Arithmetic Circuit Complexity

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Lecture - 19
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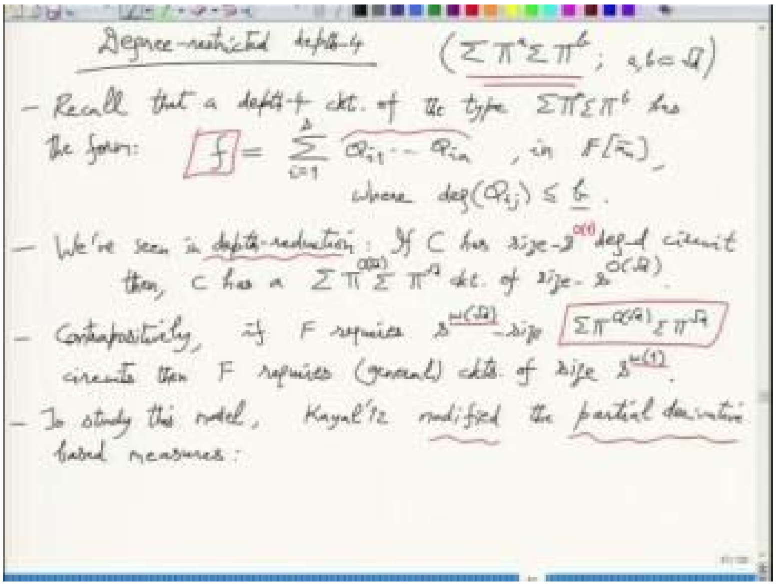

So today we will do, we will start this degree restricted depth 4 circuit model so $\Sigma\Pi\Sigma\Pi$ with the product fan-in not general but restricted to $\sqrt{degree}$. So recall that a depth 4 circuit of the type $\Sigma\Pi^a\Sigma\Pi^b$ with a, b bounded has the form $f = \sum_{i=1}^{s} Q_{i1} \cdots Q_{ia}$ there are s products and in each product you will see a factors. This is happening in $F[\overline{x_n}]$ where a deg(Q_{ij}) $\leq b$.

So you have s products, each product has a factors, each factor has degree b, this is the set of polynomials we will look at so we have seen in depth reduction to depth 4 we have seen that if f has size s, so I do not use the same f let us use something else. So if some polynomial C has size s circuit, size s degree d circuit then C has I mean that circuit has depth log(d) so that log(d) can bring it down to depth 4 and the price you will pay is the size will become $s^{\sqrt{d}}$.

So then C has $\Sigma\Pi\Sigma\Pi$ circuit of size $s^{O(\sqrt{d})}$ and moreover this depth 4 circuit is a bit special so here you get both the product fanin to be $\sqrt{d}$ so did we see this in the depth section I think we wrote in terms of t and d / t. So you take t to be $\sqrt{d}$ then you get balanced so both the Π have essentially the same form $\sqrt{d}$.

This we have seen, this is done by flattening the log(d) depth circuit at depth t. So at t you cut and make each of these parts depth to get depth 4 with the parameter square root the fanin $\sqrt{d}$ size is then $s^{\sqrt{d}}$.It was the maximum of t and d / t, which is now $\sqrt{d}$. t + d / t, yes. This we have seen before in depth production extreme depth production and what this means is conversely not conversely but more like contra positively.

The contrapositive of the statement is if a polynomial F requires $s^{\omega(\sqrt{d})}$ size depth 4 circuits then F requires an arithmetic circuit of size. So if somebody tells you gives you have an F that requires $s^{\omega(\sqrt{d})}$ size depth 4 of this type then what can you say about the circuit complexity? Can there be a poly(s) size circuit? , so look at the contrapositive you will get actually super polynomial lower bound.

