# Linear Programming and its Applications to Computer Science


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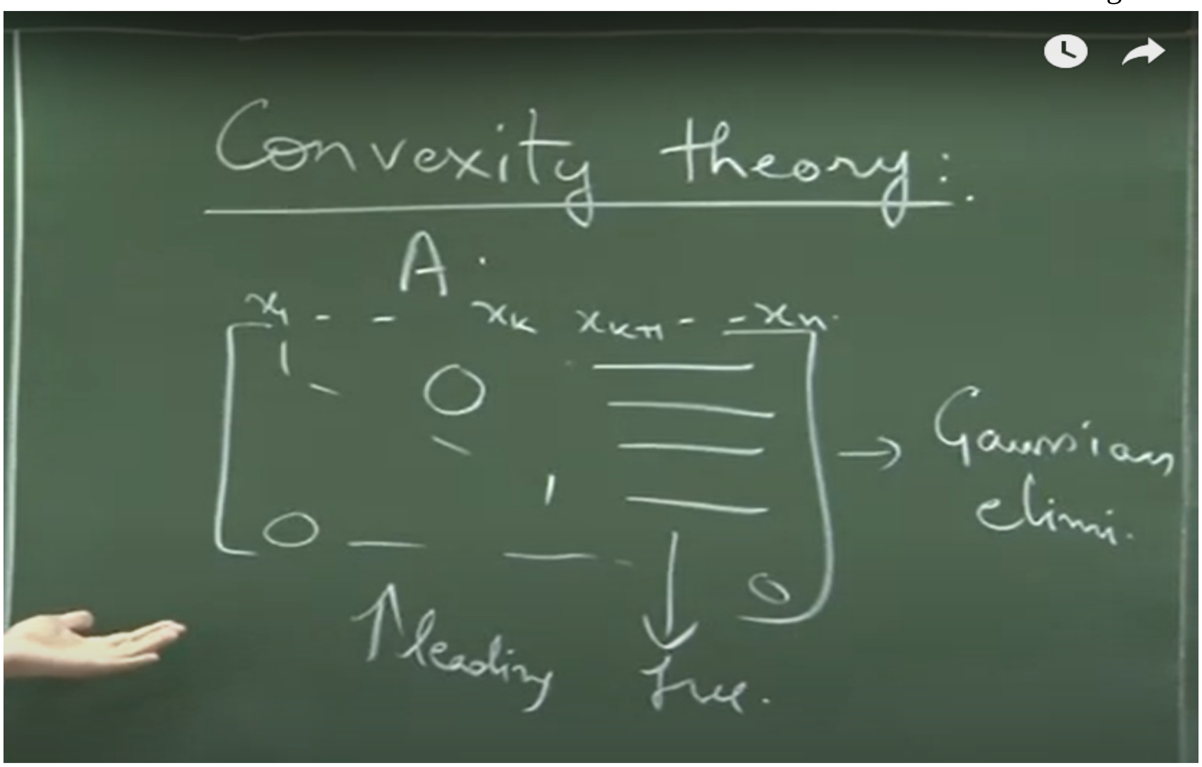
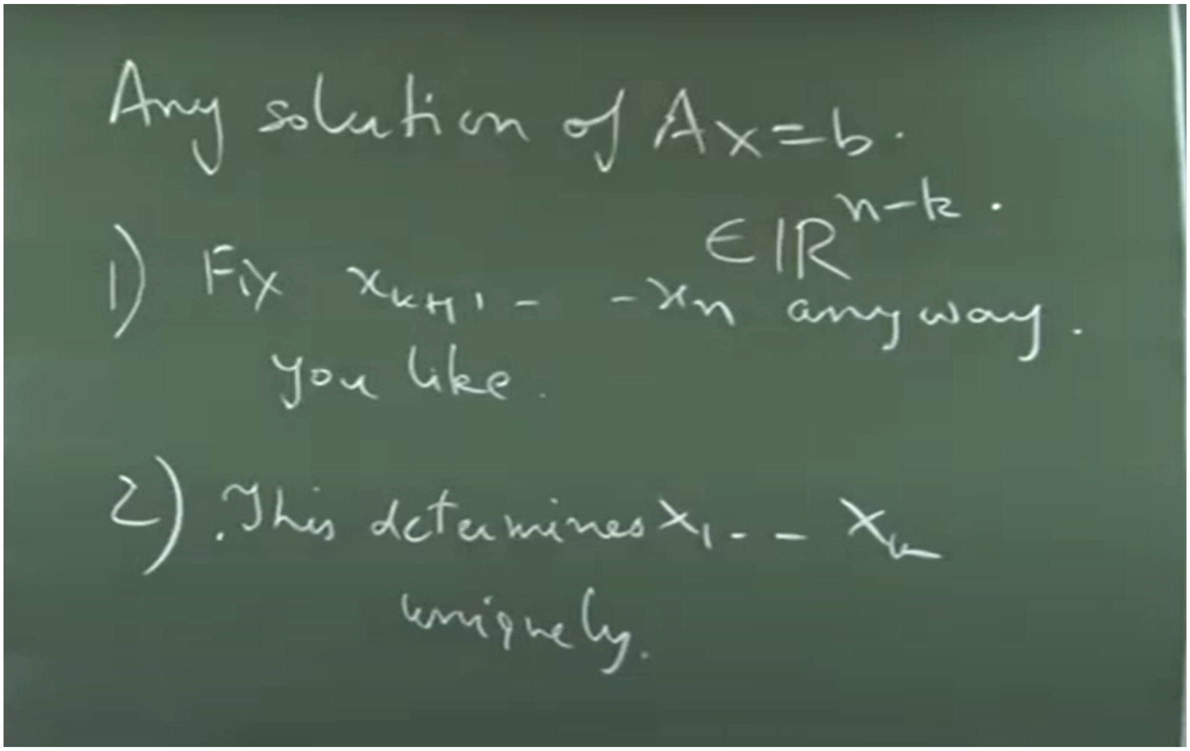
## Lecture – 11
## Introduction to Convexity

