Randomized Methods in Complexity
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Lecture - 11
Expanders

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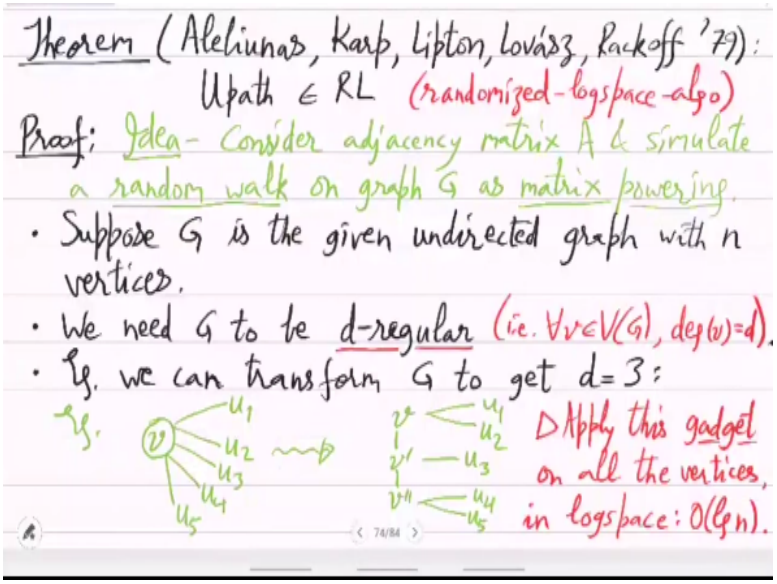

Last time we were in the middle of proving this theorem by AKLLR which will have a long proof because it involves many new ideas and many new tools in matrix analysis. So this theorem says that you are given an undirected graph on the input tape and vertices s and t and you have a very small work tape, logarithmic space. You have to decide whether there is a path from s to t.

Basically you have to solve this problem using random bits in logarithmic space.

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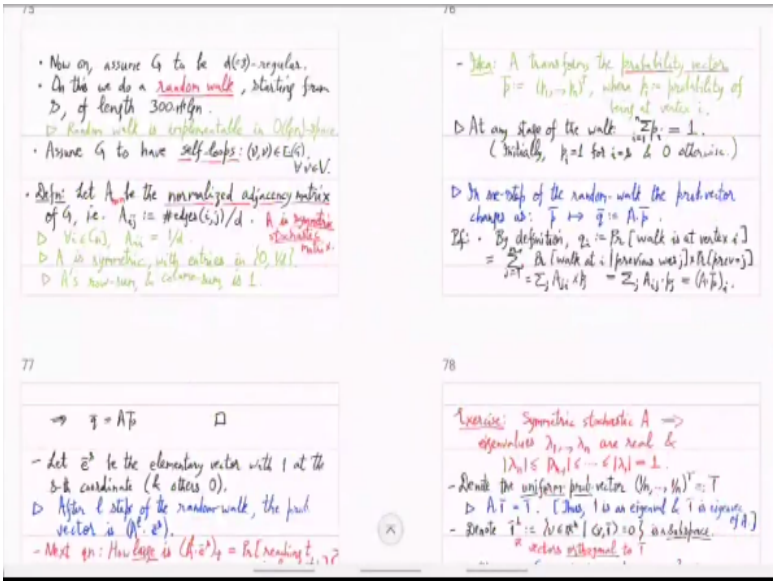

The way we are doing it is till now we have seen the random walk. You start a random walk from s. Every vertex you reach there will be three neighbors. And you will pick one randomly and then proceed. In terms of the normalized adjacency matrix we are basically doing matrix multiplication on the initial vector state and the question is how fast will this converge.

In how many steps with decent probability you will reach t. So we have made the graph d-regular with $d = 3$.

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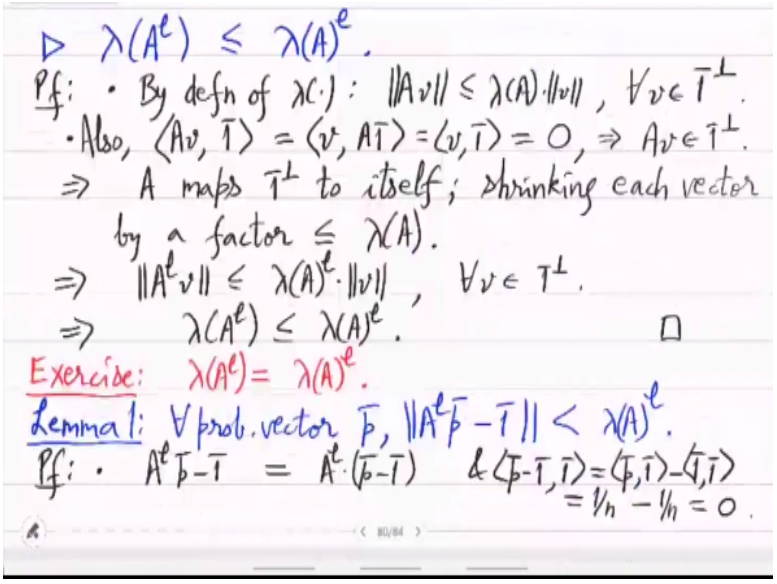

And we showed in lemma 1 that $|| A^l \cdot \overline{p} - \overline{1} ||$ is exponentially dependent on the second largest eigenvalue.