So if you have here in the exponent $\omega(\sqrt{d})$ then here in the exponent you will have $\omega(1)$ which means that there is no poly(s), it is not possible to have a poly(s) size circuit. In other words the complexity circuit complexity of F will be super poly. So the proof is just by contra positive because if it was poly s then the production would have given you s to the root d and not s to the little omega root d.

Is a notation little omega clear? This is some strictly bigger than $\sqrt{d}$, you can make it O(1) does not matter the constants that get the level of the exponent that does not change anything. So this is the statement whose contra positive we are using so to study this model then we need to study this model. Because if he studied this well enough, we might be able to design a big F that requires $s^{\omega(\sqrt{d})}$ size and then we will have a general circuit lower bound.

Next to study this model in a preprint Kayal modified or gave a modification to the partial derivative method. For partial derivative based measures, so they were modified in a major way so till now we have only looked at the given polynomial and maybe we restrict the variables and then we look at a matrix where we are looking at either the coefficients or the derivatives with that, that indexes the columns and the rows are indexed.

How did you restrict the rows? Well, in the previous multi linear model we were looking at we are partitioning the variables. So one partition gives you rows the other one part gives you rows the other part gives you columns and in this finite field measure we had the derivatives, derivative operators indexing the rows. So both the cases we are somehow looking at derivatives of restrictions of f so that will now change.

So now actually f will be the polynomial f whose measure we want to define. We will first multiply it by well we will compute the derivative but then we will also multiply it by a monomial so with the derivative operators there is also multiplication that is happening. so that is a new thing.

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So let $\partial^{=k} f$ be the set of order k partial derivatives of f. let $x^{\leq l}$ be the monomials in $\bar{x}$ have degree at most l. so we are looking at derivatives and monomials and using this the shifted partials of f which will compactly denote as $< \partial^{=k} f >_{\leq l}$ like this so this is the set we will look at. So, instead of just looking at the key order derivatives we will also allow them to be multiplied by monomials of degree upto l.

It so this is a set of polynomials, look at the vector space spanned by this they are called shifted partials of f so those have shifted partials of f is the F vector space spanned by

$$\{\overline{x}^{\overline{e}} \cdot \partial_{\overline{d}} f \mid |\overline{e}| \leq l,\ |\overline{d}| = k\}$$

monomial times derivative of f where the monomials have degree less than or equal to l and the derivatives are of order k.

So this is the shifted partials space have shifted partials of f and the rank of this gives you a new measure.

So in terms of matrix you can think of these operate this thing as an operator indexing the rows this is a derivative operator times monomial times of derivative operator indexing the rows and the columns are just all possible monomials and an entry in the matrix is the corresponding coefficient of this shifted portion, so the rank of this matrix is what is the measure. Is that clear?

Student-Professor Conversation Starts: is d and e are in some way related, **Professor:** no they are independent. **Student**: Okay, so how are you indexing the rows of the matrix? all possible? Professor: all possible $\overline{e}$, all possible $\overline{d}$. $(\overline{e}, \overline{d})$ is tupple that indexes. **Student-Professor Conversation ends**

So we are essentially taking the combination so that is why this ideal notation it is almost at a module it is almost a vector space I mean this, this set it is a vector space but it is almost an ideal. It is not exactly an ideal because you are not allowed to multiply by degree l + 1 monomials.

You can multiply by any polynomial upto degree l but you cannot multiply by degree l + 1 polynomial. It is definitely a vector space but not an ideal that is the problem so it is a new object. It is harder to analyze and the dimension of this is, the dimension of shifted partials is denoted by $\Gamma_{k,l}(f)$. So every polynomial f is associated with a number so this is the measure well you have to put some bound.

Student-Professor Conversations starts: I don't get the intuition of why multiplying with monomials of deg only $\leq d$. Professor : If otherwise it is an ideal and the dimension of ideal is infinite. **Student:** That is fine, multiplication. Will it be clear later? **Professor:** I cannot promise that it will clear later that depends on the audience. **Student-Professor Conversations ends.**

But you will see calculations that is all I can promise in the calculations you will see that both k and l will be utilized if you take l to be 0 then obviously this is a measure already used heavily in the previous proofs with l positive you can set l quite large or moderately large and get new lower bounds.