Welcome to the course on linear programming. We have seen linear programs, we have seen some linear algebra specifically how the solutions of Ax equal to b look. Today we will see that that is not good enough and we have to introduce the concept of convexity theory. But before I start what happened here right. So, the idea was that the solution set of this was easy to describe right. If you remember I take the matrix A and I can do Gaussian elimination right.

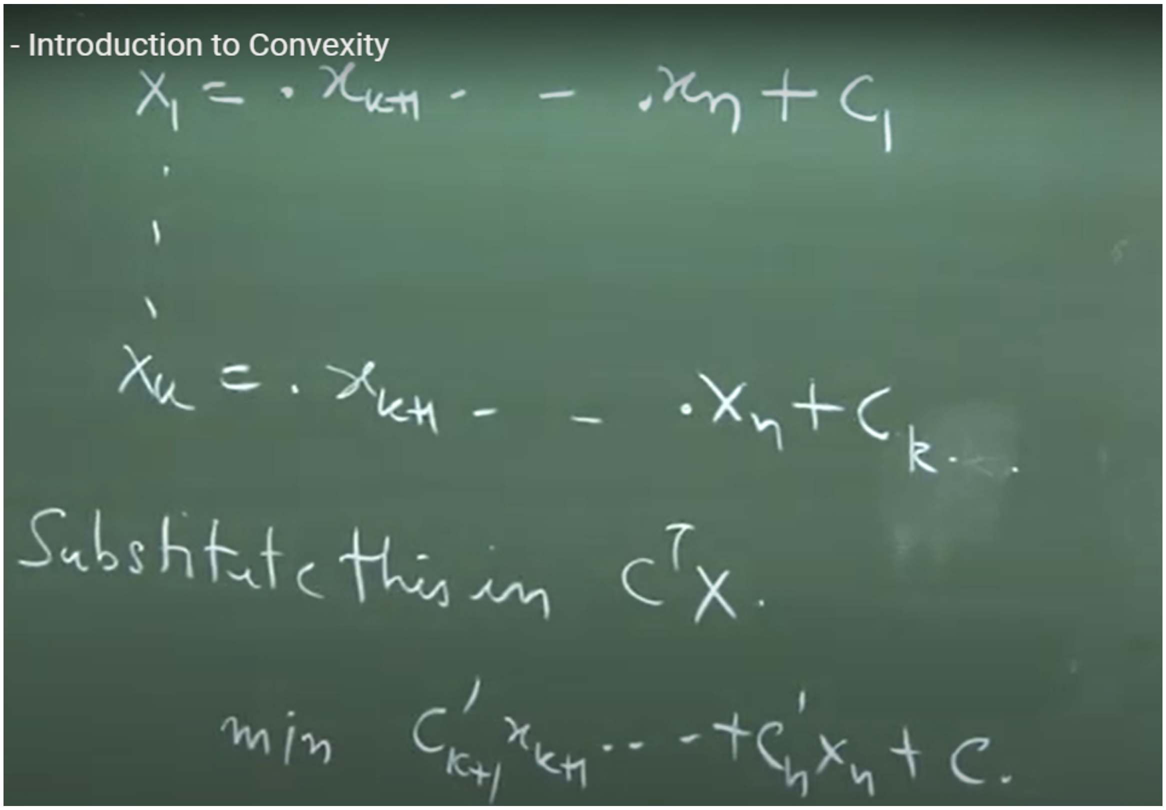



So, if I do Gaussian elimination it turns out I will have this kind of a structure where I do not know these this will be some entries. But what do I know these are my leading variables and these are my free variables right. So, let us just call the leading variables X 1 to X k and the free variables X k plus 1 to X n right. So, what do we know if I want to create any solution of A X equal to b then how can I do it? So, first fix up anything any value of free variables right.



