Linear Programming and its Applications to Computer Science
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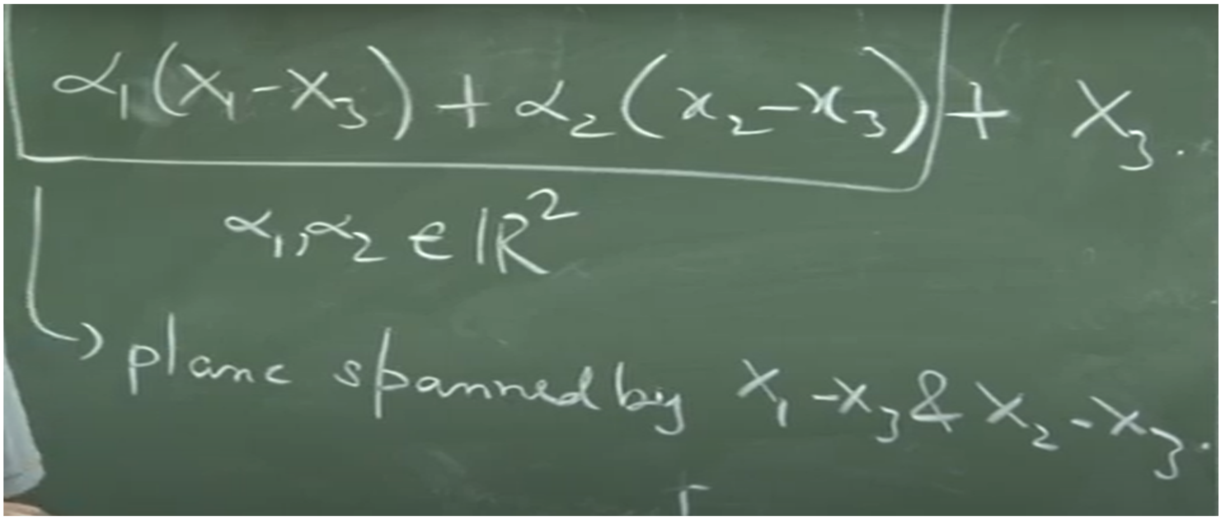
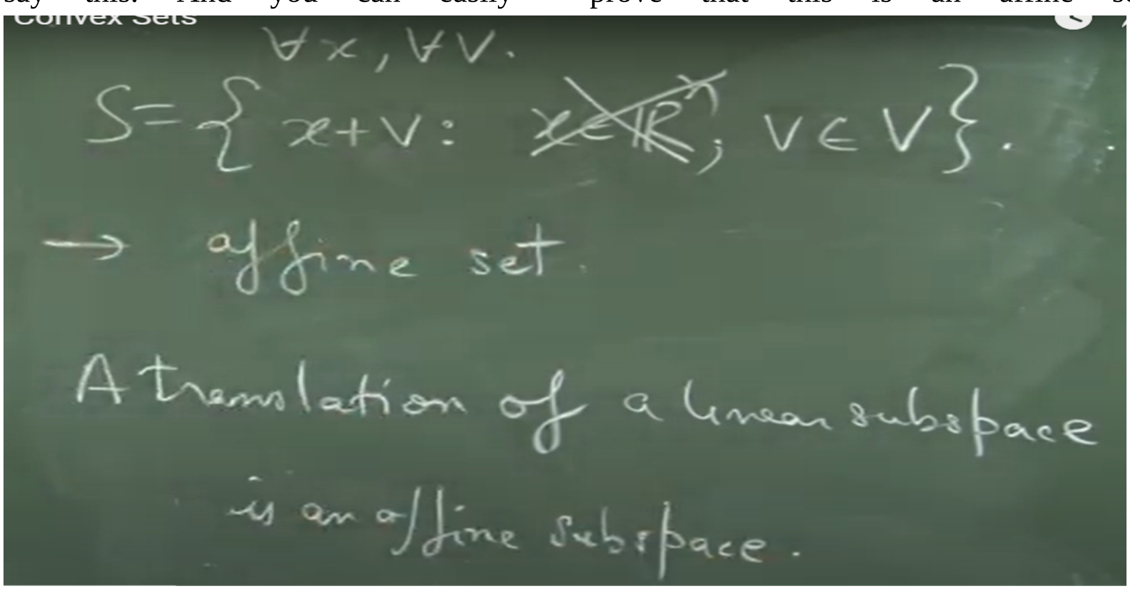
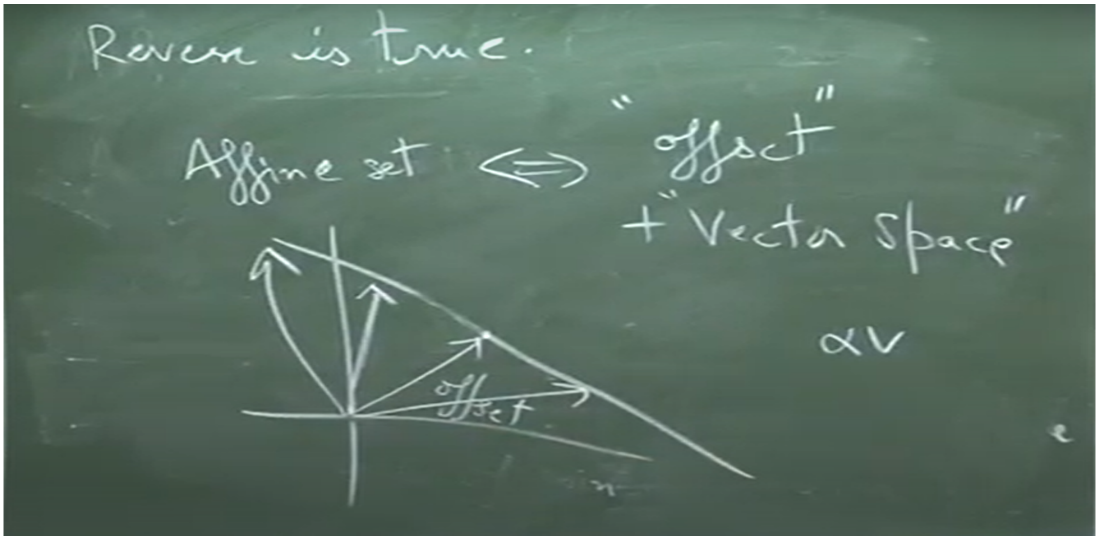
Lecture – 12
Different kind of Convex Sets