But it will not be sufficient for general lower bonds so it worked for some models. So one thing that motivates this is his powers so if you take f to be let us say Q^a with the degree of Q something small. So instead of looking at $Q_1 \cdots Q_a$, first look at the case of Q^a where Q is low degree. Let us say Q is quadratic, I mean Q you can also take linear so b can be 1 but then it is a very easy case because when b is 1.

What is the derivative space of f what is the dimension of derivative space? It is just a, so for b = 1 you can see the derivative space is enough, so in the case of b = 1 just looking at this suffices you do not need l positive. Because the derivative space dimension is already a but when b is as soon as b is quadratic when Q is quadratic or more then actually you can see that if Q was let us say $(x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2)^a$, when you start applying the derivative operator it seems to be useless because it keeps giving you newer and newer things. So you want to utilize or you want to show that Q^a is not a general polynomial just by differentiation just by the dimension of derivative space but that will not work. So Q^a with the derivatives the dimension can be as high as a general polynomial f.

But, now if you introduce l so you start multiplying by monomials you will see that they will be some advantage. So in this case l positive distinguishes f from general f. So this special f, Q^a, so this has to do with the kind of repeated roots Q^a has repeated roots in a way multiplicity a root and when you differentiate and multiply by monomials up to a certain degree the dimension will be lower than if there were no repeated roots.

But the only way to see this is by doing a calculation. This is not, this may not be immediate in case you have a background of algebraic geometry this is already done in Hilbert polynomial. So Hilbert polynomial definition actually uses these ideas so shifted partials, you can think of as motivated from Hilbert polynomial. , but if you have not seen Hilbert polynomial before then this is of no help, then you have to wait for the gory details in the calculations.

So the one thing to remember is this measure is motivated from Q^a in general it will work well when b is small and well both a and b are small when a as a and b. So I have to change this update, so this we will not work with this but we will work with this $Q_1 \cdots Q_a$. So when both a and b are small then this product is a special product but if a is large or b is large then obviously this product is not special, then it is just a general polynomial so, this will be this will be the measures limitation that you cannot hope to go to general polynomial or well big a or big b because then the polynomial becomes arbitrary, but as long as a small and b small even when one of them is small. So a small means that no but both of them have to be small so both a and b , so when both a and b are small then in some sense it is a special polymer.

It is a special product and some of these products you can hope to analyze or understand so that it works well with that so when a + b is small then this is expected to work well. Since you are trying to compute degree d polynomials, remember that ab is around, you would expect ab to be around d, so then the smallest a + b we can get a $\sqrt{d}$. So that is the maximum you can hope to get out of this measure.

We will get the maximum so we will see that strong lower bounds can be shown in all these cases but then it will stop and it does not move any further. So the matrix form is the matrix whose rank we are looking for is so you have, here you have the operators and the rows you have operators in the columns you have monomials and in the entry you will have the coefficient of that monomial in the operated f.

So the rows are the index set you can think of as $x^{\leq l} \cdot \partial^{=k}$ that is the index set of the rows and the index set of the columns is n variate monomials of degree. So what can be the maximum degree of operated f, so you started with degree d then you differentiated by degree k. So it is d – k but then you are multiplying by monomials up to l, so these are n variate columns are indexed by n variant monomials of degree $d-k+l$.

So as you change k l this matrix changes and its rank also changes and then you have to so we want to take a special look at this rank and then take a generic f look at the rank and then we want to compare this generic rank with the specific or special rank. So if the special rank is much smaller than the generic rank then we have gotten lucky we will be interested in those models, so rank of a matrix means that this measure is sub additive.