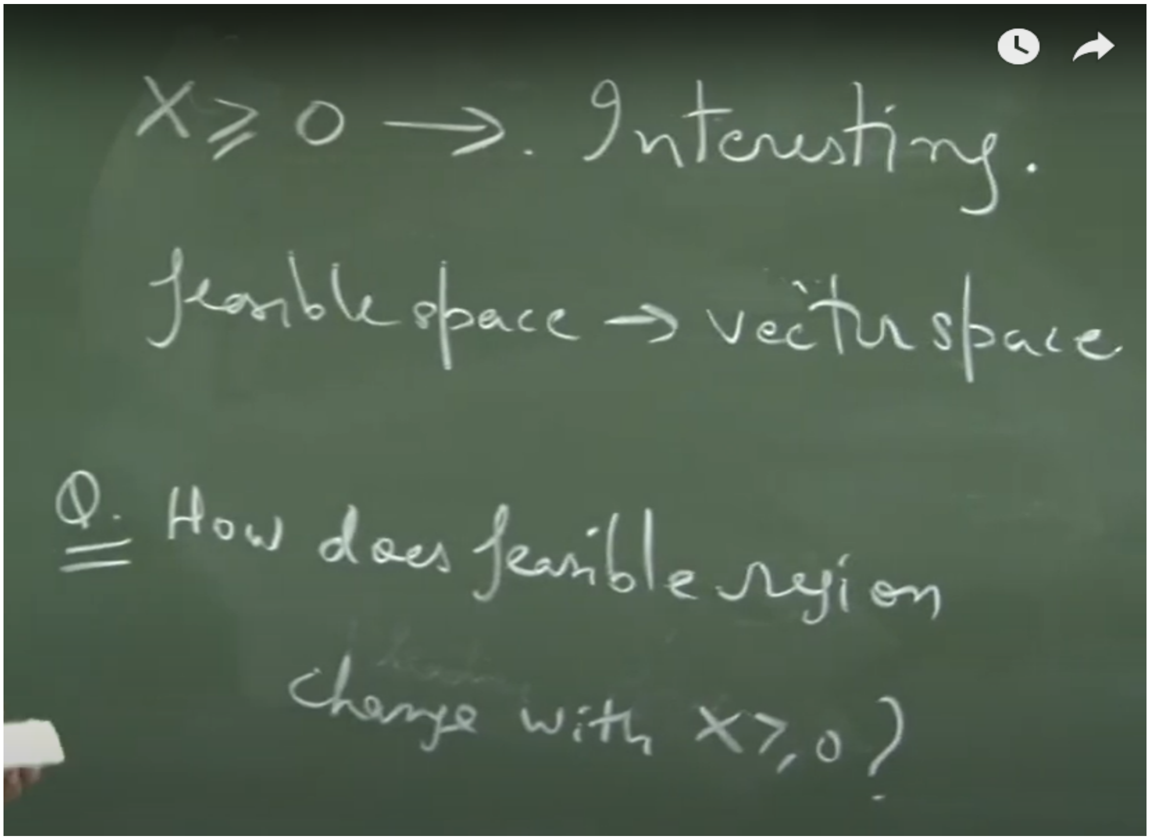

So, this can be obtained by two steps first and then once you fix up these values this will determine the value of X 1 to X k uniquely and this is given by these coefficients right. Sounds good? This means there is no restriction on X k plus 1 to X n right. This is anything in R to the n minus k. I can fix these values as whatever I like and then I am by arranging X 1

to X k in a certain way determined by these coefficients I am sure to get a solution of A X equal to b right. Other way to say it I would is write X 1 to X k as linear combination of X k plus 1 plus some constant C 1 some coefficient here some coefficient here similarly sorry C k.