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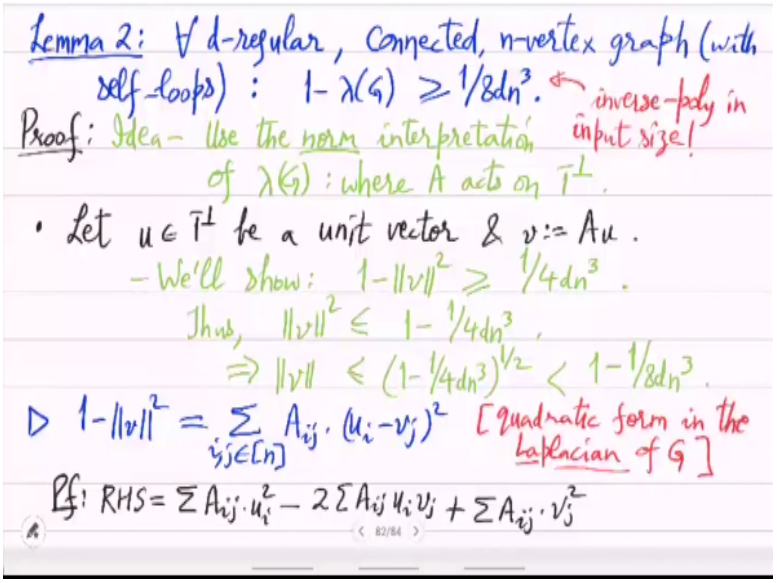

The second lemma showed that λ is basically small, it is away from 1. It is below 1 and it is sufficiently away from 1. It is not exponentially close to 1 and this we showed using the Laplacian of the graph We had $\Sigma A_{ij} \cdot (u_i - v_j)^2$.

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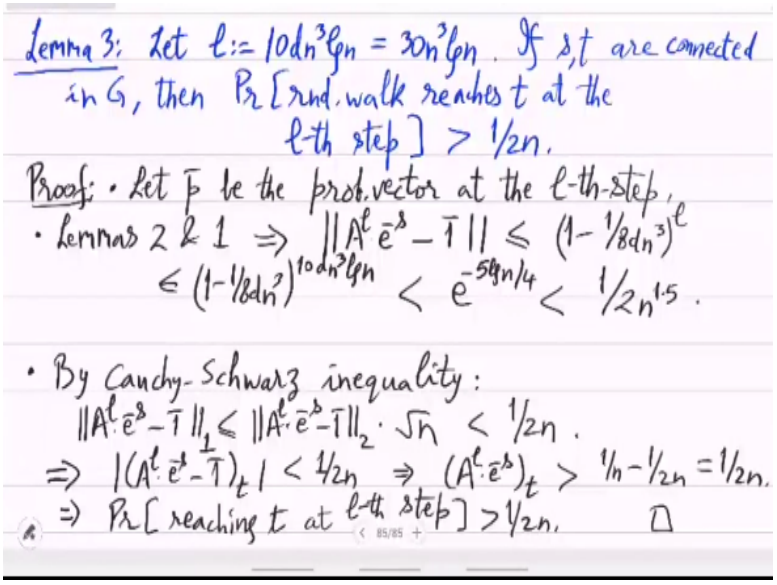

And now what we will do is the final lemma where we will show that taking l to be roughly $n^3 \cdot \log n$, if you take these many steps, then with high probability you will reach t. So let $l = 10 \cdot dn^3 \log n$. Remember $d = 3$, so this is like $30n^3 \log n$. If you take these many steps, and if s and t are connected in the graph then the probability that the random walk reaches t in l-th step, this is quite high.

So reaching t in the l-th step is probability is sufficiently high, it is $1/2n$. And to increase the success probability even more we can boost this. We can take more steps. Let us prove this. We will basically, we will use lemma 1 and 2, proof will be easy. So let $\bar{p}$ be the probability vector at the l-th step.

Lemmas 2 and 1 imply that l applications of A, which is a normalized adjacency matrix on the initial vector probability vector $\bar{e}$ which is 1 at only the s-th place. This part being away from the uniform probability vector $\bar{1}$ is is dependent on how small is the λ. That we have shown is smaller than $(1 - 1/8dn^3)^l$.

