Probability for Computer Science
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Lecture-08
Fallacies. Random Variables

Next example is what I can call Monty Hall fallacy. So, the first example you saw was this.

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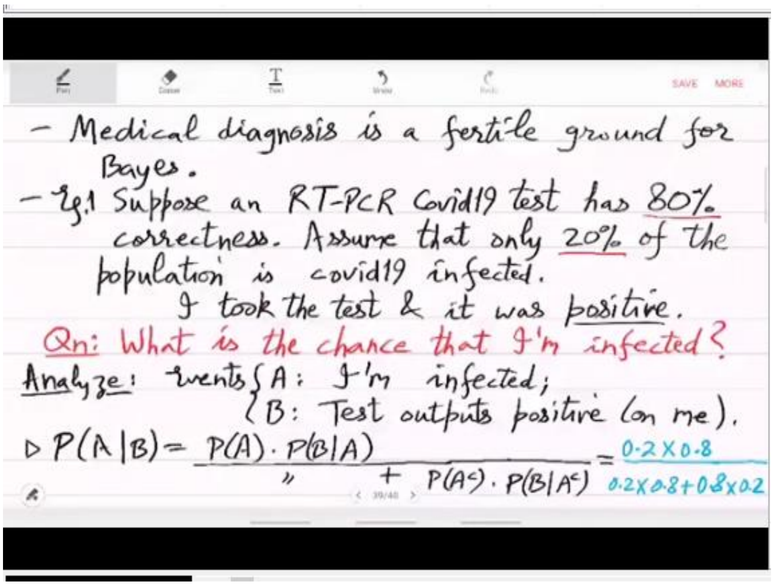

Let us call it example 1.

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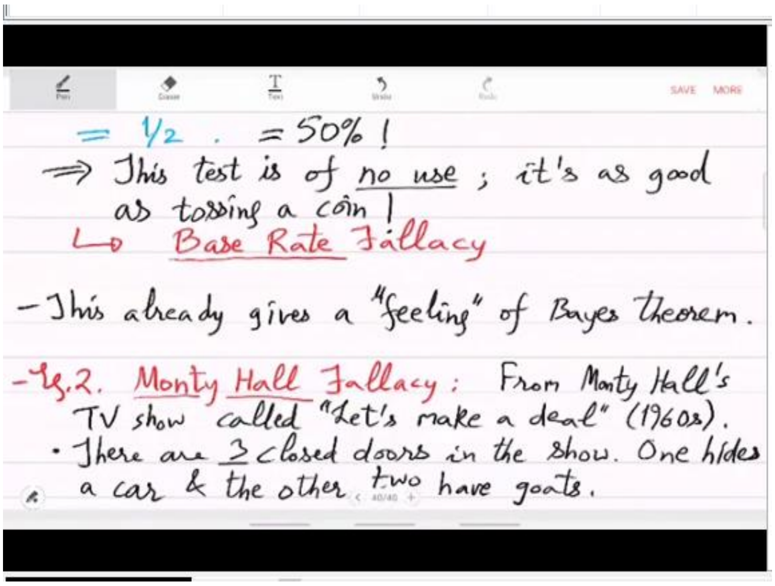

Next is example 2. So, you can call this Monty Hall problem or Monty Hall fallacy, you will see why it is a fallacy. So it is a very interesting idea for a game show. So, from Monty Halls

TV show called let us make a deal which ran successfully in 1960s. So, in one of these episodes, the game was that Monty Hall being the host; he has on the stage, there are 3 doors, one hides a car, the other 2 hide goods.

And the player has to pick one of the doors and then there is a protocol. So, let us describe what the player has to do. So, there are 3 closed doors in the show, so one hide a car and the other 2 have goats.

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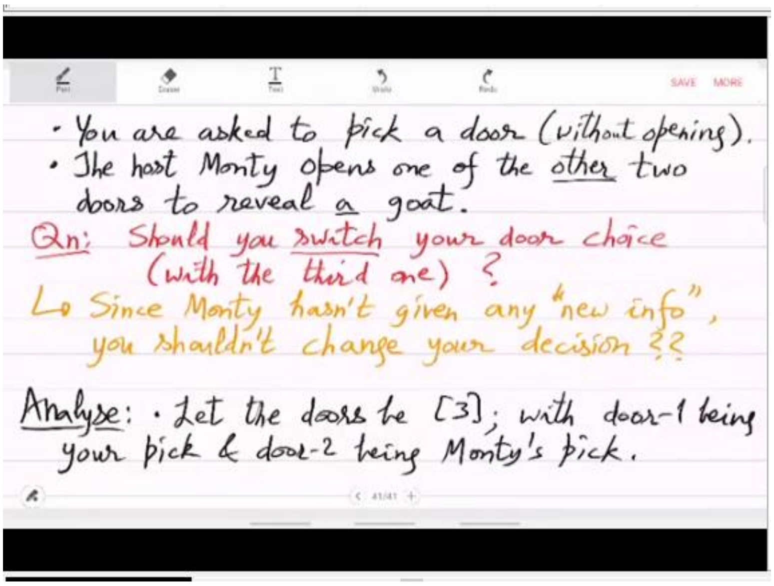

