that we can convert any A X less than equal to 0 a Y comma Y alpha comma greater than equal to less than equal to 0 to some A prime Y less than equal to 0. And then we will have a nice characterization of the convex cone sounds good. ok So, any ideas how should we eliminate you have a variable. So, let me help there are 3 kind of inequalities right everything is of the kind A X less than equal to b there are 3 kind of inequalities P where coefficient of X n is positive n where it is negative and Z where it is 0 right 3 answers which one is easy to take care of right. What should we do about this the coefficient of X in this set of inequalities the coefficient of X n is 0.

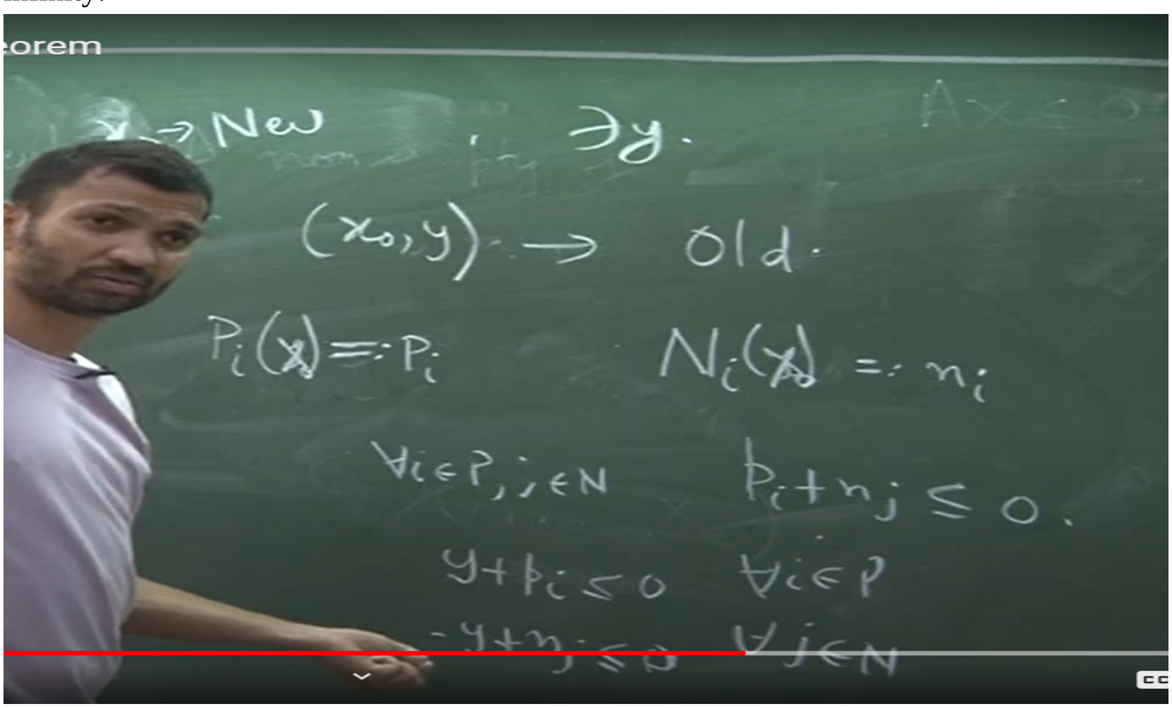
So, it does not depend on X n. So, it is already removed it is already of this form. So, if I look at all the inequalities. No, no, no, no, no, no I am dividing my set of inequalities into 3 parts. So, my total sets of inequalities are inequalities here plus inequalities here plus inequalities here right.

So, obviously, this inequalities I will just carry forward I will put it in my new system directly what should I do with this. So, what we will do is for every i element of P and j element of n we will introduce the inequality less than equal to 0 with a small change which is you take the inequality here and make. So, that the coefficient of X n is minus 1 plus 1 here you can divide the coefficients. So, that the coefficient of X n is minus 1 right. So, then here when I do this my X n is gone.

If my solution satisfies these 2 this is satisfy this yes I can do this now this is my original P i this right. Sorry if I am considering P i in the old system then clearly P i is less than equal to 0 n j is less than equal to 0. So, their sum is also less than equal to 0. So, this I get for free this was the easy direction what is difficult if this is true then can I come up with this no inequalities of second kind think about that is a very easy case. You will have like you will just keep P i less than equal to 0 and you can satisfy it for free by keeping X 0 remember that the coefficient of X 0 is the same everywhere and your X 0 is unconstrained.



