Linear Programming and its Applications to Computer Science

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Lecture – 01
Linear programming , An example

Hello, welcome to the course on Linear Programming and its applications to computer science. This is the first lecture. I am your instructor Rajat Mittal. The TA for this course is Tufan Mahapatra. Let us talk about some basic things for this course to start with. First thing is the way to communication in this course will be through the NPTEL portal.

Please remember during this course you might have multiple doubts. All these doubts might be the doubts for other students also. So feel free to ask your questions on NPTEL portal. This way in one go we can answer questions of many of you.

So please ask questions on NPTEL portal. Second thing, we will be giving you weekly assignments. I think they are very important. Please make sure that you do it on time. This will keep a check on your progress in the course.

If you are not able to solve these weekly assignments, please ask questions on NPTEL portal. Let us know so that we can help you out. There will be a final exam in April. People who have registered for this exam will have to give this exam and we can talk about it more later, but these are the ways in which you will be evaluated. And for any communication, the communication would be done through the NPTEL portal.

So with these small procedural things lined out, let us get started. Today is the first lecture. I want to keep it light and I want to talk about what is linear programming or give an introduction to the primary personality in this course. What it means is, we are going to answer three things. What, why and how.

Any time we see a new thing, these are the three questions generally which come to our mind what, why and how. So we want to answer, what is linear programming? Why should we learn it? And finally, how are we going to learn it? The plan for today is to answer these three questions. To start with the first question, the formal definition will say that this is a class of optimization problems where constraints and objective function is linear. This would be the formal definition of linear programming, but this is pretty loaded.

There are lot of words here which you would not understand. So let us not worry about it. I would say, let us start with an example. The good thing is that you might have already seen this example before in your high school. We will just see how it relates to the entire theory of linear programming.

So the example I am going to take is from this category of problems called resource allocation problems. The particular example I am taking is taken from the course notes by Robert Sedgwick from Princeton University. Let us talk about this example. Think of yourself as an owner of a brewery. So, your brewery can make two kinds of beers.

It can make either a dark beer or a light beer. It turns out these two kinds of beers need different kind of resources to be made. Ok, so generally, beer let us say needs three components corn, hop and barley. To make one liter of dark beer, I need 5 kg of corn, 4 grams of hop and 35 kgs of barley. On the other side, if I want to make light beer, it requires 15 kg of corn, 4 kg of hop and 20 kg of barley, ok.

So I can write the units here, but they are not going to be very useful. I will keep them in the same unit everywhere and it turns out that in terms of profit, a dark beer gives you 15 rupees for per liter and the light beer gives you 20 rupees. So this is clear. We can make two kind of beers, dark or light. The dark beer requires corn, hop, barley in this much amount, light beer requires them in a different amount and the profit from these two are different.

Also we have our inventory. The inventory has 480 kgs of corn, 160 gram of hop and 90 kg of barley. Obviously, as a businessman, you want to maximize your profit. Right, so, The question becomes if this is how much the amount of corn, hop, barley you need for dark and light beer and this is the profit, how to utilize your inventory in the most profitable way? This is the question. I will let you stare at it for a while, but you can realize that this is a pretty common problem.

This could arise in almost any industry where you have some resources, there are some constraints over how much resources you can use and you want to go into the most profitable situation. Right, how would we want to solve it? Now let us see. One of the situation could be that we say that let us try and see if we just make dark beer. Right how much profit can we have if we just focus on making dark beer? It turns out that if you are going to make the dark beer, barley is going to be the bottleneck. That means barley will run out the first.

It turns out for dark beer, we can make only 34 liters of it because after making 34 liters, barley will run out and this will amount to the profit of 34 into 15 which is 510 rupees. Right, this is if you are only making dark beer. If you were making light beer, then it will turn out that we are going to finish corn the first. So if we make the light beer, we can make only 32 liters of it because corn will be the bottleneck. After 32 kgs of 32 liters of light beer, my corn will get over, my profit will be 32 into 20, 640 rupees, ok.

So if we make just dark beer, we had a profit of 510 rupees. If we make just light beer, we make a profit of 640 rupees and you might feel that making light beer is the best. But what about the combination? Can we do better? Can it happen that you know in this case corn is running out and making light beer, in making a dark beer barley is running out, can we make a combination so that we can make more profit? So to actually solve the question, we have to ask a harder question. Is there a combination which gives more profit? And it indeed does, it is actually very easy to see. Suppose in the light beer, I am only going to make 31 liters of light beer.