So $\Gamma_{k,l}$ is sub additive and why is that is because this operator $\overline{x}^{\overline{\alpha}} \cdot \partial_{\overline{\beta}}$ on $f_1 + f_2$. ∂ is a linear operator and obviously also monomial multiplication is a linear operation. So you get this $\overline{x}^{\overline{\alpha}} \cdot \partial_{\overline{\beta}} (f_1 + f_2) = \overline{x}^{\overline{\alpha}} \cdot \partial_{\overline{\beta}} (f_1) + \overline{x}^{\overline{\alpha}} \cdot \partial_{\overline{\beta}} (f_2)$. Since this operator is distributed in addition and then you are looking at the rank of this matrix. So matrix for f_1 matrix for f_2 the rank sum will be an upper bound on the polynomial $f_1 + f_2$ measures. Is the definition and this property clear? So then we will go to something which ties with the motivation what is special about this measure.

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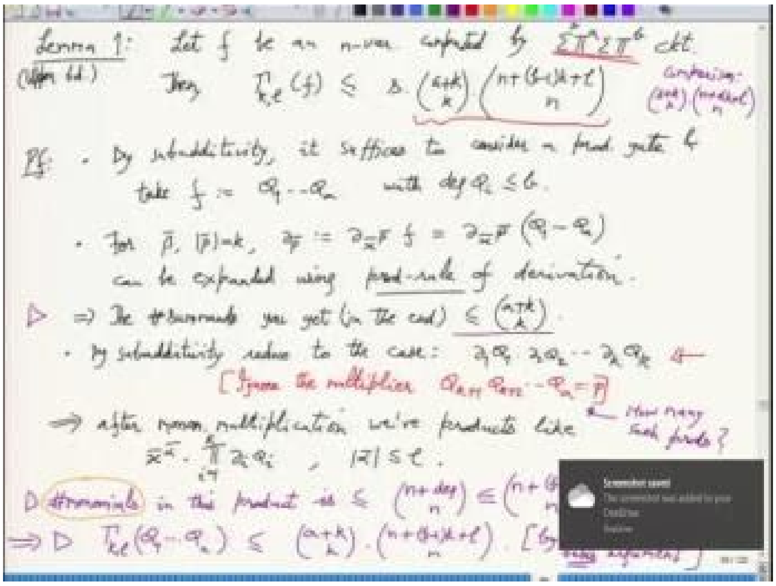

So that will be a counting calculation of monomials so let f be an n variate. So basically the upper bound that we always do this will be the upper bound lemma on the model measure on the model. So let f be an n variate computed by $\Sigma\Pi^{a}\Sigma\Pi^{b}$ circuit, then the measure for k, l parameters $\Gamma_{k,l}(f)$ is upper bounded by s top fanin let us say. so we are looking at s products so because of sub additivity s is there and it boils down to the measure on a product is $Q_1 \cdots Q_a$ product so that is upper bounded by, we will show :

$$\Gamma_{k,l}(f) \leq s \cdot \binom{a+k}{k} \cdot \binom{n+(b-1)k+l}{n}$$

Do you think that this is special, when a and b is, a and b are small? if a and b were not small then it would have been then $\binom{a+k}{k}$, itself is will be $\binom{n+k}{k}$ or $\binom{d+k}{k}$ in the second binomial coefficient also you will have this $\binom{dk+l}{n}$. So instead of d now since you are assuming $\sqrt{d}$ This may change the number a lot.

It does change the number a lot so this measure upper bound is significantly smaller than the generic measure value or the maximum measure value maximum over arbitrary f. So this model seems to reduce the measure a lot for ab and it may not be immediately clear that you get this. So we have to prove this so once we have proven this then we are in we have some hope that this will take a general polynomial, let us say determinant or permanent and show that the measure is really large on that and when we compare with this estimate it will give an exponential lower bound on s that will be the overall strategy.