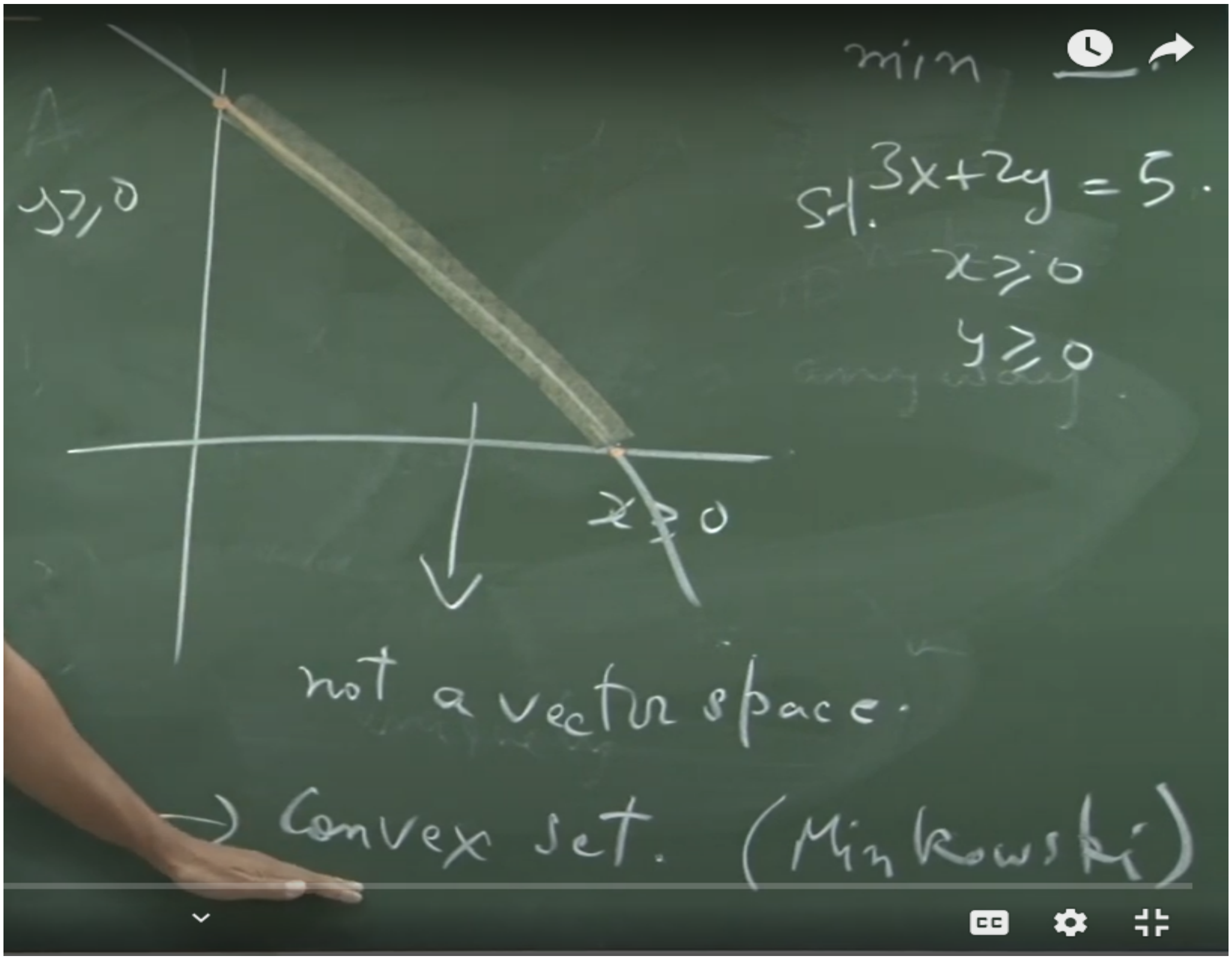

But once I know this I can substitute this in C transpose X and then I have got a function of the kind was it minimize? These coefficients will change plus some constant C and there are no constraints remaining right. It is just optimizing this linear function and now my life is easy. This is going to be minus infinity if any of these are non zero because then I can make sure that going in that direction I can make it negative. There is a trivial case you have to remember that if all of these are zero then that means for any solution the objective function is C. This is all.

This basically means this is very simple to solve because even in a very bad case you can do Gaussian elimination in order of N cube. Order of N cube is a pretty good bound of Gaussian elimination probably do better, but so that means we can solve this problem very easily correct great. So, this direction did not help. It shows us things only become interesting when I have this constraint right. Otherwise there are not many applications also it is this constraint which is making this linear program interesting.



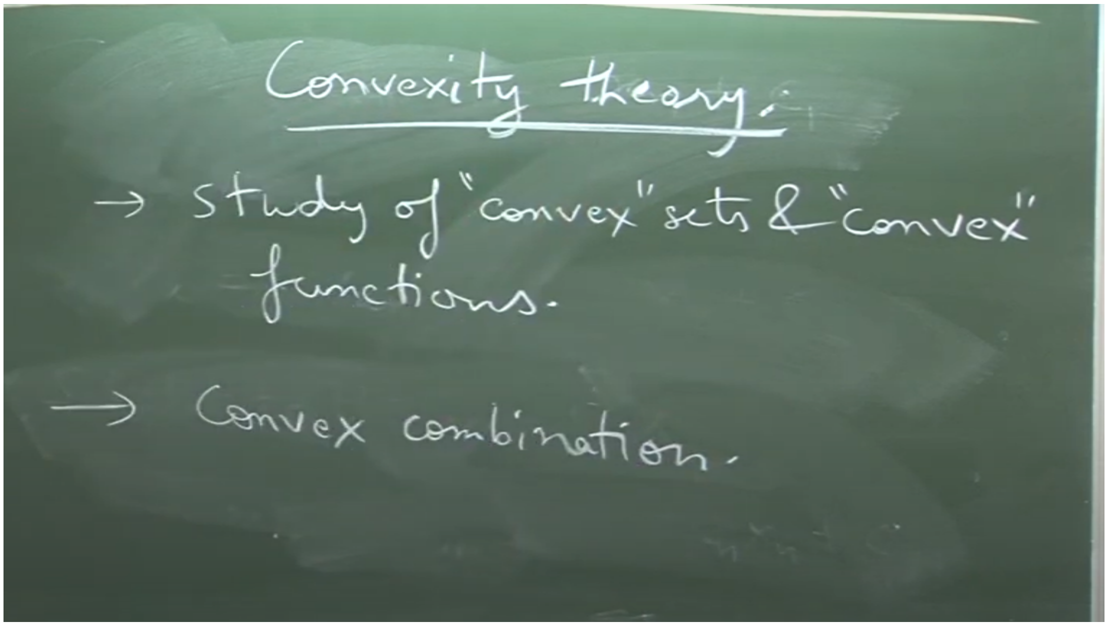

And you remember what you are trying to do? You are trying to learn linear programming and now we tried a simplification that is too simple. Now you want to see oh life is not that simple, but can you still get some more information right. So, when our feasible space was a vector space right A X equal to b the solutions of A X equal to b is a some kind of a fine space that is easy that is not very difficult, but that is not going to be the case. So, the question is there is still some structure of feasible space which allows us to solve it easily right. So, the question in other words is how feasible region changes with X greater than equal to constraint.

And when you cannot solve a problem the best thing to do is now you are not a linear algebra right we have come out of linear algebra. So, Gaussian elimination is not the answer to everything. Yes, so that we tried and generally it is good to look at small sub cases right. You want to simplify the problem there are many ways to simplify a problem, remove an assumption or now in this case since you want to understand the structure let us at least see what happens in 2D or 3D. So, now let us say in 2D that is the simplest I have program.