That will free up 15 kg of corn, 1 liter less. So I will have 15 kg of corn free. This will allow me to make 3 more liters of dark beer. So then you will actually see that I can also make, you can check yourself, but you can make 31 liters of light beer and 3 liters of dark beer. And this will give you much more profit than 640 rupees.

So this will give you 620 plus 45 rupees, this will give you 665 rupees of profit. So it is indeed true that there are combinations of these which can give me more profit. But is this the best? Probably we can make 20 liters of light beer and more dark beer and get more profit. Where to stop? Right, so to answer this question, we will go into the mathematics of it. And this is something which you have seen in even high school.

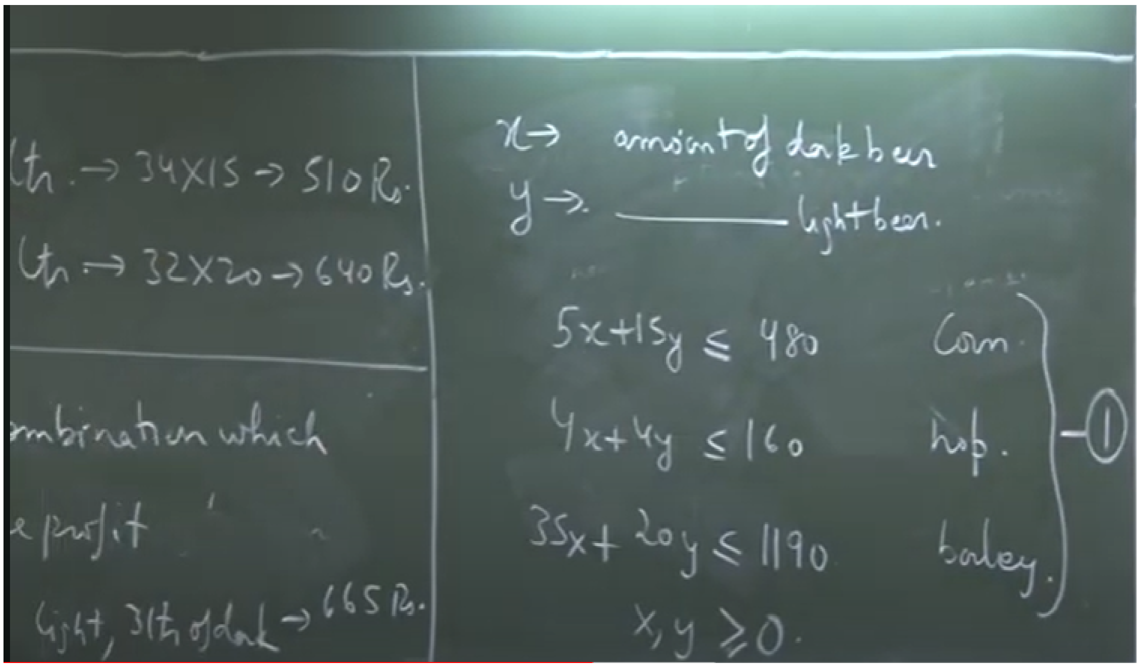
Let us write it as a system of equations. So let us say X and Y are two variables which denote how much dark and light beer we want to make. So let us say this is the amount of dark beer in liter and this is the amount of light beer. Now these three numbers will give a constraint on how much X and Y we can build. Obviously in terms of corn, 5X plus 15Y should be less than 480.

This is the constraint imposed by corn. Because we have 480 kg of corn, 5 times number of liter of dark beer times 5 plus number of light beer times 15, this is the amount of corn we will spend. And the amount of corn spent should be less than 480 kgs. Similarly 4X plus 4Y should be less than 160 g.

This is the amount of hop. And for the last one, Barley will say 35X plus 20Y should be less than 1190. And then the amount of profit is 15X plus 20Y. Great. Now we have been able to formulate our problem in terms of variables as a mathematical equation.

What is our problem? We want to make sure that our X and Y satisfy these conditions.

Look at all X and Y which satisfy these conditions such that this quantity is maximum. So we want to maximize this quantity. Though we have forgotten a small thing. If you think about it, we have another constraint. The simple constraint says that X and Y both should be positive numbers.