So by sub additivity it suffices to consider a product gate and take $f = Q_1 \cdots Q_a$ with $\deg(Q_i) \leq b$ and then we will be proving this upper bound without the s, so let us do that. For $\overline{\beta}$ the derivative operator k orders, by $\partial_{\overline{\beta}}$ are what do we mean we will mean:

$$\partial_{\overline{\beta}} := \partial_{\overline{x}^{\overline{\beta}}} f = \partial_{\overline{x}^{\overline{\beta}}} (Q_1 \cdots Q_a)$$

We are differentiating by this monomial $\overline{x}^{\overline{\beta}}$ which is $Q_1 \cdots Q_a$. The derivative operator is being applied on a product so what do you do? You expand it as a sum so this can be expanded using the product rule of derivation. So when you do that now this is complicated because $\overline{x}^{\overline{\beta}}$ is actually a monomial; it is not just a single derivative operator by a single value.

So you are actually applying this let us say ∂_{x_1} or ∂_{x_i}, you are applying this operator many times to get $\overline{x}^{\overline{\beta}}$ and every time you get a you are applied invoking product rule so the sum will be quite big. So how many summands do you see after applying after invoking these product rule of derivation again and again. So if there was only 1 variable and degree 1, k was 1, then you would have gotten a summands.

So, $\binom{a+k}{k}$ so the number of summands you get, so this is upper bounded by $\binom{a+k}{k}$ and so what happens in summands, in our summands you have many Q_i's on which differentiation does not happen you take a product of those and then there are these remaining Q_j's or Q_i's differentiation is happening take that product. So that gives you a summand, again by subadditivity reduced to the cases where you are applying $\partial_1(Q_1) \cdot \partial_2(Q_2) \cdot \partial_k(Q_k)$; by ∂_1 I mean derivative with respect to x_1 . That is kind of the most typical case that you picked k of these Q_i and you differentiated each of these by the simplest partial derivatives and took the product.

Student- Professor conversation starts: So ∂_1 and ∂_2 could both be in x_1, right? **Professor:** Sorry? **Student:** ∂_1 and ∂_2 could both be in x_1. **Professor:** No ,by ∂_i I mean ∂ with respect ∂_{x_i}, th, **Student:** But the derivative is with respect to $\overline{x}^{\overline{\beta}}$ **Professor:** Sure, but then this is the worst case. **Student- Professor conversation ends**

So we reduced to this case the counting and moreover here we have ignored the Q_i's which were not differentiated; so ignore the multiplier $Q_{k+1} \cdot Q_{k+2} \cdots Q_a$.

So there is this part that remains differentiated and derived in a summand because you have only k simple derivatives. So they will act only on k of the Q_i's and the remaining which is a – k will remain untouched. But we are ignoring this multiplier for now. Well, they will also multiply and produce monomials so is it fair to ignore them?

Student-Professor conversation starts: For degree small we can right? **Professor**: No nothing is no those things are not small. **Student-Professor conversation ends**.

The reason is that this contribution is already counted in $\binom{a+k}{k}$, so in the end it will already be counted within that count so for now we ignore this we just focus on this. Yes, in the end you will see that this was justified so you have this product and you multiply by a monomial. So after monomial multiplication we have products like $\overline{x}^{\overline{\alpha}} \cdot \prod_{i=1}^{k} \partial_i Q_i$ where $|\overline{\alpha}| \leq l$.

So, the number of monomials here is so the number of monomials in this product is less than equal to. So this product when you expand it out how many monomials do you get, what is the support size? So there are n variables, so this will be whatever is the degree of this product $\binom{n + deg}{n}$ is an obvious upper bound. So, if you look at the degree upper bound that is l + b – 1 so the support size of this product is at most $\binom{n + (b-1)k + l}{n}$

Is this clear? So now these 2 purple conclusions do you see that you get the lemma statement using both of these. So the number of these is a - k Q_i's that remain undifferentiated that is counted under $\binom{a+k}{k}$ and when we fix 1 of such sub products what is happening in the in Q_1 to Q_k the remaining Q_1 to Q_k is differentiation a lot of differentiation on a lot of monomial multiplication which produces at most these many monomials right so if you multiply the number of these monomials with $\binom{a+k}{k}$ you get the obvious upper bound on rank.