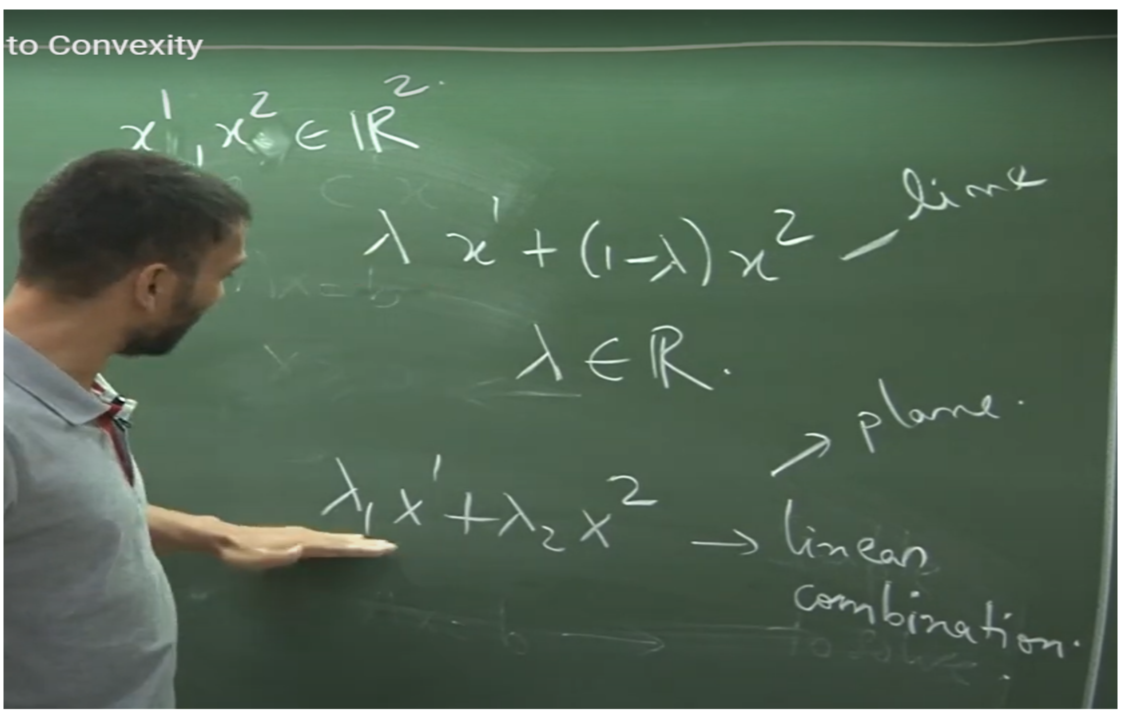



So, minimize something I do not care I am interested in just the feasible region. Let us say this is the problem right. If this is the case and there is some line like this right. What is my feasible region? No, just the line segment right. My feasible region is just this right and now this is not a vector space, but then what is it? It is still nice right still a line segment it is not completely unstructured or something.

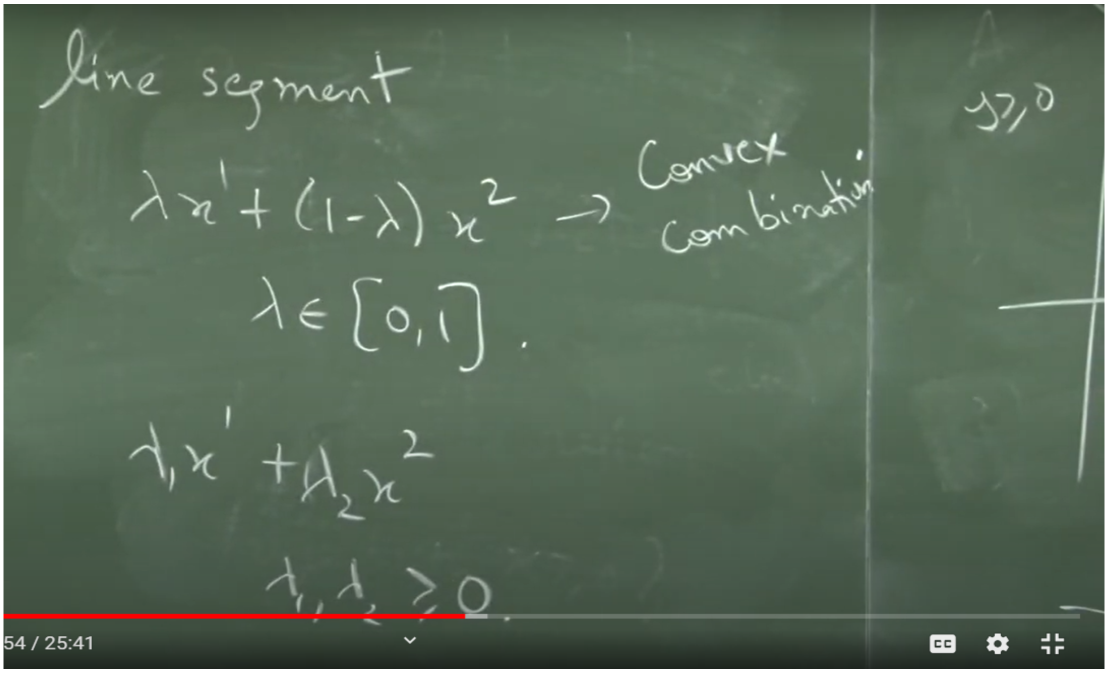
So, what do we know about line segment? It is actually a convex set and once we observe this we remember Minkowski and say in 1900 he has done so much about convex sets. Let us learn from him, let us see what has been done and see if we can apply those principles in our problem and this is what we are going to do in next week. So, you need to learn convexity you need to learn about convective theory and immediately we will see that an algorithm pops out because of the structure already known of convex sets. Sounds good, that is the plan. Our target becomes learning convexity theory.