And not surprisingly you can do this trick with any affine set right. X is some point in R to the n and V belongs to some vector space. So, this is for all x and for all vector spaces. So, you fix an x you fix a vector space you define a subspace like this sorry I should not say this. And you can easily prove that this is an affine set.



So, a translation of a plane or a sorry I should say a linear subspace is an affine subspace. And since I can do this trick for n number of points it is going to turn out that even reverse is true. So, what is an affine set? It is offset plus a vector space. And in my favorite two dimensions it looks like this. This is my offset and then if I translate this line to here that is going to be my vector space.

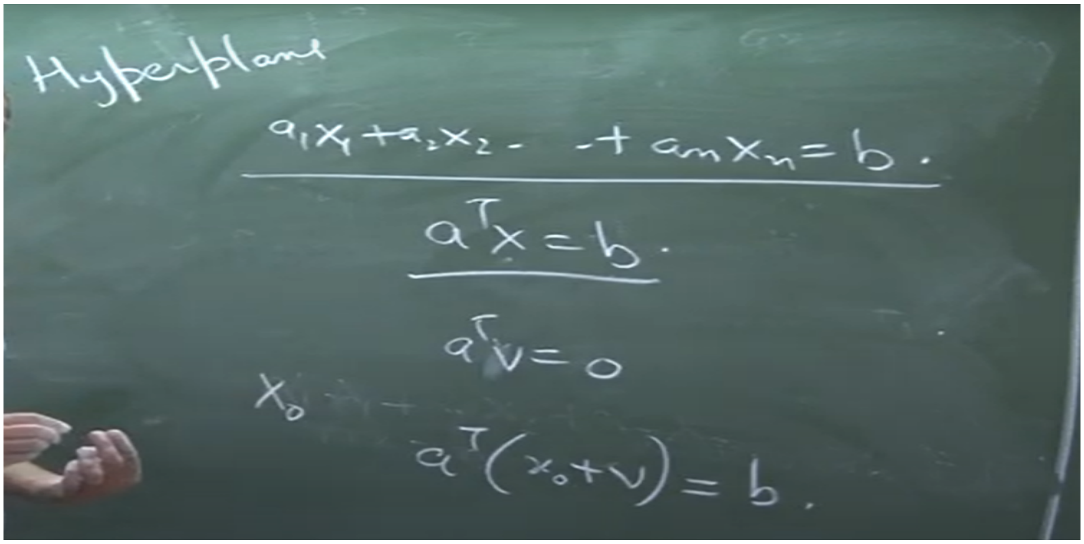


Remember this is not a vector space if I pass it through the through 0 then it is a vector space right. That you can easily easily see that if you look at all points you know alpha V this is a vector space. But if I move that line that does not remain a vector space right. But all your affine sets look like this. And now if you look at this definition these two things seem important it turns out actually not.

The offset is not really important. The vector space always remains the same you can actually choose this offset you can choose this offset you can actually choose any point in the affine set as the offset. And this is true I could have played the same trick with x 1 right I could have played the same trick with x 2. So, even though it looks like it is offset plus a vector space it is any offset in the affine set adding the vector space. And that is because of the linear combination structure.

