Probability for Computer Science
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Module - 6
Lecture - 22
Page Rank Algorithm: Ergodicity

Could we turn this into a random walk on the graph? This formula, p i equal to Sigma p j by m j; p j by out-degree of j, which is rank of j over out-degree of j; could we turn this formula into something about random walk and then do this random walk many times, for a long time and then get the rank? This is what we will do.

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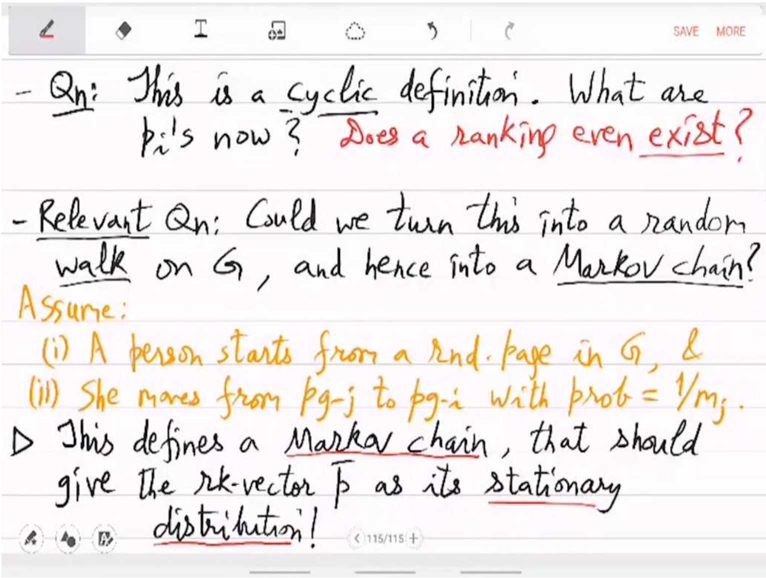

This is how we will implement it. So, assume that a person starts from a random page in G, and she moves from page-j to page-i with probability 1 by m j. So, look at this 1 by m j as given the, defining the matrix. The entries of the matrix, think of it as 1 by m j. It will be a stochastic matrix, a transition matrix with transition probability as 1 over m j. And then, you can see this as a Markov chain.

This defines a Markov chain that should give the rank vector p bar as its stationary distribution. So, we have done many things. We have rehashed the equation into a Markov chain by using transition probability is 1 over m j. And then from the stochastic matrix, you get the stationary distribution, in the limit. The only question is whether it exists. If the chain is not regular, then this may not exist.

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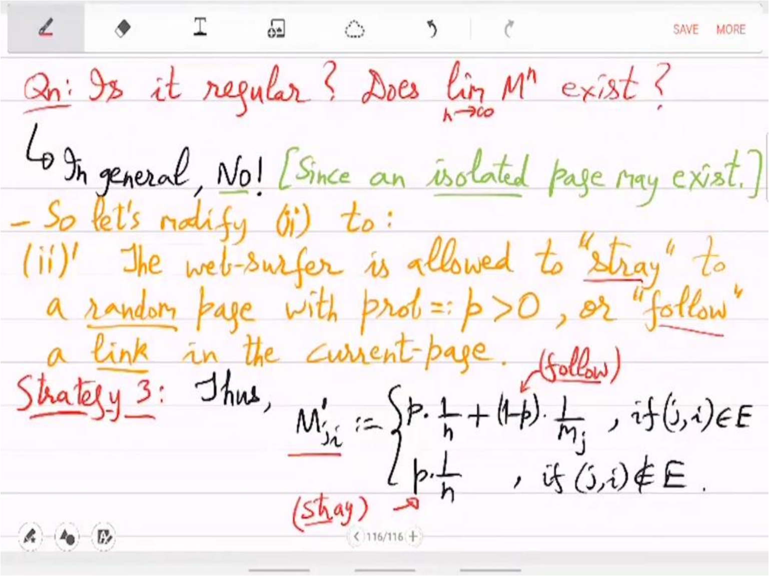

So, first of all, is it regular? Does this M to the n exist? The only way we know limit exists is by regularity, right? But if it is irregular, then we do not know whether the limit exists. So, these questions are outstanding for the Markov chain that we just defined. So, actually, in general, the answer is no, because it is quite possible that you have an isolated webpage; since an isolated page may exist.