How many people have studied convexity theory before some idea? Ok So, convexity theory is study of convex sets and convex functions. If you read this what you realize is most important thing is this word convex right and the first thing we will start with is convex combination and again the idea is to describe this line segment instead of this line right.

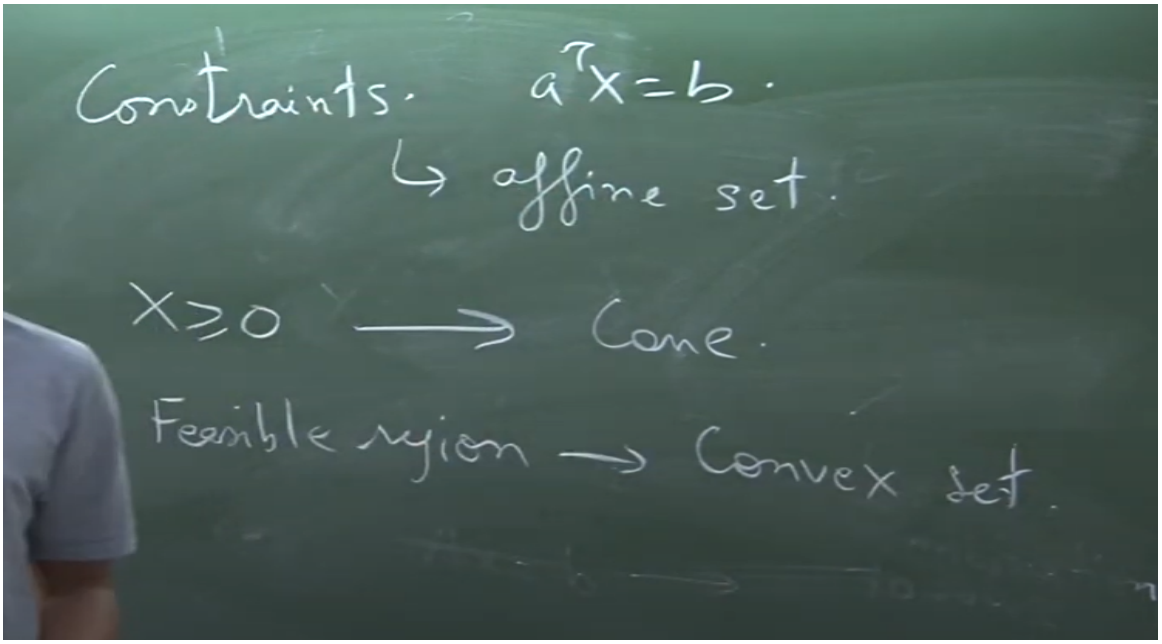



So, in a plane how do you write if you are given vectors let us say x 1 x 2 or let me these are two vectors in r square what is the line passing through it lambda where lambda is that ok with everyone correct. What is this? This is a linear combination right and then that means in two dimension this is what is this? What is the shape? Plane mostly a plane right




except some trivial cases when these are linearly dependent right good. So, first thing was from linear combination if I wanted to come from the entire plane if I want to restrict my attention to a line I do this.