Sounds good do we have a good understanding of what an affine set is now? Because before you do convex sets you understand affine sets know. And we have to understand convex sets because linear programming requires convex sets. The feasible region is going to be a convex set. So now this is one kind of a for the lack of the better term duality here in the sense there are two descriptions of a affine set right. One description is sorry right no, but that end this yeah that is true that is kind of a way to define it, but I do not think we ever use it that is not very useful definition to use.

So, one is this offset plus vector space and the other is you know how to describe a line you write a equation for it. In affine set it is going to be what we call a hyper plane which is.

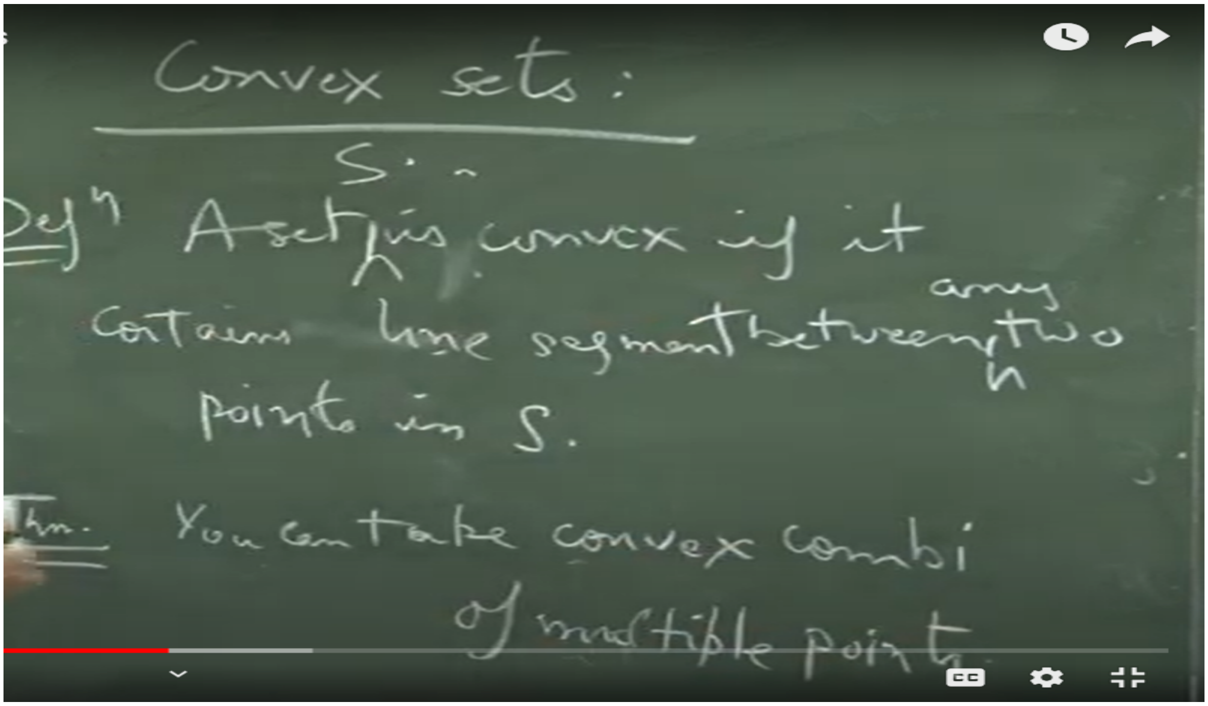


So, again this both natures I can write it as an equation I can write it as a geometric object. Now this is an affine set is not hard to prove other way is also not hard right. Because what is this saying? This is saying that every x which has a constant inner product with A correct right.