So, because of isolated webpages, no matter how much you walk, you may never be able to reach those webpages, and hence, when you look at the transition matrix or this M to the n, certain entries will be 0. It will not be possible to make the entries all positive. This is quite possible in the internet. So, this is obviously an irregular Markov chain, because there might be no link to a page.

So, what you do is; so, let us modify this second condition that we had, moving from j to i; to 2 prime. What is 2 prime? This is, the web-surfer is allowed to stray to a random webpage with some probability. So, now, since these isolated pages are there, by just following the link, the web-surfer will never reach those webpages. So, the only possibility of reaching there is by starting from there, right? So, by just jumping to that without using a link.

So, that is to allow that. So, I mean, we have to allow straying to random webpages, and let us say with some probability, positive probability, or with a higher probability, follow a link. So, either stray or follow, allow both. So, this solves the problem of isolated webpages, right?

Now, straying will land you in that stray webpage with some positive probability. And usually, you will just follow the link of wherever you are.

So, this is a good mix of strategy-2 and trying to make it regular. So, correct. So, now strategy-3, we get to formally. Thus, let us define M prime ji, the matrix M prime to be P times 1 over n plus; so, going from j to i, what is the probability? Either you reach i by straying; so, that is probability p; and so, decision to stray, probability p. And then, out of the n pages, why did you land in i?

That probability is 1 by n; or decision to follow which is with probability 1 - p, and then look at the links out of m j links. So, this is if j links to i. If there is a link, then this formula; otherwise, the only way to reach is by straying; because, from j, you can never reach i, but by the straying decision you can. So, that is a good solution. So, you can see by physical interpretation that this is; let me write that this is by straying and this is by follow. So, but the physical interpretation already tells you that M prime will be a regular Markov chain, it is a transition matrix.

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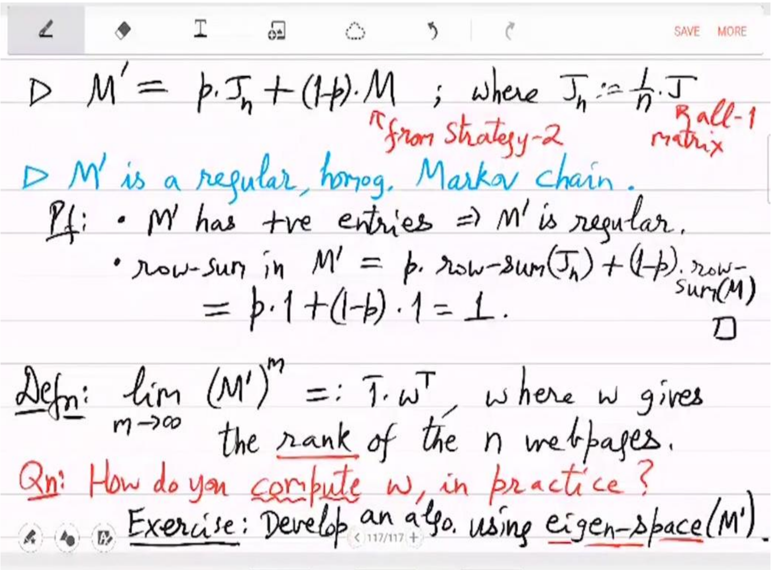

Now, let us study a bit more. So, M prime is actually, if you look at the equation, it is p plus 1 - p M, where J n is 1 by n times j. j is the all-1 matrix. So, this just follows from the definition. You are basically taking this p is to 1 - p linear combination of all-1 matrix or this all-1 by n matrix, 1 by n entries; every entry is 1 by n. This J n matrix or this strategy-2 matrix, which is M. So, that is our third strategy.

Slowly we have been improving, we are getting better rank, a better way to rank the webpages on the internet. So, that is by definition. And then, M prime is a regular homogeneous Markov chain, because, in fact, you can see that every entry is positive. M prime has positive entries, so, it is regular. Also, you could easily check why it is stochastic, right?