Basically after this l applications of the matrix you are very close to $\bar{1}$, the error is only this much, which is less than equal to $(1 - 1/8dn^3)^{10dn^3 \log n} < e^{-5 \log n/4} < 1/2n^{1.5}$. And now there is this small thing that the difference vector we have taken is the Euclidean norm

From this now you can extract information about each coordinate - how small is each coordinate? That can be done by the Cauchy-Schwarz inequality. You will get that $|| A^l \cdot \bar{e}^s ||_1 \leq || A^l \cdot \bar{e}^s - \bar{1} ||_2 \cdot \sqrt{n}$.

The 1 norm is related to the Euclidean norm by the factor of $\sqrt{n}$ number of coordinates. Whic is still less than $1/2n$. This is the information you have about each coordinate, the sum of the magnitude of each coordinate that is less than $1/2n$. Which means that if you look at the t-th coordinate, which is the probability of reaching t, that is also smaller than $1/2n$.

This is the difference between the t-th coordinate of $A^l \cdot \bar{e}^s$ and t-th coordinate of $\bar{1}$. This implies that the coordinate of interest exceeds $1/n - 1/2n$ because the difference cannot be more than $1/2n$, which is still $1/2n$.

This means that the probability of reaching t at l step is greater than $1/2n$ promised in the lemma statement. This is just a simple application of what we have already shown.

Basically if you take these l many steps in the random walk then you are almost in a random place. In the connected component of s, you are everywhere with equal probability, which means that if t was there you are also at t.

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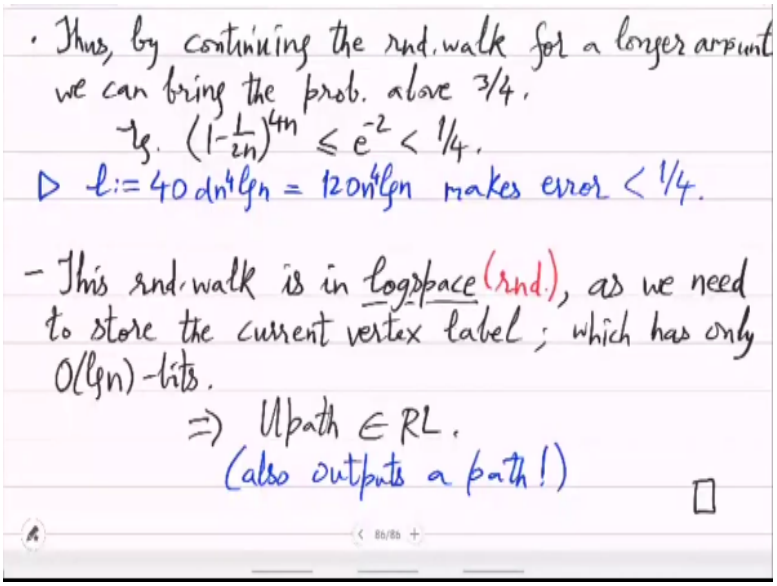

This also means that by continuing this walk, the random walk for a longer amount, we can bring the probability above let us say three fourth or two third - sufficiently high probability. You can reduce the error probability to quite small. For example, if you do this calculation, the error probability currently is $1 - 1/2n$.

So if you repeat this 4n times, you can see it is smaller than e^{-2}, which is smaller than $1/4$. What you have deduced is $l :- 40dn^4 \log n = 120n^4 \log n$ makes error smaller than one fourth. Error has become quite small and if you want it further exponentially small, then you can go up to n^5. If you take n^5 random steps then the error of not reaching t is exponentially small.

This is a very good procedure. And every step you can do in logspace, because you only have to look at 3 neighbors, you can identify this from the input graph and the number of steps is even if it is n^5, to keep track of each of these iterations you only need logspace.

So this random walk is in logspace as we need to store the current vertex label, which has only log n bits. Which means that we have shown that the Upath is in RL. And

notice that since you are actually going to vertices you can also keep outputting these vertices, so the whole path can be outputted as well.

The working space is only O(log n) space and you can bit by bit or vertex by vertex you can write the path on the output tape. So the next question is can we improve on this. This is a randomized logspace, can we derandomize this algorithm? Can we make it exactly the complexity class L?