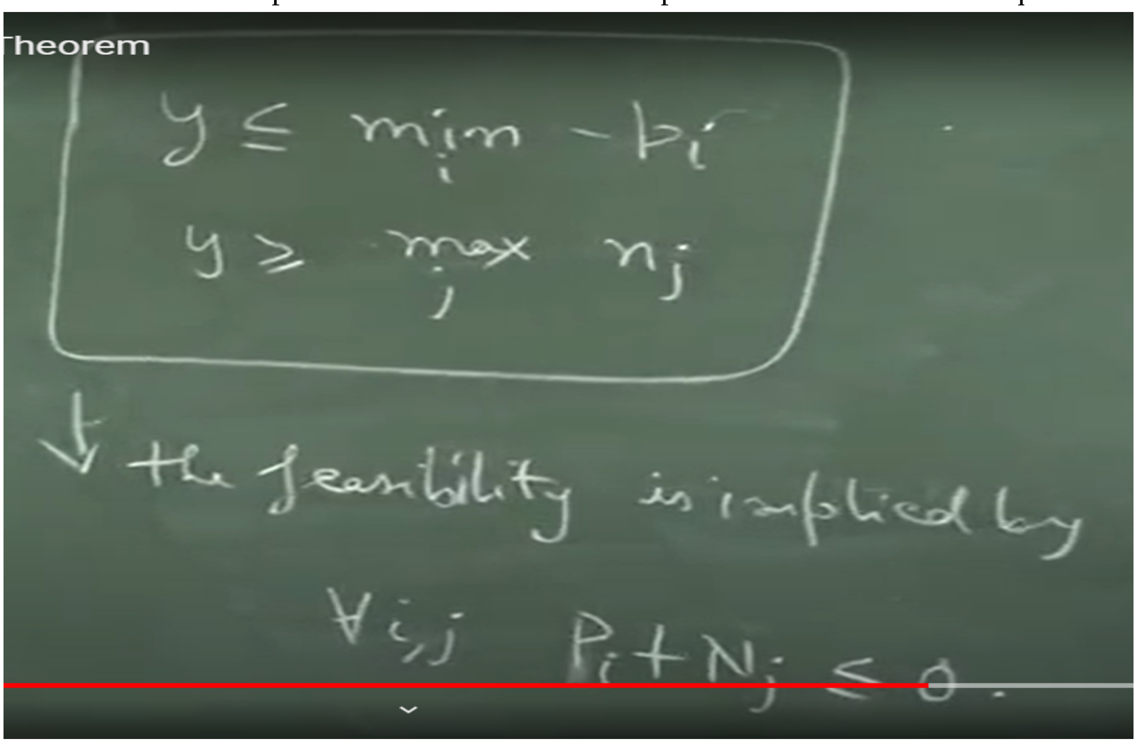
So, there is a choice of X 0 which will trivially satisfy all this. By if in all your inequalities X 0 appears as positive you can take X 0 to be minus infinity and all the things will be satisfied right. You are saying that you only have inequalities of this form right. So, you have only inequalities of plus X 0 less than equal to 0 plus X 0 less than equal to 0. So, this I can trivially satisfy I will just put X 0 to be minus infinity or something right big thing and if it is all minus minus X 0 I will take X 0 as positive infinity.



So, then these things will be automatically satisfied. So, I do not have to worry about that. So, what do you want to show? We want to show that if I have a solution Y or what did I call it? I call it X nougat right system there exist a Y. I want to fix my final variable such that my old system is satisfied yes comma Y is a solution of my old system sounds good. So, let us say with abuse of notation now I will say now P i are my inequalities in the new system where you know your X 0 is kind of removed. You can call them P I prime if you want, but this is what did I call it define it to be P I and sorry this should be the other direction.

So, P I is defined as the value at Y of this inequality right. So, look at the old inequality right remove X n from it. So, I am abusing notation I am also calling that P I. So, P I Y is some value.

We are eliminating X n X 0 is the solution of the new set of inequalities right. So, I am writing Pi X0 as P I and X 0 as and basically whatever are the values right. So, what I want is for all I element of P j element of n, I know that P I plus n j is less than equal to 0 right. This I know and now I want to come up with a Y. And then what should happen is I want to come up with a Y such that Y plus P I is less than equal to 0.