When you look at a row; so, row sum in M prime equals p times the row sum in J n + 1 - p times the row sum in M, which is equal to; so, row sum in J n is 1 and row sum in strategy 2. In strategy 2, matrix M was this summing up 1 over m j, in the jth row. So, that is again 1. So, you get 1. That finishes the proof. It is a stochastic matrix, it is also regular, every entry is positive. And so, you will have the stationary distribution.

So, limit n tending to infinity of this M prime to the n will give you the w row vector, where w gives the; I am using the same n, that is unfortunate. Let me change that notation a bit. So, w has these n entries that gives you the Page Rank, gives the rank of the n webpages. That is the definition of Page Rank. So, this is a good Page Rank algorithm or at least conceptually, it is a good way to define Page Rank, but you still have to work out what is the algorithm.

So, that is still a question. How do you compute w in practice? And how fast can you do this? How easy or difficult this is, because this seems to be requiring m tending to infinity; you cannot wait for infinity. So, how do you actually compute it? So, I will leave it as an exercise to do this via eigen-spaces. So, develop an algo using eigen-space of M prime. So, that is another advantage of studying the right action of this matrix M prime, right?

This is actually related to eigen-values, eigen-spaces. And then you can study M prime to the m, small m, as applying the same linear transformation again and again, and see how the eigen-spaces and eigen-values change. And then, from that approximate w. So, I will not go more into this algorithm. I leave it as an open question. Let us see one more strategy, although it is not needed now, but still maybe it has some other use.

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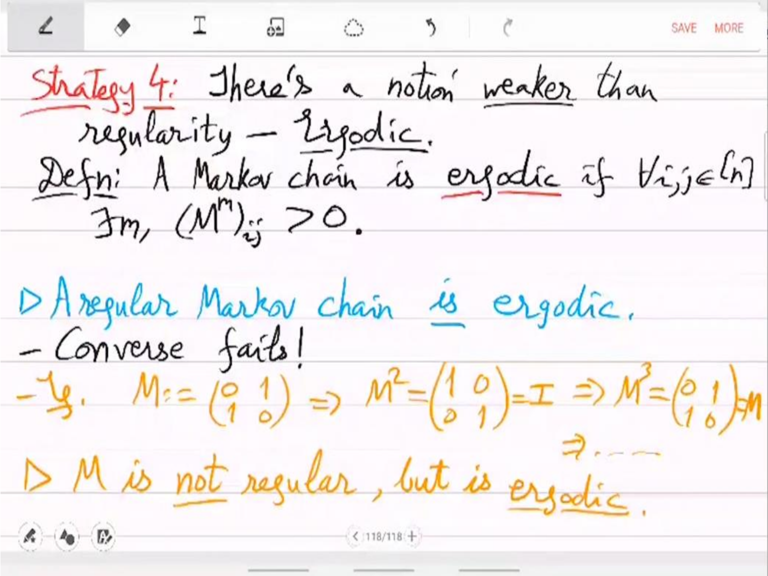

So, strategy 4, right? You have seen strategy 1, 2, 3. So, in 4, what we will do? So, there is a notion weaker than regularity and that is called ergodic. So, what we can do is, that matrix M, instead of trying to make it regular, we can try to make it ergodic. And that as well will give you a stationary distribution. So, what is the definition? So, Markov chain is ergodic if, for all ij there exists an n, says that M to the n ijth entry becomes positive.

So, instead of asking for all the entries of the power M to the n; let me change this to small m; instead of asking all the entries to become simultaneously positive, we just focus on 1. And then do this independently for all these entries. So, let us compare regularity with ergodic. So, clearly, regular Markov chain is ergodic. Regular Markov chain is always ergodic, because, well, if a Markov chain is regular, then M to the m, every entry is positive, this definition goes through.