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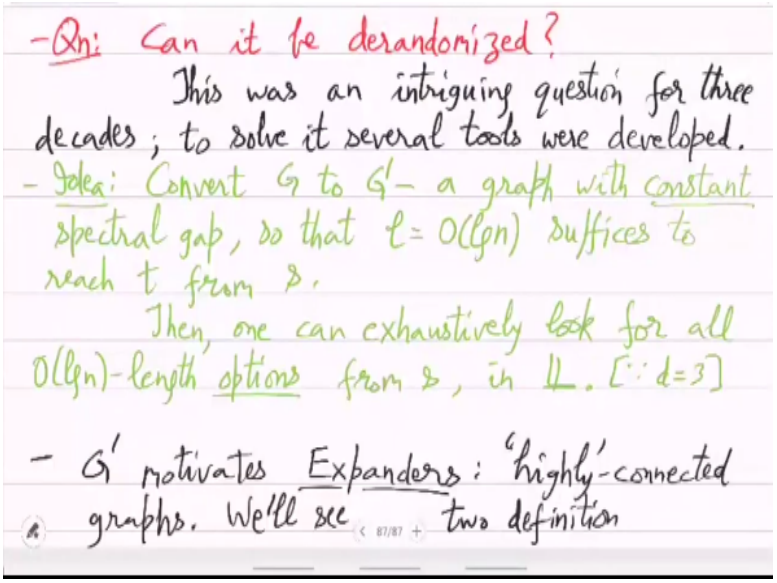

This was an intriguing question for three decades and to solve it several tools were developed. One idea could be that you transform the input graph G in such a way so that your random walk converges to the vertex t in the fastest possible way. And what is the fastest possible? In the algorithm we took l to be n^4 or n^5. What if you can take l to be log n.

In log n steps you are already at t. Can we achieve that? If we can achieve l log n, so logarithmically smaller l then what will happen is we can actually explore all possible l length paths from s in logspace and this we can do because every vertex has only 3 neighbors. You can just store which neighbor to go to 1, 2, 3 of s and then in the next which neighbor to go to 1, 2, 3.

You just have to search over l length strings of 1, 2, 3. So convert G to G' a graph with constant spectral gap so that l equal to log n suffices to reach any vertex from s.

Basically it boils down to having constant spectral gap. Because remember lemma 1, if λ is constant let us say it is $1/10$ then you just have to make $(1/10)^l$ steps, which is something like $1/n$.

Just reduce the error to be inverse polynomial in n. So for that l log n will suffice. Once this is achieved, then one can exhaustively look for all log n length paths from s in logspace, because the degree is constant. Since the degree is constant you can actually try out all possible log n length strings in 1, 2, 3. That is how you will be able to go over all the paths.

Hence solving connectivity or directed path questions in G in L. This is the key challenge to convert G into this highly connected graph G' and these graphs are called Expanders. So G' motivates Expanders. These are highly connected graphs. For expanders, there are many definitions that we can use and especially the connections between the definitions are highly interesting. We will now define it in two ways.

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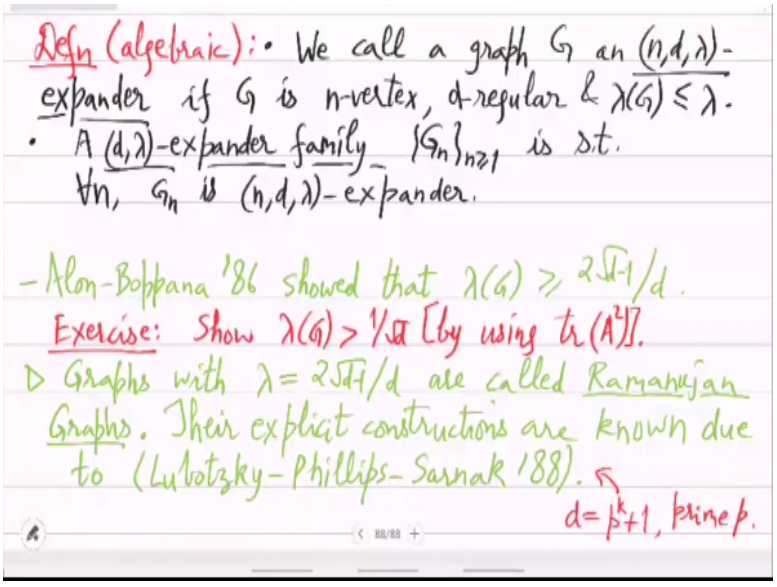

First is the algebraic definition. We call a graph G and n, d, λ expander if G is n-vertex, d-regular, d is the degree and the second eigenvalue of G which is $\lambda(G) \leq \lambda$. The spectral gap is bigger than $1 - \lambda$. You want λ to be small. The smaller λ is the better is the algebraic expander G.

Instead of looking at only one graph we will obviously or we will be more interested in actually graphs that are growing. A family of infinite graphs. So a (d, λ) − expander family, so graphs $\{G_n\}_{n \geq 1}$. You can take d to be 3 and in general you can ask the question. There are two parameters in expander family, d and λ.

For a given d you can ask how does λ behave as a function of d. Can you make λ extremely close to 0? Or is there a limit? Is it away from 0? Is it large in terms of d? Actually it depends on d pretty strongly. There is this result by Alon and Boppana which showed that $\lambda(G) \geq 2\sqrt{d-1}/d$. So if you take $d = 3$ then this is $2 \cdot \sqrt{2}/3$.