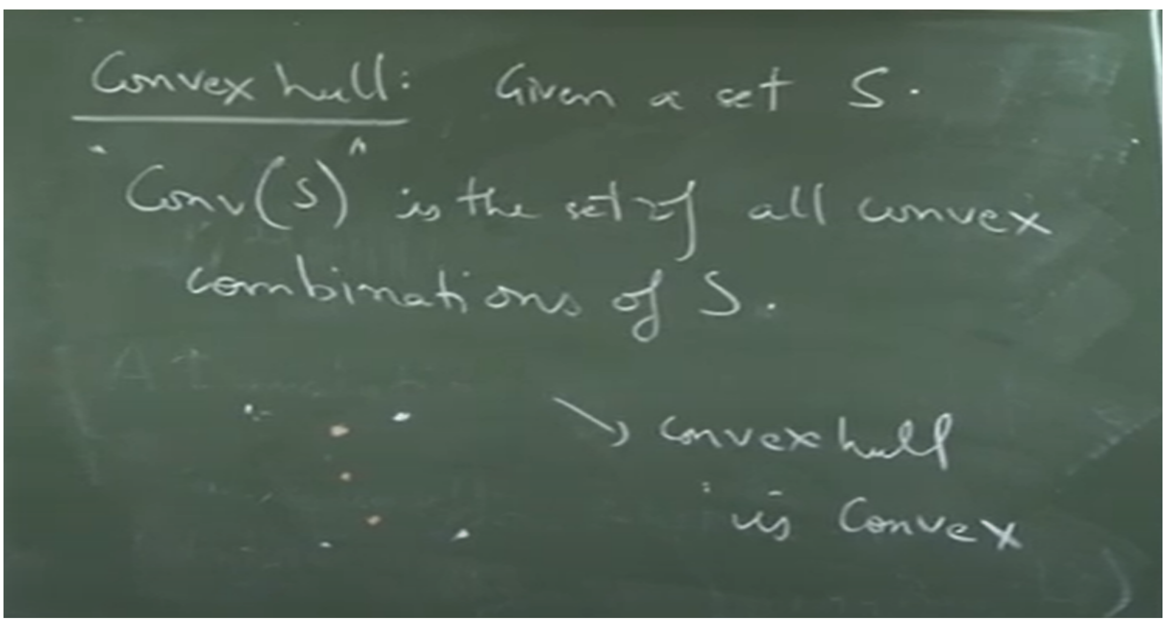
This is saying that my inner product with B is constant. Now again this would have been easy what is this? This is the things which are perpendicular to A right. This now let us say X0 is a point in this then this is X0. So, let us say this was the case you have an X0 an offset A transpose X0 will give you B you can add any V to it such that A transpose V equal to B right. So, this is the vector space which is perpendicular to A moved in some direction.

And again the same thing the vector space remains the same you cannot change the vector that is the important part it is the perpendicular vector space to A X0 pick any point such that A transpose x equal to B right. So, in multiple ways we see the same thing, but this is the important trick this is the trick which will play again and again for almost all the geometric things we will have a geometric picture and an equation picture right. Sounds good

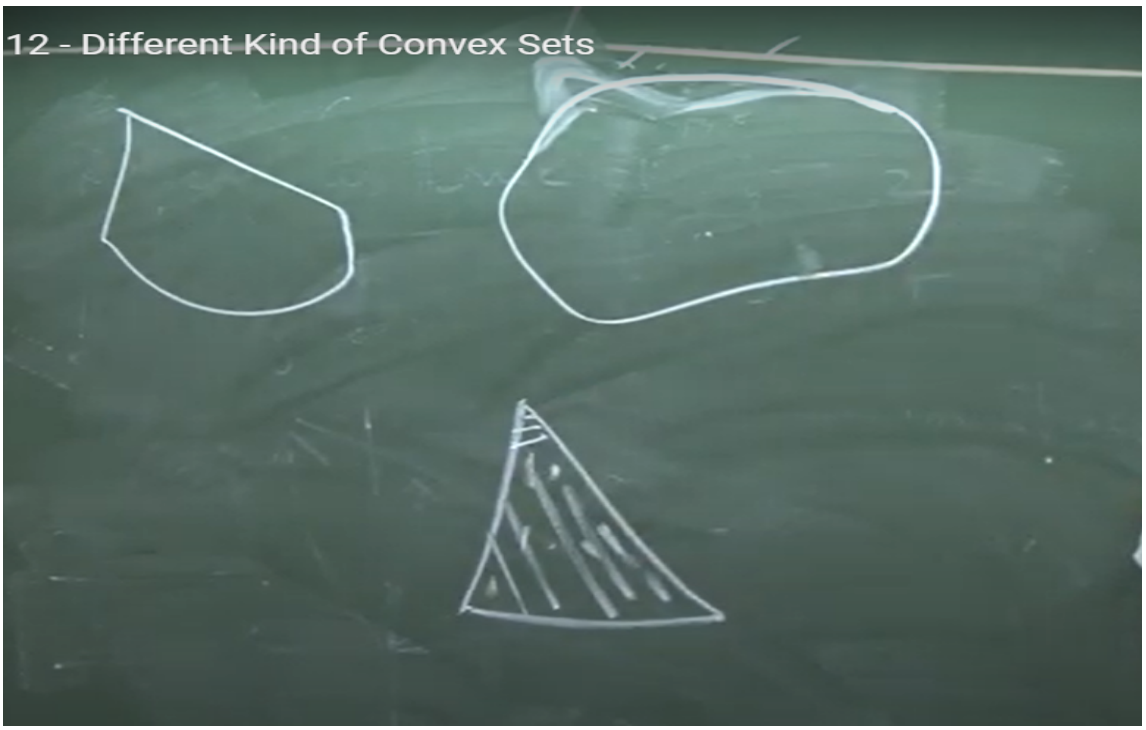
Now we move to convex sets I should talk about conic sets also, but that is not very important I will talk about cones later they are slightly difficult you have to be very careful while you define conic sets of cones that we will do later today we understand convex sets any questions about affine sets good. What is the definition? A set is convex if it contains. The lines are points natural the way we did the affine right and no price for guessing theorem yeah you can take convex combination of multiple points also.