So both these observations imply the observation that $\Gamma_{k,l}(Q_1 \cdots Q_a)$ this is upper bounded by $\binom{a+k}{k}$ that is the number of a - k of the is that you will not touch and the remaining k you will differentiate and multiply by a monomial producing at most these many things:

$$\Gamma_{k,l}(Q_1 \cdots Q_a) \leq \binom{a+k}{k} \cdot \binom{n + (b-1)k + l}{n}$$

So we have not gone completely down to the level of monomials because we are also keeping track of these Q_{k+1} to Q_a product. So the remaining part we have gone down to the level of monomials so monomials times these products of a - k and we have that many such products and their linear combinations will give you any polynomial in the space of shifted partials.

So this is a basis of shifted partials by shifted partials basis argument. So we have actually identified a basis and the upper bound on the size of the basis is at least this when your f is $Q_1 \cdot\cdot Q_a$ and you are in the last step you multiply by s.

Student-Professor conversation starts: I don't get the part Q_{k+1} to Q_a part. So it is accounted in how many such products are there, how many such products are there. But, first you count the number of summands, then you look at each summand. Each summand looks like product of $\partial_i Q_i$ times Q_k k-i many things and then you want to count the number of, the dimension of, the number of monomials that span all of them. You want an upper bound on that space. So the point is, if you did that, you have to account for Q_{k+1} upto Q_a right? because they would increase your second thing, number of monomials in this product, that upper bound and that space would be, I don't know, another b times k -1 the degree.

Professor: But on Q on this product let us call it P no operator acts once, you See that this is my P then P is divide of any operation so why should it contribute anything. Once you fix P when you fix P the operators are not acting on P the operators are acting on P complement. So what is that action on P complement that we have accounted for P is not playing any role now, P will just play a role by how many P's there are. That is $\binom{k+a}{k}$. **Student:** You can choose the P in $\binom{k+a}{k}$. **Professor:** that is why you are multiplying, **Student:** No, no. P is fixed for a particular summand right? **Professor:** what do you mean by summand? Studen: When you write $\partial_{\bar{x}^{\bar{\beta}}} Q_1 \cdot\cdot Q_a$ apply the product, you get a lot of summands. That count is $\binom{k+a}{k}$ **Professor:** You collect all those summands where P appears, **Student:** But then their summands are divided into blocks right? different blocks for different P's. **Professor:** Collect all the summand were which are divisible by P in that what is the basis? Student: In that, that would be the basis. Professor: you get $\binom{n+bk+l}{k}$ **Student:** But then there is different P. **Professor:** So how many pieces are there, that is $\binom{k+a}{k}$. We are identifying a basis this way it may be an overestimate, but that is fine.
Student-Professor conversation ends.

I think it is not bad overestimate can get close to this number, we are overestimating because we are seeing number of monomials in this product this, we are seeing number of monomials in this product that much of basis is needed but why basis as he is saying that it may even be 0. So just by giving this support size as an upper bound for a product is an overkill but I think overall if you see as you go over all $\overline{\alpha}$'s and $\overline{\beta}$'s .

I do not think it is so bad, I think this is quite close to the correct bound in general. So as already mentioned, this bound is good because you should compare it with the, should compare it with the $\binom{d+k}{k}$ and $\binom{n+dk+l}{n}$. For a general polynomial you would have gotten that and here you will get for ab something significantly smaller and you also have a flexibility of changing k and l.

So in the end it will be an elaborate calculation to actually fix k and l, fixing k and l is not obvious. So it will really depend on the intricacies of what 2 expressions are you comparing and then you have to optimize k and l so many of these things are actually difficult to predict at this point, it just gives you new space to study.