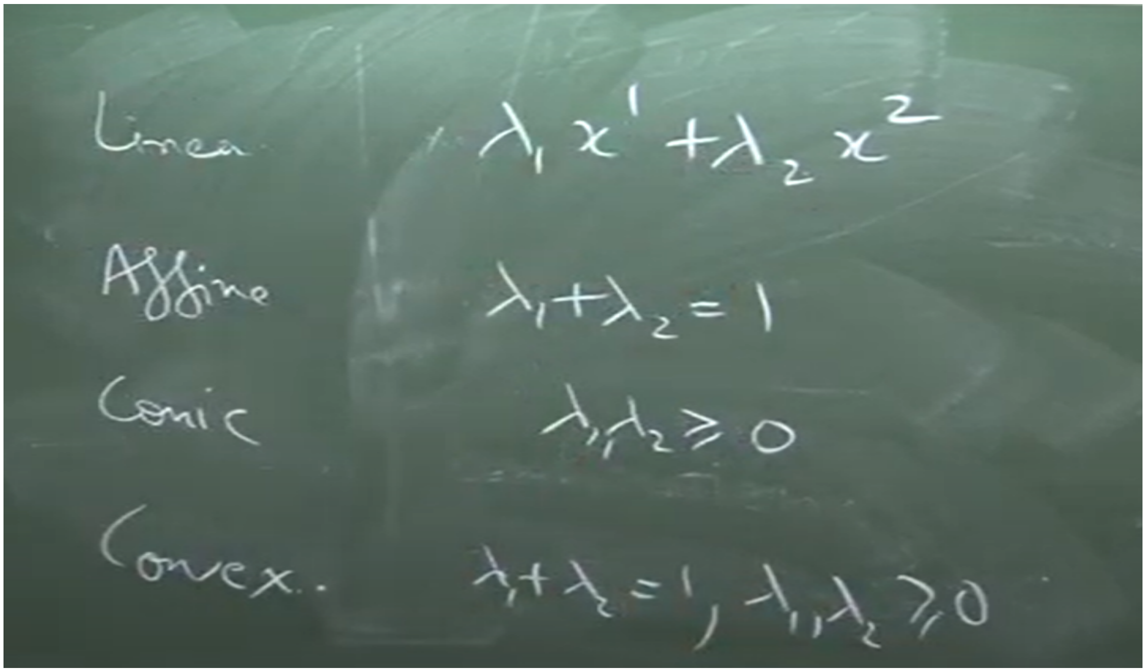
So, this is a line this is a plane and how do I get a line segment anyone except Dev? Thank you right. So, it is lambda x 1 plus 1 minus lambda x square such that lambda is in 0 right and now when you look at all this there should be one more combination which should come to your mind what is that? This is called convex combination this is going to be important. What is the remaining combination if you see these cases do not think of geometrically look at these three. So, now I started with any general lambda 1 lambda 2 I put one kind of constraint I put other kind of constraint what kind of constraint is remaining? So, this was a plane this is the constraint that summations are up to 1 this is the constraint that both are positive if you mix these you get this. This is called conic combination ok and how will the conic combination look? Right suppose this is a point x 1 this is the point x 2 the conic combination what will be that set? What will be that set? You can tell me definitely this point is going to be there this entire line is going to be there you can convince yourself of that you can you can get this.




And now because I talked about convexity theory convex sets and everything it seems like this is the most important thing, but I assure you all four are going to be important for us all four kind of combinations are going to be useful for our study of linear programs. The reason is pretty simple if you look at constraints something like A transpose x equal to b this is going to be an affine set. If you look at this is a cone. So, all these shapes all these combinations are going to arrive when you when we are going to study linear programming, make sense. And this is what we are going to study these combinations that is the plan for today. Just to make sure that you understand what is going on which statement is correct.

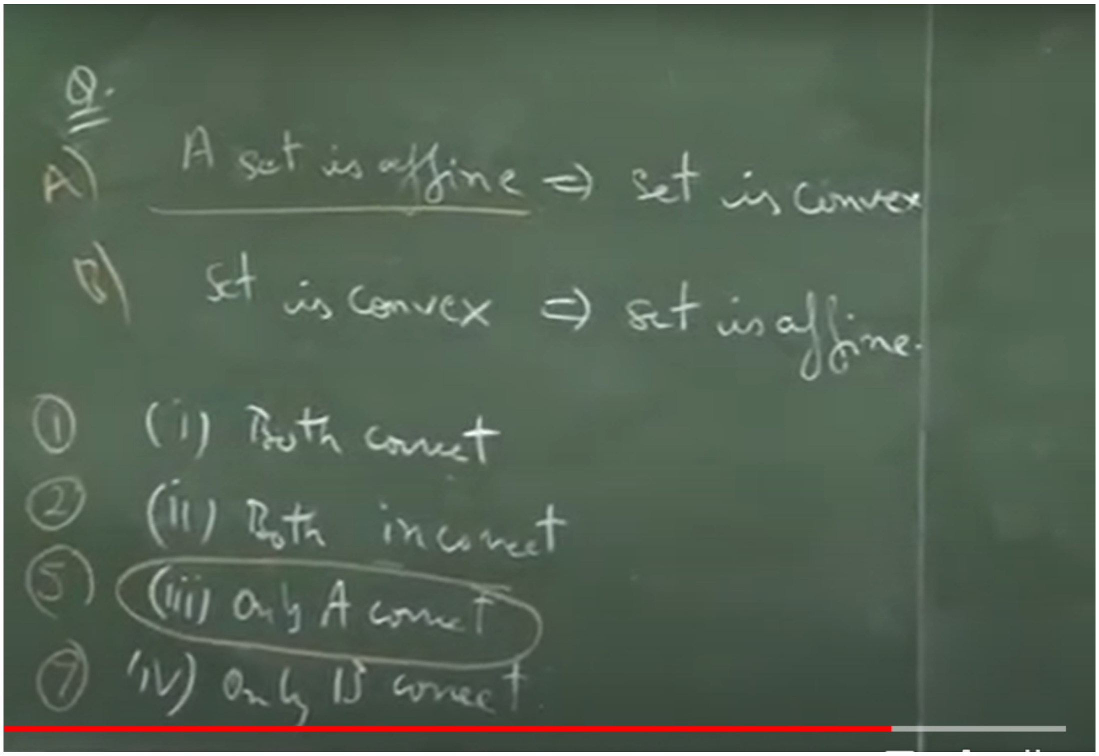
discuss let us take a vote four options none of them are correct, both of them are correct, first one is correct, second one is correct that is why India does not have a good political system. Because nobody votes like out of twelve people only three people voted what the hell think about it I will give you some time, but everyone has to vote. bring out your pen and paper. Good very good. So, the first question to ask is we have seen an example of an affine set I told you that line segment is convex a line is affine. So, at this point you can only answer this in terms of examples, but an affine set is something which has all its affine combinations.