So, you were asked to pick a door without opening it. So, the host Monty who confuse you what Monty does is he opens. So, now there are 2 other doors, one you have chosen to remain. So, clearly out of those 2 one, at least one has a goat and Monty knows it. So, he opens that door and shows the goat. So, the host Monty opens one of the 2 doors to reveal a goat.

So, now you have 1 door picked and there is another door that is left unpicked and unopened. The question is looking at the goat, should you swap your choice? Should you change your choice or not? That is the question. So, should you switch your door choice with the third one. Should you stick to your ground or should you flip? That is the tough question for you. Now, it may seem that this new information that Monty has shown you presence of a goat was known to everybody.

So what is the point of switching? You can as well stay your ground. And that is where the fallacy is which Bayes' theorem will resolve mathematically? Since Monty has not given any new info you should not change your choice, but if you think this then you are very wrong. So, let us do the analysis. So, let the doors be labeled naturally 1, 2, 3 with door 1 being your pick and door 2 being Monty's pick.

So, this is post facto labeling of doors, the one you pick is 1, Monty picked second, third is unpicked and closed. So, you can assume that since you had picked your door randomly without any information, so the probability of the car being behind the door 1 is one third. And before Monty picked door 2 actually probability of car being behind any of these was one third. So, in fact, let me first define the events, let me first define the events.

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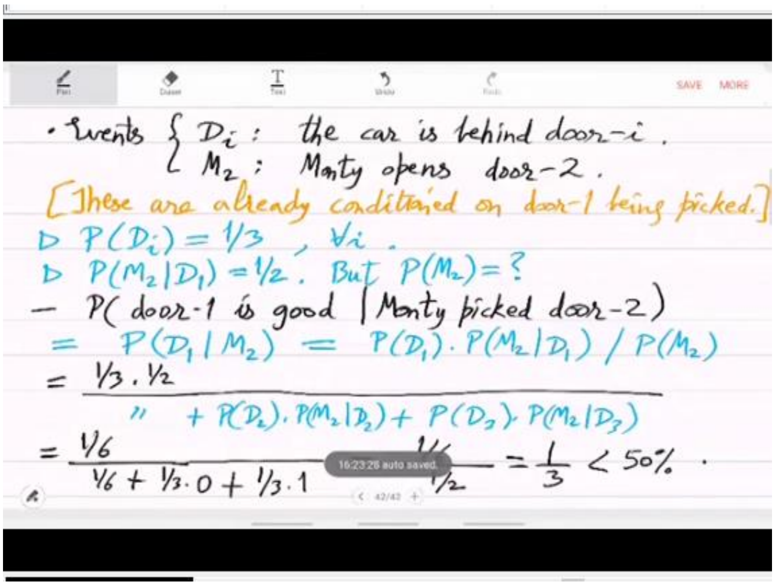

So, the events are so Di is the event, the car is behind door i. And M 2 is the event that Monty opens door 2. And these are already conditioned on door 1 being picked. So, the event definition happens after you have picked door 1. So, basically, that sets the scenario mathematically. So, you have picked door 1 and after that Monty has 2 choices, either to pick door 2 or door 3, he happens to pick door 2 with probability half.

And just conditioned on you picking door 1, the probability of car being behind door i that is one third for all these i's because no information has been released to the public by users picking door 1. So, you can assume it to be uniformly distributed. So, that is one third probability. So, probability Di is one third for all i and probability of Monty picking the second door is half and now what do you want?

So, you are interested in evaluating the probability that door 1 is good. Given that Monty opened door 2, this is what you are interested in which is symbolically it is probability of event D 1 given M 2, which by Bayes theorem is probability of D 1 times probability of M 2 given D 1 divided by probability of M 2 which is. So, let us write this down probability of D 1 is one third time M 2 given D 1, M two given D 1 is half.

And so I should call this M 2 given D 1 if door 1 has the car, so, let me correct something here. So, given that first door has the car, second and third doors, both of them have goats. So, Monty could pick any of those 2, so that probability is half. But probability of just without any extra information probability that Monty will pick the second door, that is not clear that is not at all clear.