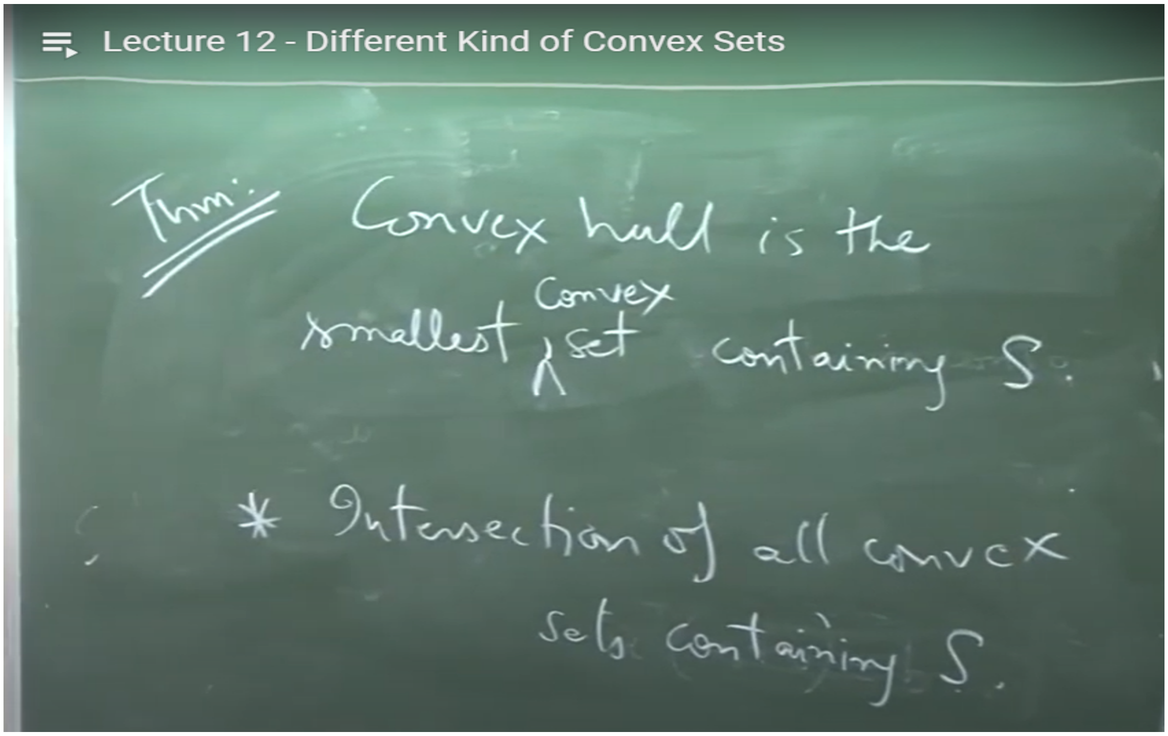
I won’t go through it because that is basically an exercise in induction, but once we define this another important thing is what we call convex hull.



So given a set S convex hull of this is the notation is the set of all convex combinations of S ok. So, what is the convex hull of 2 points line segment? That ok right. This we have seen multiple times that is the simplest case, but in general I can have multiple points and if I take convex combination of everything everything in that set S that is called the convex hull of S right and also now what can happen? Suppose with these 3 points I take a convex combination I get a point here with these 3 points I get a convex combination this is here. Now, I can possibly take convex combination of these 2 points again what do you think? Is this a convex combination of these 4 points? Yes it is do I need to prove that you can all do it right you have 2 set of coefficients each of them you kind of multiply by let us say half and half or 1 by 3 2 by 3 still they will remain positive they will sum up to 1 what does this prove? Convex hull is convex too many convex thing, but yeah convex hull is convex right even though I just took the convex combination of S these are going to be vertices, but this is fine and this might again look complicated, but you have seen these things like affine set turned out to be just vector space and offset. Similarly, convex hull see generally a convex shape could be very complicated very hard to describe a circle is a convex set because you take 2 points inside the circle the line segment stays inside. Right this is convex or at least the shape I intended to draw was convex right convex, no oh sorry right and to be able to describe it in you know every possible way this is going to be it is convex in my eyes.