You cannot make the second largest eigenvalue arbitrarily small. It will really depend on the degree d parameter and this expression is a bit more non trivial to show. But, as an exercise you can show that $\lambda(G) > 1/\sqrt{d}$ by using $tr(A^2)$.

But Alon-Boppana bound is better and it is optimal. It is saying that actually it is more than almost $2/\sqrt{d}$. Graphs with $\lambda = 2\sqrt{d-1}/d$ are called Ramanujan graphs. And their explicit constructions are known. That result is due to Lubotzky, Phillips and Sarnak. So these objects are very important.

They are heavily studied and using analytic number theory there are these constructions due to LPS of Ramanujan graphs. The degree there is actually prime power plus 1. So it is not for every degree, but for this prime power plus 1 degree these constructions are known and they achieve this Alon-Boppana bound. This is an optimal bound. Point being that these infinite families are constructible.

They not only exist but they are constructible and we very well understand the spectral gap. This is some advanced topic, independently, we will not go deeper in this. Instead what we will do next is define expanders combinatorially.

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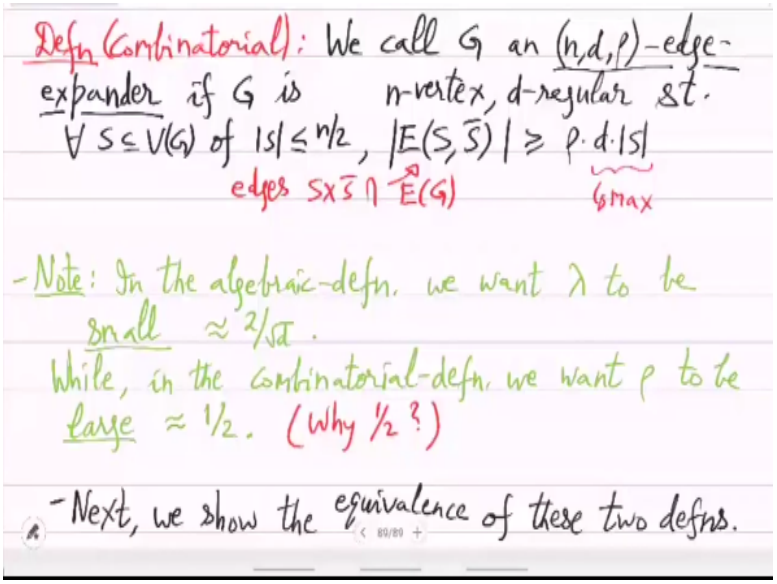

In combinatorial definition of an expander we will call these now edge expander instead. So (n, d, ρ) − is an edge expander. If G is an n-vertex, d-regular graph. This will be more pictorial than the algebraic definition. In the algebraic definition, it was not clear why you pick the second largest eigenvalue, unless you recall lemma 1 that we have shown where if λ is smaller than the convergence is faster of the random walk.

But the combinatorial definition will be even simpler and more directly related to connectivity. So here what we will say is that, if you take any subset in the graph, then there are many edges going out. If G is an n-vertex d-regular graph such that for any subset of vertices of size small, let us say smaller than $n/2$. Say you take an n, you take half the vertices as S then count how many edges are going out.

That is the connectivity information or a way to measure connectedness in the graph. The edges that are going out of S, that is S to S' is large. So this $E(S, \bar{S})$ is basically these edges going from S to $\bar{S}$. This is all undirected edges. And $d \cdot |S|$ you can think of this as the max that could happen, because they are |S| vertices and the degree is d. There cannot be more than $d \cdot |S|$ edges going out.

And ρ is the fraction that is the parameter given to you for edge expansion. Let us say $\rho = 1$ then you are actually saying that maximum possible edges are going out of S for any S. And if ρ is let us say 0.99, then still many edges are going out. So this is a

very good level of connectivity inside the graph. In the algebraic definition we want λ to be small.

So close to $2/\sqrt{d}$. While in the combinatorial definition we want ρ to be large. But how large can it be? Can it be 1? You can show that it cannot be 1. The maximum it can be is actually half, because you want it to be true for every subset. If you want it to be true for every subset of size n/2 or less, then you can only afford half. That is what we want optimally.

Now what we will do is next we show their equivalence. Why are these two very different definitions of expansion? Why are they equivalent? This is not at all clear although intuitively we have some feeling. The combinatorial one is the natural meaning of well connectedness and the algebraic one comes from lemma 1 because lemma 1 says that the convergence of the random walk will be rapid if λ is small.

But see what is the quantitative relationship between that λ parameter and this ρ parameter. So here it is.