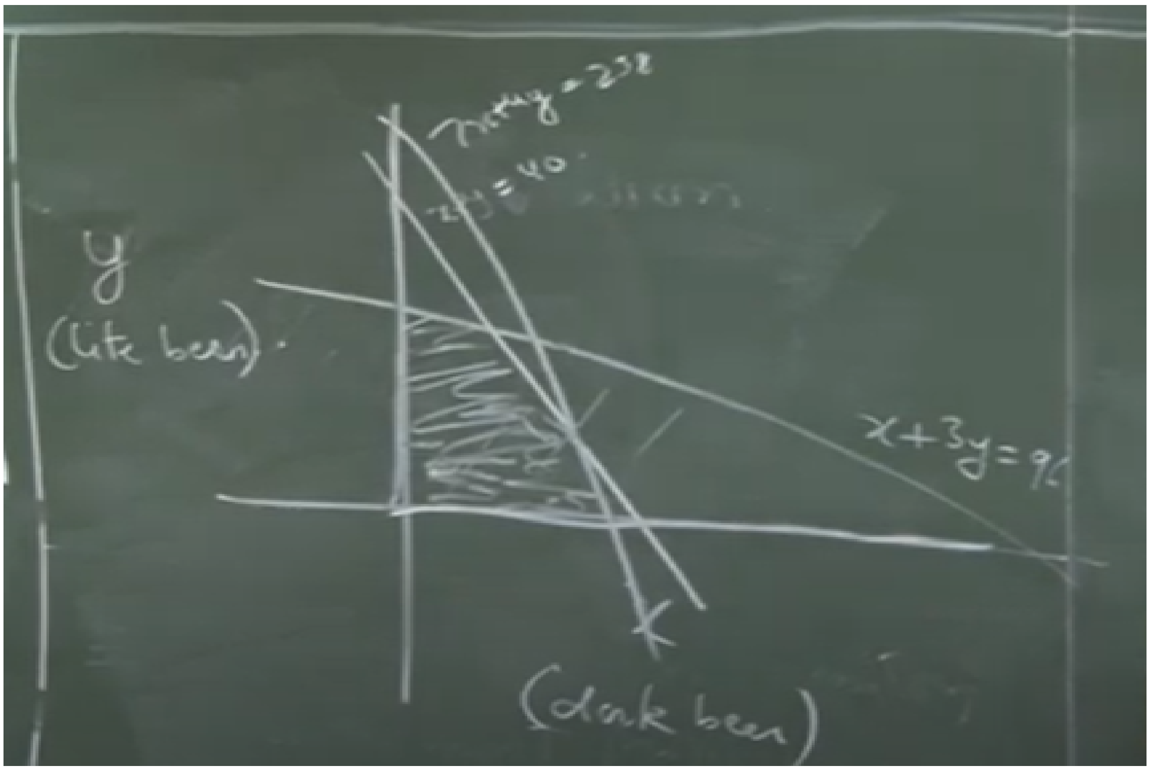


Because X is the amount of dark beer, I cannot make minus 2 liters of dark beer. So then I should say X and Y greater than equal to 0. Let us say this is my set of constraints. Now the problem is quite clear mathematically. We want to maximize 15X plus 20Y such that it satisfies these 4 equations.

Look at all possible X and Y in this world which satisfy these equations and we ask which of that X and Y actually maximizes this profit. How should we solve this? Given your background in these equations, you remember that we can plot these equations on a plane. So let us plot these equations on a plane and see if we can get an answer. In this 2 dimensional plane, let us say X is should I say Y a nd X, remember this is the amount of dark beer we are making, this is the amount of light beer we are making. And actually these meanings are not really important right now.

The only thing which is important to us is how to maximize 15X plus 20Y such that these equations are satisfied. Their internal meaning of what is X, what is Y, why is this equation there, we can forget about it. So if we concentrate on solving these equations, then let us see how does this equation look like on this diagram. We have done this before; we have seen how to draw this equation before. So the first equation can be simplified to be X plus 3Y less than equal to 96.

I first draw the line X plus 3Y equal to 96. If I write this line X plus 3Y equal to 96, you observe that everything in this part, everything below this line satisfies our first constraint. So that means we are looking at all X and Y's which lie below this line. That is great, we can write other equations also. Again this is X plus Y less than 40 which looks like this.



So this is

$x+y <= 40$ and the last equation becomes

$7x +4y=38$.

I can plot all these lines and I know that I am interested in X and Y's which fall below these lines. But obviously X and Y are positive, so they should lie in the what we call as the first quadrant that means in this space. So everything in this quadrant which is lying below all these lines that is easy to identify, that is this area. So at least geometrically our problem becomes slightly simpler.

Look at all X, Y in the shaded region find point where 20 Y is maximum. You see that we had a word problem, we first converted it into a set of equations. Once we convert it into a set of equations, we can draw it and now we have a geometric picture. What we want to do is look at all X and Y's in this region and ask where is this quantity maximum. This is interesting, how should we find this out? There are few key observations which will really help us out.