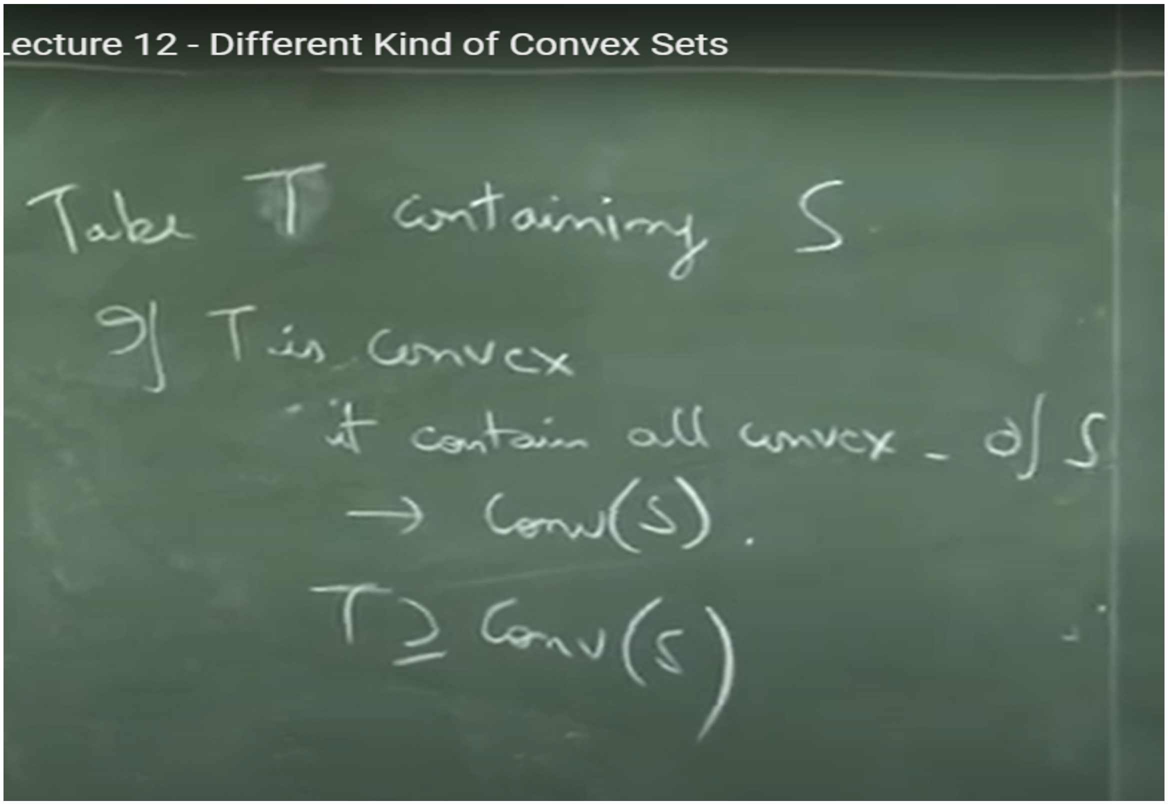


So, but it is very hard to describe it if you want to describe it by equations we will take the simpler task we will talk about convex hull of finite many points and that is going to be reasonable. Ok so, now I have 3 points how will the convex hull look like? Why? Start this line has to be there this line has to be there this line has to be there is it convex yet no I can take these points good I was hoping that you ask this question because then I can tell my next theorem. So, this is the smallest set just or I should say do I want to write it as intersection yeah is this is the smallest or if you want to think about it take all possible sets which are convex and contain S take their intersection that is going to be convex. So, let me write it all these are equivalent definitions.



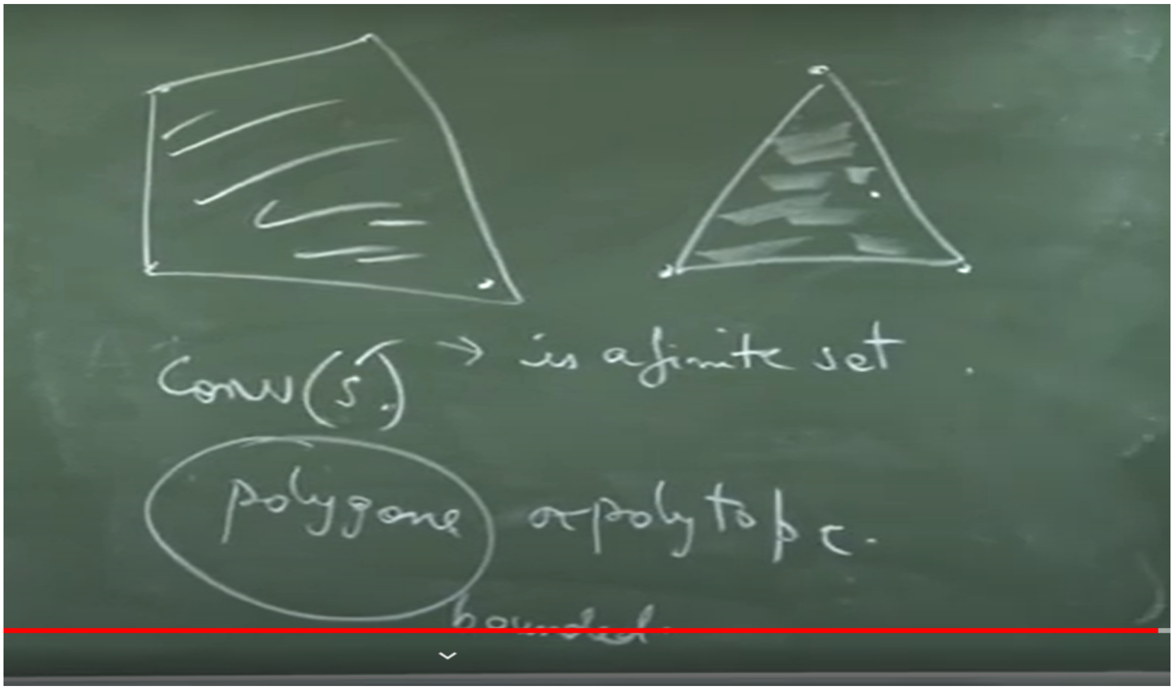
So, this is a valid point what is the guarantee that I have not left out anything this is a convex set this contains 3 points that is the that is enough. So, that means I just need to prove this theorem take 2 minutes and prove this it is not hard it is like a 2 line proof it basically says that you are not confused between all these convex. That is the next thing I am going to talk about, but again I am not going to talk about it because you will prove it is like the simplest thing right again if there are 2 points if they are in both the sets their line segment will be in both the sets it will be in the intersection. So, then right so every comp E. So, if it is S 1 S 2 S 10 S 1 will contain the line segment S 2 will contain the line segment S 10 will contain the line segment. That's all but we go there later.