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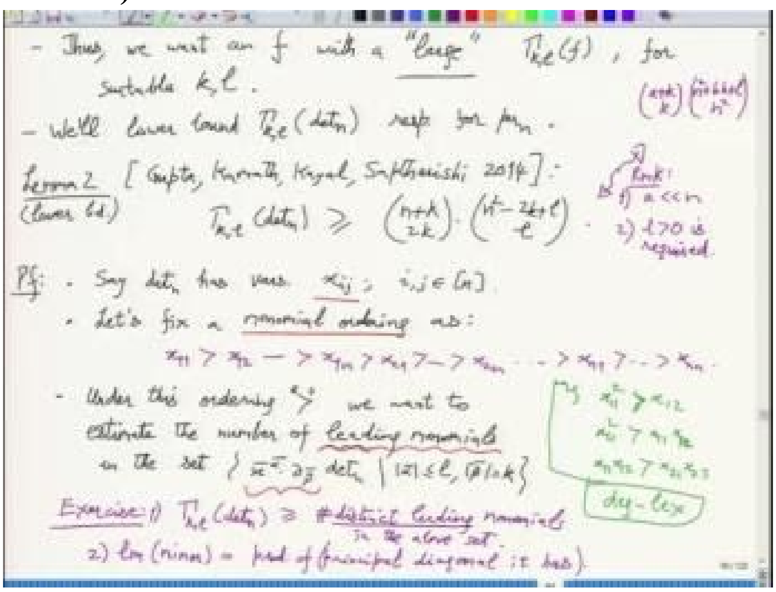

So now we want an f with a large gamma so relatively large $\Gamma_{k,l}(f)$ for suitable k l parameters. So it should be larger than the bound that we have just shown up and that will suggest and ultimately prove that f does not have that expression. So yes there is no surprise that we will take f to be determinant and we will show this. So we will lower bound $\Gamma_{k,l}(det_n)$, determinant on $n \times n$ matrix n^2 variables and respectively for permanent.

The proof will be applicable to both so that is the statement so working with the determinant this is due to GKKS [Gupta, kamath, kayal and saptharishi]. So that is the lower bound so they showed that $\Gamma_{k,l}(det_n) \geq \binom{n+k}{2k} \cdot \binom{n^2-2k+l}{l}$.

So how does it compare? So what did we have before? The upper bound we had was this. Now we want to compare these 2. Now the number of variables has become n^2.

And degree of determinant is obviously n so if you allow a to be n then actually they are the they are very close then you will not get anything special for the model. But if you allow it to be significantly smaller than n then you can see that the first term in this Lemma 2 first term in lemma 2 can be significantly larger than what we got in lemma 1 so the remarks are 1 is that take a to be significantly smaller than n.

And second is that if you take l to be 0 then in lemma 1 this estimate is very large, because the second term is $\binom{n^2+bk}{n^2}$. So it will be a huge term on the other hand in lemma 2 for l = 0. The second term is nothing, it is just 1 so for l = 0 you will get nothing so l positive is required. Now once it l has to be taken at least 1 then it becomes a question which l to pick, so it will be an optimization from k.

So we have to take, we have to optimize k and l such that the ratio of these 2 gives you the best result is that clear? Yes,that is the quantitative motivation for shifted partials. Since you want to show this ran to be large for determinant you will obviously use the fact which we have seen before that the derivative of determinant is a minor or it is 0 so we have to, we are basically computing.

We are looking at these miners and we are multiplying the miners by monomials that is the LHS and rank of this then has to the rank lower bound will follow from identifying a basis. So we will say that okay this is the set of miners which are linearly independent so we have to identify these miners. So it will be a specialized proof that may not finish today. Let us start. It is not very difficult but it is intricate.