We will do that actually using the partition formula. So, let me continue with that. So, probability of D 1 is one third and M 2 by D 1 M 2 given D 1 is half and then I will break up probability of M 2 like this, I will have this, which is with respect to D 1, then I have with respect to D 2. And then I have with respect to D 3. So, I hope this is clear. This is because I am using the partition given union D 2 union D 3, because it is a unique, it can only be in one of the doors.

So, this is an actual partition of the sample space. And then M 2 I am conditioning on these 3 and then I have the partition formula. So, let us now work this out. So, I have one sixth divided by one sixth plus probability of D 2 is of course one third. Now, if the car is in door 2, then Monty will never pick that door because Monty has to reveal the goat, so this probability is 0.

Next is probability of D 3 is one third. Without any extra information car could have been in the third door, it could have been in any of the doors equally. But once the car is in the third door, Monty is forced to pick the second door. There is no choice. So, this conditional probability is 1. Now, what do you get? So, one sixth divided by half, which is one third. So, you see that the probability of door 1 being good is quite low; it is much less than 50% which means that you will actually have an advantage if you change your decision.

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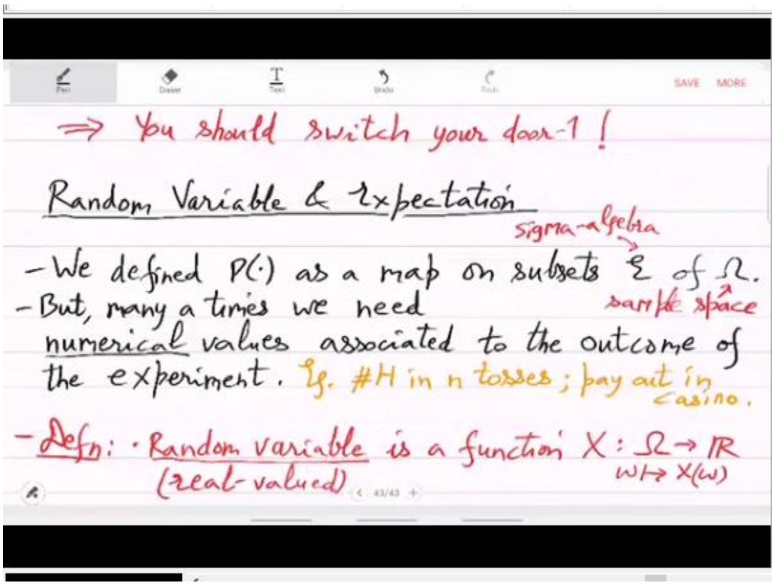

So, this means that you should change your decision, so you should switch your door thanks. **(Video Starts: 15:00)** So, that is the amazing consequence of Bayes formula, which was completely not clear before we started this analysis**.** So, just by Monty showing you that door 2 has a goat puts more probability or puts more likelihood on door 3 having the car, not door 1, the probability is kind of Bayes towards door 3.

So, this is called the Monty Hall fallacy or Monty Hall problem. **(Video Ends: 15:38)** And with this, let me go to our new property related to probability distribution function which is random variables. So, recall that we defined the probability map on collection of subset called E of omega. So, what was E, E was this sigma algebra and omega was sample space. So, point being that probability map was defined on the subsets in the most general form.

But sometimes, in fact many times in experiments, we want the sample space elements to be numerical values; we are actually interested in the numbers not just subsets. But many times we need numerical values associated to the outcomes of the experiment. For example, when you toss many coins, what is the number of heads? So, head is being seen as 1, tail is being seen as 0.

So, number of heads in tosses or payout of a lottery in any of the casino games, you will be interested in the payout. So, which is a number, you do not care about subsets, you actually care about these numerical values. So, we have to expand our definition of probability distribution to make sense of numbers inside omega. So, let us do that. So, random variable is a function x from omega to reals, that is it maps element omega 2 X omega.

And this is a real valued random variable. We will only be interested in real valued random variables. So, it is nothing but a function from sample space to real numbers. And the value is X of omega for every omega. That is all.

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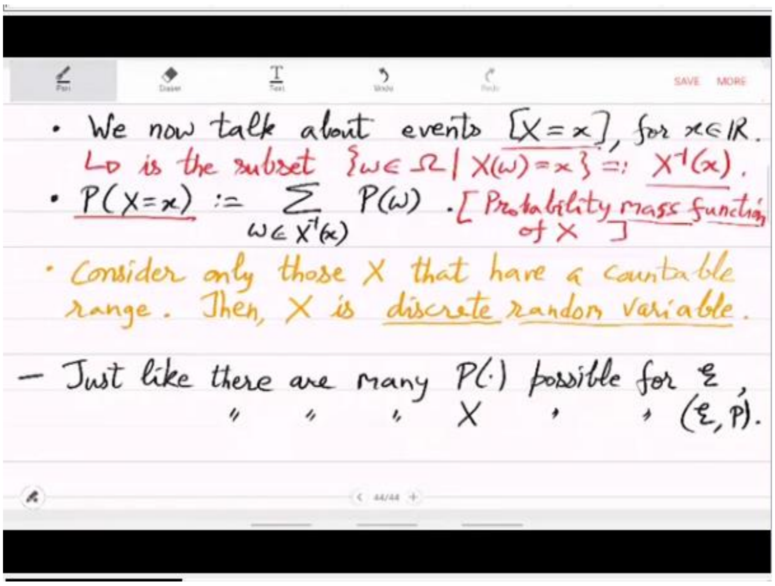