So please pay attention. Look at 15X plus 20Y, in itself the value of the profit gives me an equation. So I can think of what are the points where 15x +20y =0. What are the points where 15x +20y =5, is equal to 100 so on and so forth. It is easy to see 15X plus 20Y equal to 0 is a line like this, because it has to pass through the origin with a certain slope. My diagrams are not really accurate, but they are just to give an idea of how we will proceed towards the solution.

And now you will notice that as I move this line, as I draw a parallel line to it, my value of 15X plus 20Y increases. So this was the line at the origin, if I move it like this, its value increases. It will probably be 15x +20y =90 or something. If on the contrary, if you move on this side, keeping the line parallel, the line the 15X plus 20Y will become negative as to find mention. So great, now we are one more step ahead in our solution.

We had this region where we want to find the solution. This line when it moves up, it increases the profit. This line when it moves down, it decreases the profit. What do you want to do? How far can we move this line such that it still intersects our region? That will be the answer to our problem.

So notice that this is a continuous region. So what we will do is, we will start moving this line up until it keeps intersecting with our feasible region. When it stops intersecting or when is the last time it has an intersection with our feasible region, that is the maximum profit. So let me just write this observation because this is kind of an important observation. So, move line 15X plus 20Y, first quadrant up till it has at least one point of intersection. Good, the thing now to notice and which is again a very important observation is that at the point when this line leaves this area, it will be intersecting with the one of the points which we call the vertex of this area.

When you move the line, it cannot leave where it is just intersecting with any other internal point. It will only leave with the, when it is leaving a vertex. So that means, we notice the line leaves the feasible region at a vertex. And my claim is we are done now.

We have solved our problem. How? Previously, we wanted to find out among all such points, which one is the, which is the point which maximizes 15X plus 20Y. We had infinite points here and want to figure out where is 15X plus 20Y maximum. But by noticing the geometry of the solution, we realized that the maximum has to lie at one of these 5 points. And this observation is critical not just for this problem, but for solving such problems in general. The best point where the profit is maximized is one of the vertices of the shaded region.

Once you notice this, your problem is simple. You have equation of these lines. You can find all these 5 points. Just calculate the profit at all these 5 points and compare. Now it is very easy.

They are intersection of our constraints. And they are easy to find. Compare their profit. And actually, you can ignore this point. You would know that the maximum would not lie here.

This is, I think, I can leave it as an exercise for you. If you do this exercise, you will find out that X equal to 12 and Y is equal to 28 is going to be one of the vertices. And in this case, your profit will be much higher than 5, 10 or 6, 40. It is going to be 740 rupees. Remember if you are only making dark beer, you are making 510 rupees of profit.

If you are making only light beer, you are making 640 rupees profit. But here we get much more profit. And again this was possible. We were able to find this because we had this nice geometry. We had this nice observation that the best solution will be on one of the vertices and you can calculate this vertex. This is generally called the graphical method to solve linear programming.

Before I erase this figure, let me ask you a question. Is the best solution always unique? Here we can see that there is only one vertex where the leaving line will touch the feasible region. So it will be like this and there is only one point of intersection when my line leaves the feasible region. Is that always true? So as Tufan said this is not necessary. It might happen that the feasible region was like this and my leaving line would have looked like this.

In that case every point on this would have been our best solution. Any point on this line would have give us the maximum profit. In this case we were lucky there was only a unique solution, but there might be a geometry where you might have more than one solution. But let us not worry about it. Let us just see what does it mean for this course.

We had a resource allocation problem. We solved by a graphical method. This was your first example of a linear program. If my knowledge is accurate about the current curriculum, many of you have seen this in your high school. So we have a linear program. We have a way to solve it. Why do we need a course on it? All of you or most of you have solved linear programs using graphical method before.

So then great we are done. What else to teach in linear program? Let me point it out. The equations we solved were very special. We had only two variables and we had very few equations. Most of the time when you encounter a resource allocation problem in real life, you are going to have many variables and many equations.

When I say many variables, I mean thousands of variables. Then the picture I drew just now on the board, you have to draw it in thousand dimension plane. I do not think any of us can do that. So the real world problems are not as simple as our brewery problem. There you will have thousands of variables, more than thousands of equations and then the graphical method is not sufficient to solve anything. So this course is going to tell you how to solve linear programs even if you have lot of variables and lot of equations and not just that what are the real life applications of solving such things. Thank you.