So say determinant has variables $x_{ij}\quad i,j\ \in [n]$ and so let us fix a monomial ordering. So monomial ordering because this basis that we will produce of minors will try to produce the leading basis. So for that we need a monomial ordering which will give the most weight to x_{11} that is the leading variable, then the next is x_{12} and so on. So the first row is the leading row and then you go to the next second row.

Finally the last key that gives you these lifts to a strict monomial ordering all the monomials are distinct and ordered completely and under this ordering " > " we want to estimate well maybe you have not seen monomial orderings before. So once we have ordered the variables how will you compare 2 monomials, so how do you compare ${x_{11}}^2$ with x_{12}. So I should also say degree, so this this will be deg-lex.

So first you look at the degree and then you look at the order of these alphabet letters. So ${x_{11}}^2$ will be considered bigger than x_{12} and if you have ${x_{11}}^2$ and $x_{11}\,x_{12}$ then will x_{11} is common so remove that. So you basically have to compare x_{11} with x_{12} and that is defined so it is bigger or you may have $x_{11}x_{12}$ and something very different like $x_{21}x_{23}$. So what do you do in this case?

So in this case x_{11} is bigger than both x_{21} and x_{23} alone and the degrees also match so this has to be greater so that is the standard way so when we want to give a monomial ordering we just give it in the most basic in the base case which is variables and that lifts so this is called deg-lex. That is the way to order monomials you can also show that this is the only way I mean if you know.

So degree assumption we can assume it is natural to say that a bigger degree monomial is bigger but assuming that that axiom the only way to order the monomials is by lex you have to order the variables and then you have yes that is that it is again a natural action. I mean so both these things are very natural so with those 2 axioms you can see that it suffices to work with variables. Yes, a monomial ordering always means what Abhibhav said that if $m_1 > m_2$ then any multiple $m_1 \cdot m > m_2 \cdot m$. so that is the definition of monomial ordering in fact and usually we also respect the degree so then you just have this way. The number of leading monomials in the set $\overline{\{x}^{\overline{\alpha}} \cdot \partial_{\overline{\beta}}$ acting on $det_n | \, |\overline{\alpha}| \leq l, \, |\overline{\beta}| = k\}$. So apply all these operators on determinants you will get polynomials which are essentially products of minors.

And suppose we identify, so every polynomial has a leading monomial since, the monomials are ordered. look at the highest monomial. So basically we are mapping every this operated determinant maps to a monomial so we have a set of monomials and how many are distinct. It suppose we know that t of them are distinct so will this give you a lower bound? How is this related to the measure?

So take it as an exercise that if you show that the number of these leading monomials is t, then the measure is at least t in fact something stronger is true that is then you have the if 2 polynomials have leading monomials different then they are linearly dependent and if a set of polynomials have distinct leading monomials then they are linearly independent. So that immediately gives you a rank lower bound.

So we will use that, will invoke that so $\Gamma_{k,l}(det_n)$ is greater than equal to the number of distinct leading monomials in the above set. Yes we have to now identify a number of distinct leading monomials here like. So if $\overline{\alpha}$ was not there if you were only differentiating then you are basically getting these minors and the easiest way to show that the miners are linearly independent is by looking at the leading monomials.

So in a minor what do you think is the leading monomial? The principal diagonal. So that is another property worthy of an exercise. So lm of a minor is product of principal diagonal. So wherever the minor is it has this principal diagonal and just multiply these entries that is the leading monomial that is how the monomial ordering is, that is by design. So that is a small exercise and now but we do not have minor.

So we have minor multiplied by a monomial so what is the leading monomial of that, exactly, so natural for people that monomial times principal diagonal that is the leading monomial. So we have identified leading monomials for every operated determinant. So, we have identified these and then. So once you have seen this showing that these things are distinct or a lot of them are distinct. That it will boil down to studying basically $\bar{\beta}$. $\bar{\alpha}$, $\bar{\beta}$ how is this or we will identify $\bar{\alpha}$ and $\bar{\beta}$ such that these leading monomials are a lot that will give us the lower bound.