But then it start giving you many related things, new things like we can talk about events X being equal to a value. So, this is the event that so we are interested in those elements in the sample space that evaluate to this number X. That is an event. This real number X. So, this is the subset of those omegas in the sample space, which evaluate to this element, this number x, little x under big x function, we also can call this x inverse of x.

So, once you have this event defined you can extend now the probability distribution function for this which will be big x equal to small x probability is sum of P omega where omega is in the preimage of little x. All this omega is that are relevant, just some of their probability. So, now, there might be some issues when you go to infinity. So, for that, I will just say that we will consider only those x that have a countable range.

So, range meaning the values that the function x takes overall that is a countable set. We do not allow a whole real number set, but it is something like integers or natural numbers or rational numbers or sets that are in bisection with this. So, the next is called discrete random variable. So, we are only interested in these discrete random variables and then hopefully this sigma the sum of probability will make sense, but will not go too much into that right now and this has a name.

So, this is called this action of P on x equal to little eggs, this is called probability mass function of a random variable So, mass function because it is telling you how this random variable big X is distributed. So, for a particular value small x, what is the chance that it is equal to this? This is called a probability mass function and this we have defined. Now, clearly this random variable given a probability distribution function there may be many possibilities to define big X.

Just like there are many P possible, there are many X possible for omega P. So, this given omega and E maybe I should call this E instead of omega. So, given a sigma algebra there are many ways to define probability function P and once you have defined it there are many ways to define x the random variable. So, these things are not uniquely defined. So, shall we see an example? Let us see a quick example.

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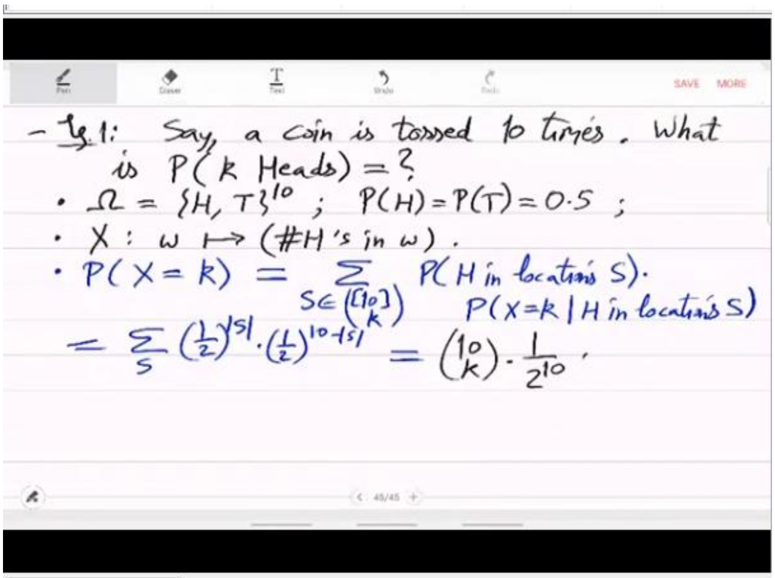

So, see a coin is tossed 10 times. So, what is the probability of getting k heads? Give you are interested in this question. So, in this case omega is head or tail 10 times. So, this Cartesian product is the sample space, probability of head, probability of tail is obviously half and what is the random variable. So, you want to see H as 1, T as 0. So, it sends little omega to little omega is a sample space element.

So, this is a string of 10 heads or tails. So, just the number of heads in omega. That is the random variable for every instance of the experiment and now what is the probability of x being k. So, this is your look at basically the places where head appears. So, let S be a subset of 1 to 10 k sized. So, probability that each happens in locations S times the probability of x being k.

Given each is in locations S, which is what, well once S locations are head. So, it means that x is k. So, that probability is 1 and what is the probability that H is present in these locations? Will that probability is just 1 over 2 raised to for the head and the remaining 10 - size of S which is so, what you get is number of subsets there that 10 k times 1 over 2 raise to 10. So, that is the mass distribution function. That is the mass function in this experiment.