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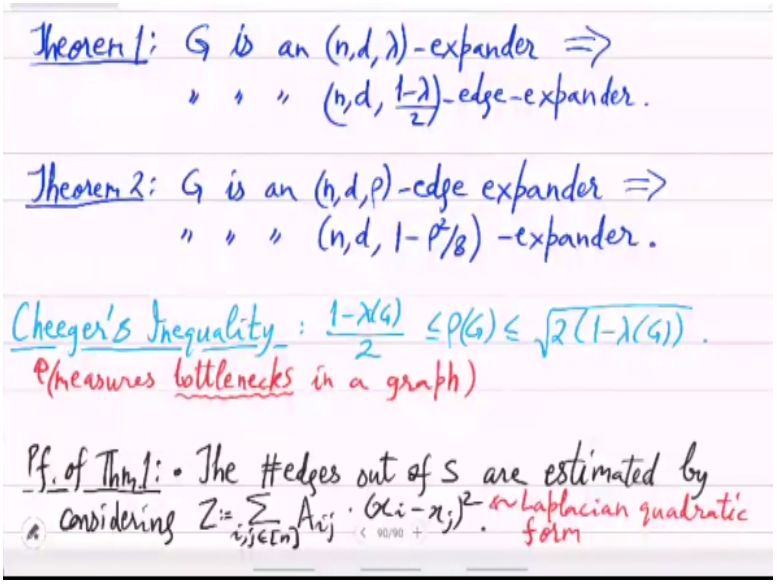

We start with showing if G is an (n, d, λ) −expander; so algebraic definition. Then G is also an $(n, d, (1 - \lambda)/2)$ − edge expander. If G satisfies the algebraic expansion definition then G also satisfies edge expansion definition. So λ you want to be close to

zero and you can see in the implication that then edge expansion will be close to half and theorem 2 is the converse.

Let G be an (n, d, ρ) −edge expander. Then what can you say about the second largest eigenvalue of edge expanders? It will not quite be the converse of theorem 1, it will be slightly weaker. What you will get is $(n, d, 1 - \rho^2/8)$ −expander. Again you can check if ρ is half which is the best edge expansion possible.

Then in the bound for the second largest eigenvalue or for the spectral gap, you are getting is 1/32. That is the relationship. So this anomaly is not if and only if, which is why we have to write actually two theorems instead of just one theorem with the, if and only if. We can also state it as Cheeger's inequality. Basically, it tells you what ρis for a graph in terms of the λ parameter, the second largest eigenvalue.

There is this gap that you get for edge expansion. If in the best possible case λ will be 0, I mean ideally speaking it could be 0, then this is telling you that ρ is between half and $\sqrt{2}$. That is a wide range.

The point is that this is not precisely telling you what ρ is but it is giving you a decent lower bound on the edge expansion. As you keep making λ smaller, edge expansion improves. It matches our intuition. You can think of Cheeger's inequality as a way to measure bottlenecks in a graph. This is the only inequality which kind of connects both theorems, and combines theorem 1 and 2.

But we have to prove these theorems individually. Proofs will be sort of related but still they will look different. Let us start with the proof of theorem 1. We have to connect this algebraic structure with the or concept with combinatorial concept. We want to estimate the number of edges that go out of S. You are assuming that G is a λ expander.

The second largest eigenvalue is smaller than λ in magnitude. From this you want to deduce that if I take a subset S in the graph, how many edges are going out. For that

actually we will consider a function. It is called the Laplacian quadratic form. It will measure this exact thing. Its evaluation will measure how many edges go out of S.

The number of edges going out of S are estimated by considering this Z quadratic form, which is

$$Z := \sum_{i,j \in [n]} A_{ij} \cdot \left(x_i - x_j\right)^2.$$

This seems like a weird function, but it has a name, and it appears in many places.

We have actually used something like this also before, when we wanted to use the connectedness of a graph to deduce that the second largest eigenvalue is sufficiently away from 1. There as well, we used something of this type. This is in particular this is called Laplacian quadratic form. Now you have to realize that this could measure the number of edges that go out of S.

Basically the vertices which are in S when i is in S, you give them value 1 and the vertices which are not in S you give them values 0. Then this difference square, this is 1 only when you are looking at an S that crosses. Within S and within $\bar{S}$ it is 0. Let us formalize this. This is the bridge between algebra and combinatorics.

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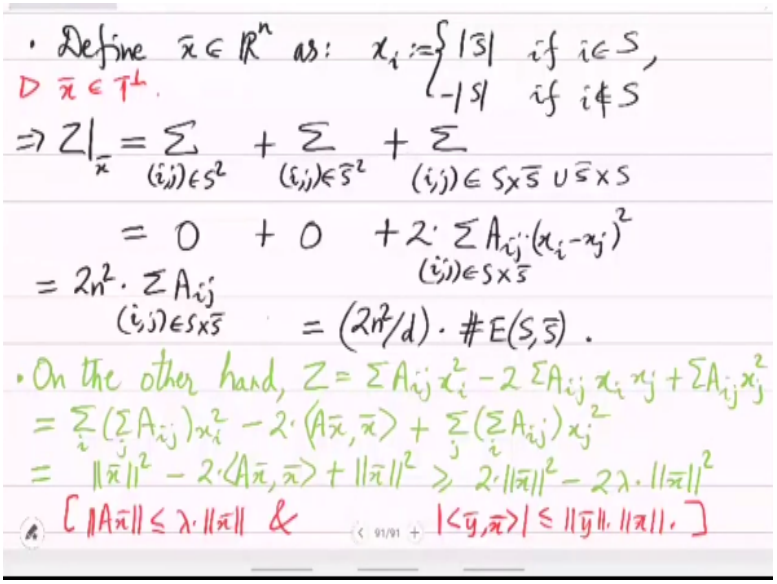