So, the intersection will contain the line segment that is all, but we go there later. So think of any set which contains S right now it contains S it is a convex set that mean that that mean it will contain all the convex combinations of S. So, take S I am sorry take T containing S take any set which contains S. Now if T is convex it contains all convex combinations of S that means it contains convex hull of S that means T is bigger than convex hull of S simple. Again this definitions might seem out of line or slightly confusing, but there is something geometrically it is very simple.



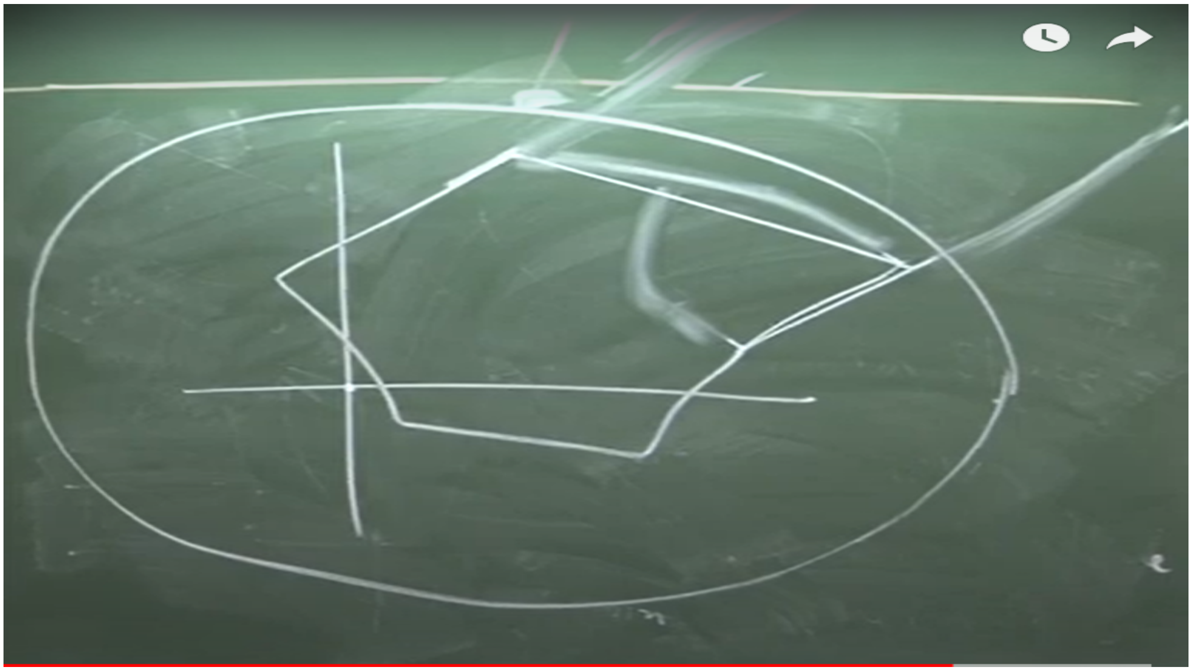
So, it is just to make sure that you align your geometric intuition with these definitions. And if there are any questions I would be happy to answer or give them as exercise right. So, we had affine sets affine sets were linear subspaces by an offset right. Now we are talking about convex sets look like a triangle if I take 4 it will look like a mostly a quadrilateral sometimes a triangle right. So, if I take 4 points great this is the convex set right when I mean all this.