On the other hand, what may happen is, this little m that you are getting for ij, for different ij, it may be different. It may not be, there might not be a single little m that works for all ij. So, the converse fails. Let us see an example. So, example we have seen before actually; so, if you look at 0 1 1 0 transition matrix, then what is M square? M square is 1 0 0 1. So, identity and M cube is again M, and so on, right?

So, it switches between identity and M, depending on what little m is order even. So, you can see that for every ij, at some point, that entry becomes positive, but any power that you look at, you have 0 entries. So, M is not regular, but is ergodic. That is why I said it is weaker than being regular. So, the good thing about ergodicity is, there is a natural way to transform M, so that it becomes regular. And hence, there is a stationary distribution.

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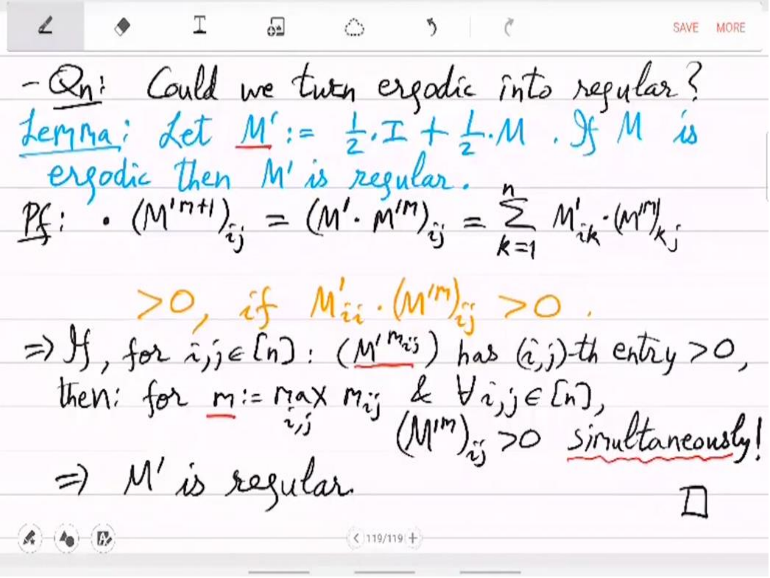

So, could we turn ergodic into regular? So, here is the lemma, which says yes. So, define M prime as identity plus. So, it is a combination of identity and the matrix, ergodic matrix M. What you are basically doing is, you are just looking at the diagonal, and there you are making a change, you are adding half. You have also obviously scaled all the entries by half. And then, in the diagonal ones, you are adding another half.

So, that is basically in terms of a random walk what you are doing or in terms of walking on the internet graph. What you are doing is, you decide to stay on the same page or you use a link. So, if M is ergodic, then M prime is regular. So, this is another way to get to a regular matrix with a very natural physical interpretation on the internet graph; either you stay where you are or you take a forward link.

So, note that the problem that we were facing of this isolated webpages, right? That somehow gets solved here. So, let us see a proof. So, if you look M prime little m plus 1, if you look at its ijth entry, so, what is that? It is M prime times M prime to the m ijth entry, which is go over all k's and take the sum. So, ikth entry of M prime and kjth entry of M prime to the m. Now, this is positive as M prime; if you fix k to be i, then M prime ii times M prime to the m ij, this is positive.

So, you are assuming that the i for ij, the m, basically small m is defined by this ij, as it was in the definition; should have said that. So, for ij, if you have this small m, says that ijth entry there in that power is positive. Then, it is also positive in M prime to the m + 1. So, this is positive if M prime ii, this is positive. So, what this means is that, if for ij you know that m ij power has ijth entry positive, then for m defined to be the max of these m ij's and for all ij, M prime to the little m ij is positive simultaneously.

So, if for ijth entry, making it positive, you have to go to exponent little m ij. Then you, by the previous property; so, once you have made ijth entry positive, next power, it will remain positive. That is a property of the way we define M prime. This definition of M prime gives you that. So, next time it will remain positive. And hence, if you just take the maximum of all these m ij exponents, that is an exponent which works for all ij's simultaneously.

So, this means that M prime is regular. So, what do we do next? So, now, for this Markov chain, you can get a Page Rank algorithm. That is the fourth strategy. We will finish this next time.