Define vector $\bar{x}$ as, the i-th coordinate is $|\bar{S}|$ if i is in S and it is $-|\bar{S}|$ otherwise. That is the definition of x_i, $\forall\, i \in [n]$. Now you can see the value of Z at $\bar{x}$. Let us break

the sum into three parts. So one is within S. Then within $\bar{S}$ and then across, that is, the rest.

$$Z|_{\bar{x}} = \sum_{(i,j)\in S^2} + \sum_{(i,j)\in \bar{S}^2} + \sum_{(i,j)\in S\times\bar{S}\,\cup\,\bar{S}\times S}$$

For i in S the sum is 0; i and j both in S it is 0; i and j both in $\bar{S}$ bar it is 0.

And now let us see the last thing. Since, S to $\bar{S}$ or $\bar{S}$ to S it is the same thing, there is a factor of 2 and what is the sum?

$$Z|_{\bar{x}} = 2 \cdot \sum_{(i,j)\in S\times\bar{S}} A_{ij} \cdot \left(x_i - x_j\right)^2.$$

You can notice that $\left(x_i - x_j\right)^2 = |S| + |\bar{S}| = n$. So you get this equal:

$$Z|_{\bar{x}} = 2n^2 \cdot \sum_{(i,j)\in S\times\bar{S}} A_{ij} = (2n^2/d) \cdot \#E(S, \bar{S}).$$

This is what this Laplacian evaluates to and so this is one way to evaluate the Laplacian. And next what we will do is we will do this calculation in a different way.

We will relate this to λ, the second biggest eigenvalue. We have this Laplacian quadratic form Z and we have shown that value at $\bar{x}$ is giving you essentially the number of crossing edges. Let us now recalculate it in a different way so that this connectivity information relates to the eigenvalue. You can expand Z as:

$$Z = \sum A_{ij} \cdot x_i^2 - 2\sum A_{ij} x_i x_j + \sum A_{ij} x_j^2.$$

Now the nice thing is that this $A_{ij} \cdot x_i x_j$ term is this you can see as the action of A on $\bar{x}$, that is $< A\bar{x}, \bar{x} >$. That is the nice thing about this calculation and hence this expression of Z becomes:

$$Z = \sum_i \left(\sum_j A_{ij}\right) x_i^2 - 2 < A\bar{x}, \bar{x} > + \sum_j \left(\sum_i A_{ij}\right) x_j^2$$

Now clearly this calculation we have done before that ΣA_{ij} over all j. This is just the i-th row sum which is 1. You will get:

$Z = ||\bar{x}||^2 - 2 \cdot < A\bar{x}, \bar{x} > + ||\bar{x}||^2.$

And furthermore, you know something about $A\bar{x}$. Note that here $\bar{x}$ that we have chosen it satisfies this property. So $\bar{x}$ is actually orthogonal to 1. Because if you added up Σx_i, what do you get?

$|S| \times |\bar{S}| - |\bar{S}| \times |S|$ which is 0. That is a very useful thing in the second green calculation, because you know that on such vectors the action of A scales by at most λ factor. So $||A\bar{x}|| \leq \lambda \cdot ||\bar{x}||$. So what we can write is this as:

$Z \geq 2 \cdot ||\bar{x}||^2 - 2\lambda \cdot ||\bar{x}||^2.$

The other thing we are using above is $| < \bar{y}, \bar{x} > | \leq ||\bar{y}|| \cdot ||\bar{x}||$. This is simply saying that inner product is bounded by the product of the vectors, length of the vectors This is clear by geometry because we are working in the real Euclidean space. So that gives you a nice relationship of lower bound on Z.