If I take a point inside still remains the same right. And now what do you think intuitively can you describe to me what is convex of S if this is a finite set. What is again I do not want it to be mathematically correct in your mind what is the picture for 4 it was a quadrilateral something which has kind of a boundary lines right. It might happen that it the number of the number of vertices number of extreme points are not equal to S because some of the points can get inside. But in some sense I look at all the points I look at all the outer points and connect them by straight a line that is called a thank you polygon or polytope. Ok.

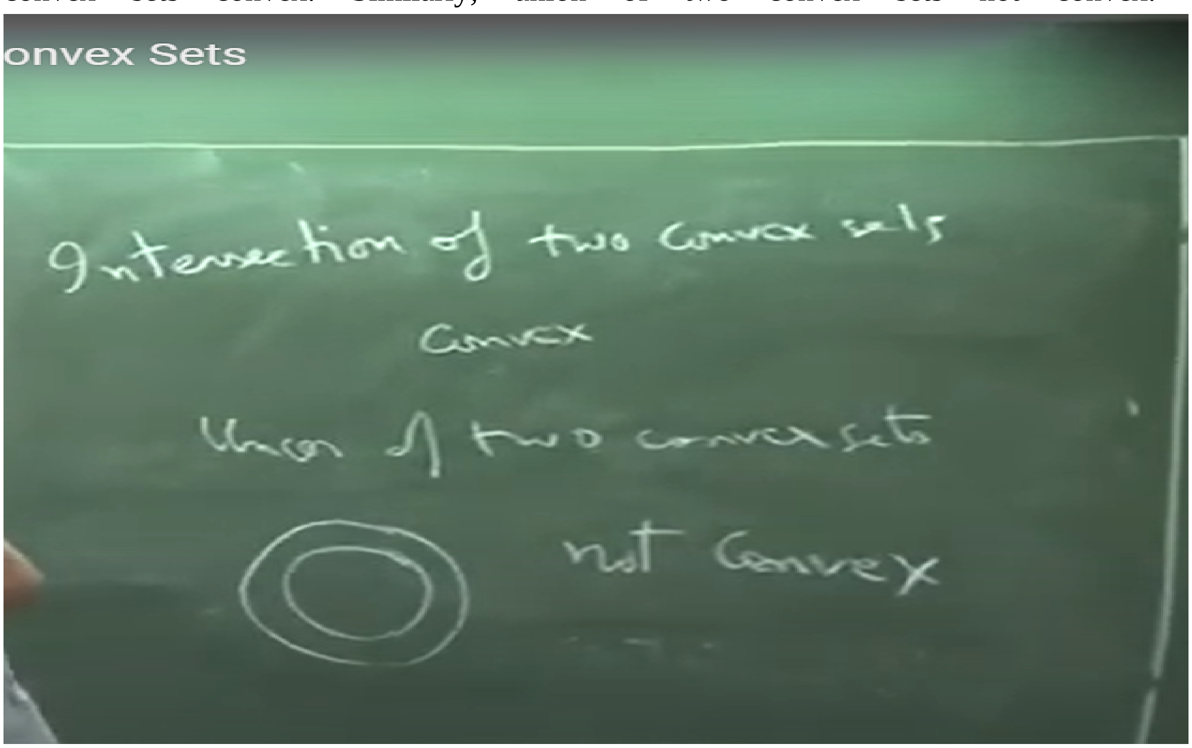


So, the difference is this is bounded. So, what do I mean to say? So, actually for finite set of points this will always be a polygon ok. That means it will be bounded, but we will be interested sometimes in convex sets which are slight extension of polygons right. So, polygon is right polygon like this ok, but you might sometimes be interested in convex sets which calling the previous thing was bounded this is unbounded right. What is the mathematical definition of a set being bounded what do you think it should be? Good.

So, you have seen it before. So, there is a big ball. So, there is a big ball you can say from origin which contains the entire set of finite radius. There is a sphere of finite radius which contains my set. So, in case my set was like this there is a big ball. So, basically you are not going to infinity right.



So, again simple intuition I have to give a mathematical definition to all these things bounded ok. So, polygon is bounded polytope is unbounded, but something which whose boundaries are these straight lines and it is a convex set. But again we will see these two things coincide. And before I proceed further and forum objects intersection of two convex sets convex. Similarly, union of two convex sets not convex.



Each of you draws an example. Why cannot you draw in your paper? I do not think that is a good enough excuse everyone keep a pen and a paper with you it helps in my in my class at least please try these things. I know you are ultra smart as I always say what is the best thing about UGs? They give very give very quick incorrect answers right. So, let us try to make them correct and the simple thing is to think before you are answering. So, everyone can draw two convex sets such that their union is not convex right good. Simply put you take any two disjoint convex sets their union is not going to be convex good.

Sorry union is convex and intersection is also convex. So good. Sorry, it's completely right.