A convex set is something which has all its convex combinations. Oh I just told you I had written down these ones let me just write it. So, the way to remember it is linear combination is affine the constraint is lambda 1 plus lambda 2 equal to 1. Conic is lambda 1 lambda 2 greater than equal to 0, convex is which basically means it lies between 0 and 1.ok The A transpose X equal to B no the space you get is not a plane you get a line right for example, when it was 3 X plus 2 Y equal to 5 in the two dimension did you get a line or did you get a plane you got a line.



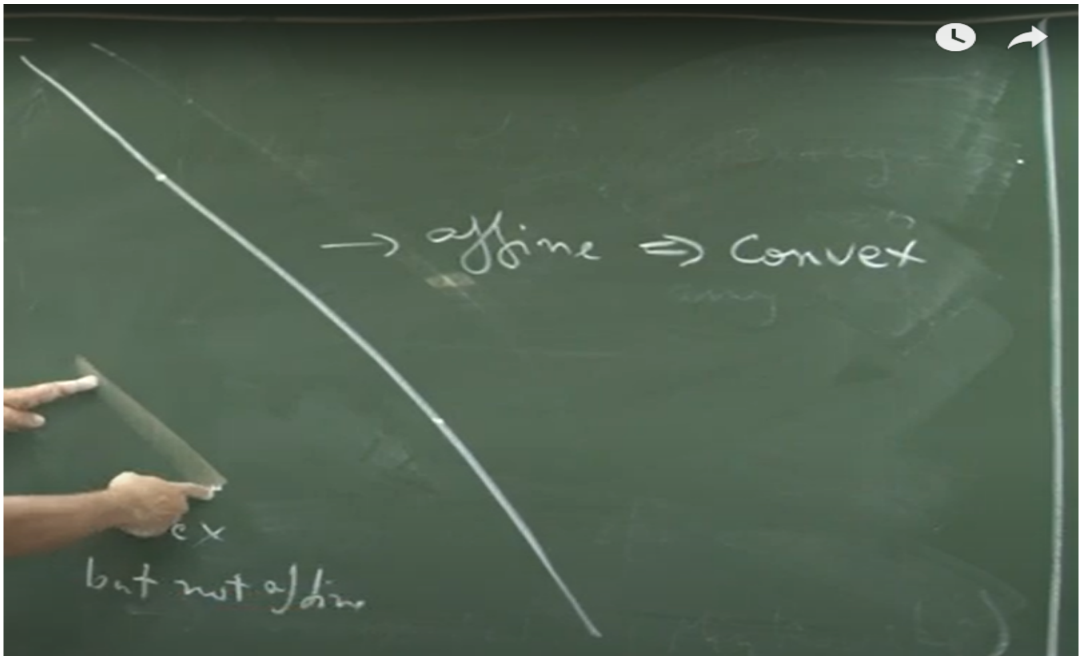

So, it is only an affine combination do not get confused by when I say A X equal to B are you thinking of it in terms of and you are saying that is why this is a linear combination no here what you are saying is your any X 1 to X n with satisfy this equation are going to be in your feasible set. It is not saying that you have X 1 X 2 and X n are vectors here and you are taking any possible linear combination here that is not what you are saying right. So, do not get confused and this is actually very important it is a very nice question because you will see this duality in terms of it is oh a linear combination affine combination or I can describe it in terms of equations. Both sides are very important for a geometric picture oh it is a linear combination or an affine combination, but if I want to study it then I also want to see the equivalent way in which it can written it can written down as an as a bunch of equations and this we will see. Ok, so, this is very nice this confusion is natural to arise, but we should remember.

So, all these sets you will describe in almost two ways. So, now you know an affine set is something which contains every affine combination a convex set is something which contains all the convex combinations. Now, let us vote. How many for one? One vote both incorrect people are guessing, how many for three there is influence you are not allowed to either you would not vote or you will vote twice something wrong with the youth in this country one two three four five. You voted already I know and you said that is obvious no.




No. This is the correct answer. Given that out of fifteen people only five got it right  it is definitely not obvious right. So, what is happening here a line segment remember  a line segment a line was affine is it convex why not take any two points take any anything here it is convex combination is inside that line yes no.  How does convex. Convex combination look it is the line segment  between the two points right that we describe remember I said what is the line segment is  the convex combination of those two points right.

So, if you take a line you take any  two points the line segment between them is part of this. So, it also contains convex  combination. So, affine implies convex and a line segment this is convex, but not affine  why because the affine combination of this point these two points will be the entire  line. And you can make an argument you know the affine combination condition is stricter  convex combination is weaker and always get confused. You can do that and you can figure out and say that oh the condition for this is weaker than the condition for this sorry the other way this is the stronger condition this is the weaker condition.




So, the weaker condition is true this is fine right, but if you get confused line segment line is your friend this weaker and stronger is always confusing for you. But yeah if you are very very precise and if you are not drunk then even the weaker and stronger will make it work and for the record I am not drunk now good. So, affine combination all those are making sense now.