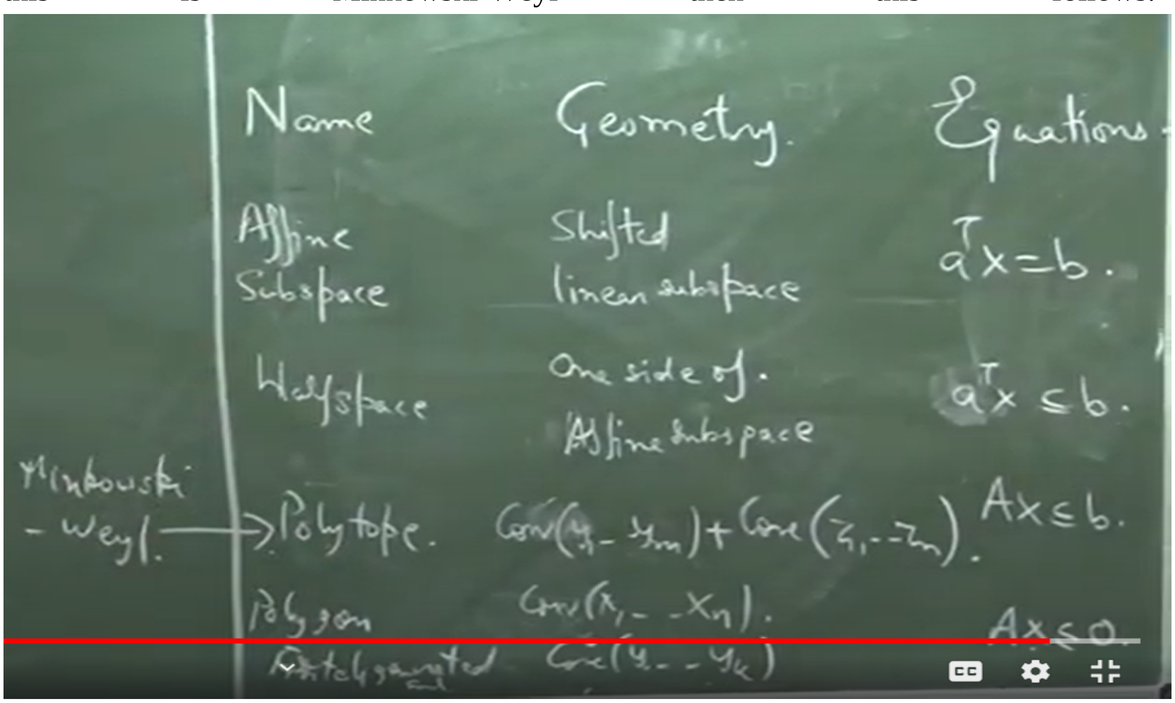
So, what does this mean? What does this equation mean? This is for all I. This means Y should be less than min of minus P I. This is P I these are numbers. No Y is just 1 value right it is X n.

What are we doing? We have new solution we want to show. So, that X naught Y is a solution of old. So, Y is the value of X n we just want to set 1 variable right good. So, now, why is this feasible? Because of this, right what is this same any minus P I is bigger than any n j. That means, the minimum here is going to be bigger or at equal to the maximum here.

So, the feasibility is implied by sorry. Then the way we constructed it we made sure that RHS always remains 0. We did not have any non trivial constant variable right. So, this shows is that a finitely generate cone is specified by equations A X less than equal to 0. And now we can make a very nice table.

Depending on the kind of person you are you might like this column or this column, but for this course you need to like both the columns. So, what do we know about affine subspace? It is a shifted linear or vector subspace and in terms of equation vector set is like this. I can just write less than equal to b because less than equal to is covered correct. This is a we defined it like this and now knowing we have the definition of cone it is basically convex hull of cone of Z 1 to Z N. Ok. What does plus mean here? Take an element here take an element here.

So, this is called Minkowski sum simply put S 1 plus S 2 is you take element of any element of S 1 and it when element of S 2.ok This I will not prove this is Minkowski-Weyl theorem which is outside the scope. Ok. Then you have polygon which was bounded polytope right. So, this is convex hull of X 1 to X N clearly it is bounded it cannot have this right does not have this that it is become this. So, if you assume this line this is Minkowski-Weyl then this follows.



And what is the last one? Convex cone finitely generated cone. Convex cone is slightly more general you can have this ice cream cone right which is completely circular that would not be finitely generated right. So, for what we are interested in finitely generated cone which is cone of Y 1 to Y k. So, for the course the important thing is understanding the shapes and their equivalent transformation. The proofs are not important I just covered one.

So, that you at least have a flavor of how convexity and all these are used. They are obviously, because this is of this kind yes. So, this is a subset of this yes right this is the subset of this yes.