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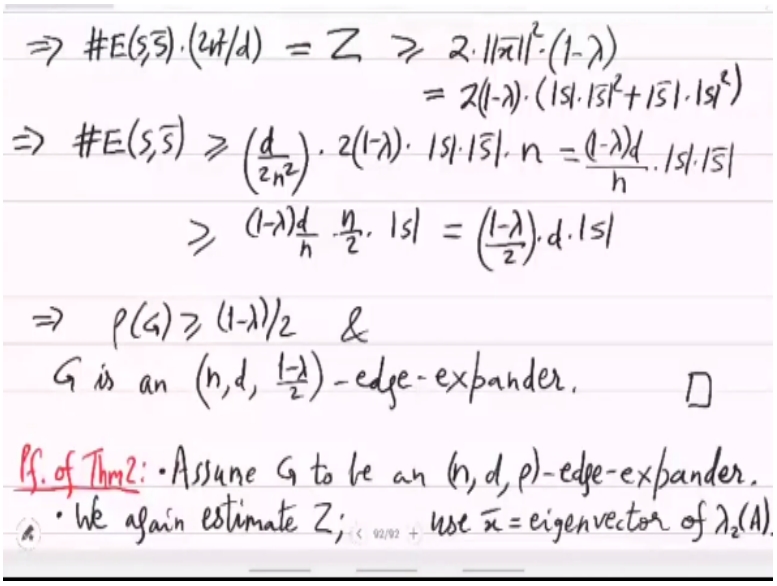

So this means that Z is at least

$2 \cdot ||\bar{x}||^2 \cdot (1 - \lambda).$

And it is exactly the number of crossing edges. That gives you a relationship. Further what is this equal to?

$Z \geq 2(1 - \lambda) \cdot (|S| \cdot |\bar{S}|^2 + |\bar{S}| \cdot |S|^2)$

Which implies that the number of crossing edges is at least $\left(d/2n^2\right) \cdot 2(1 - \lambda) \cdot |S| \cdot |\bar{S}| \cdot n$. Further, |S| you have assumed is less than equal to n/2. So $|\bar{S}| \geq n/2$. Then you get

$\#E(S, \bar{S}) \geq (1 - \lambda)d/n \cdot n/2 \cdot |S| = (1 - \lambda)/2 \cdot d \cdot |S|$.

So this in particular tells you that $\rho \geq (1 - \lambda)/2$. And G is an $(n, d, (1 - \lambda)/2)$ − edge expander. And this we were able to achieve because of the bridge between algebra and combinatorics - Laplacian quadratic form. We actually evaluated it at $\bar{x}$ and that gave us the first equality Z equal to basically the number of crossing edges.

Then we did a different calculation on Z to get a lower bound. So that is a brilliant connection. Here we are 50% done. Next we will do the almost-converse of this, proof of theorem 2. Now assume G to be an (n, d, ρ) −edge expander. And given this combinatorial property now we want to estimate the second largest eigenvalue of the graph.

Again we will use this bridge called Z. We again estimate Z, but we evaluate at a different $\bar{x}$. Now $\bar{x}$ will actually be an eigenvector. Use $\bar{x}$ equal to an eigenvector of λ_2. So the second largest eigenvalue of the matrix became eigenvector $\bar{x}$ and now estimate Z afresh.

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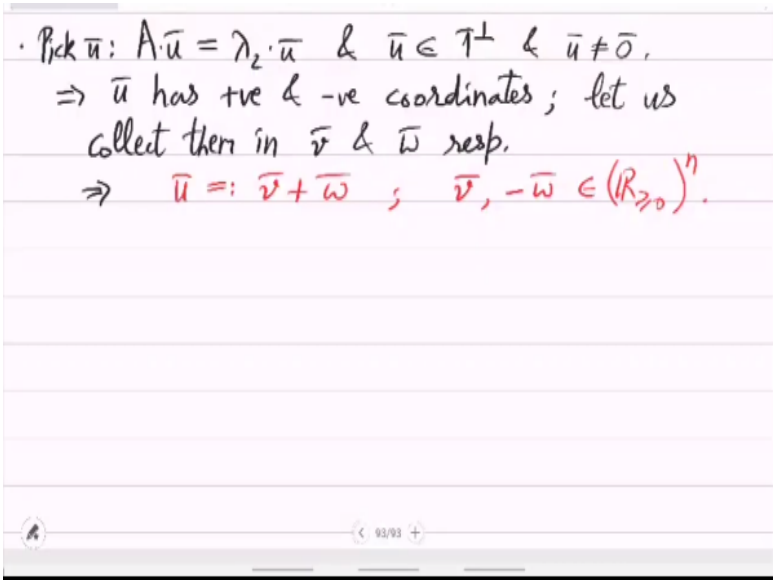

What you have is $A \cdot \bar{u} = \lambda_2 \cdot \bar{u}$ and you have that eigenvector $\bar{u}$ is orthogonal to 1.

In fact, let us take this as a definition. Pick a nonzero $\bar{u}$ such that this happens. This means that $\bar{u}$ has positive and negative coordinates. Let us collect them in $\bar{v}$ and $\bar{w}$ respectively.

What we have is we have basically written $\bar{u}$ as $\bar{v} + \bar{w}$ with both $\bar{v}$ and $-\bar{w}$, every coordinate is non-negative. We have separated into positive locations, and negative locations. Based on this now we will estimate Z in the next class.