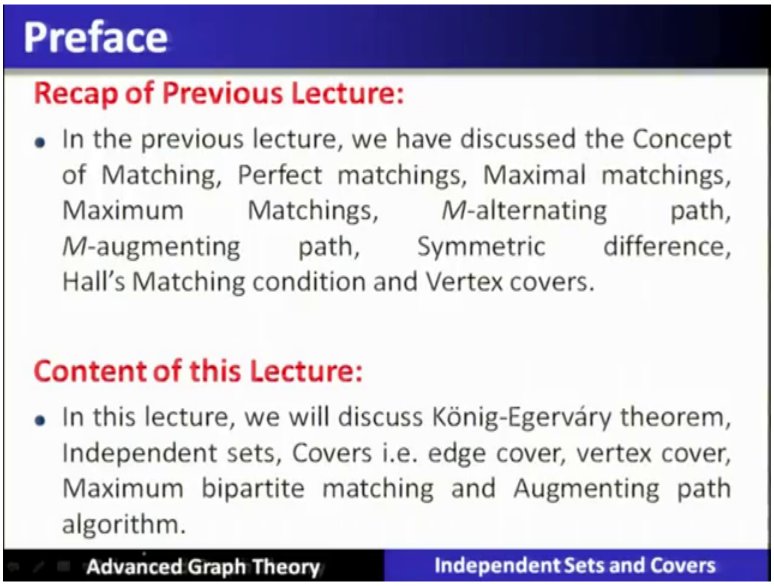
Advanced Graph Theory
Dr. Rajiv Misra
Department of Computer Science and Engineering
Indian Institute of Technology, Patna

Lecture – 08
Independent Sets and Covers

Independent sets, covers, maximum bipartite matching.

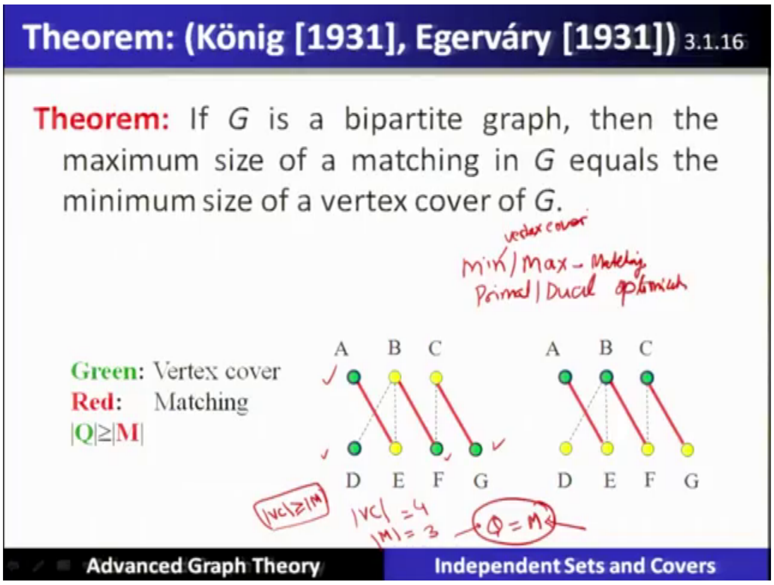
(Refer Slide Time: 00:19)



Recap of previous lecture, in previous lecture we have discussed the concept of matching, perfect matching, maximal matching, maximum matchings, m alternating path, m augmenting path, symmetric difference, halls matching condition and vertex cover. In this lecture we will discuss Konig Egervary theorem that is for the connection between the maximum matching and vertex cover.

We are going to also see independent sets, covers, for example, edge, vertex cover both kind of covers we have will be including up, maximum bipartite matching algorithms we are going to see in this lecture, one such algorithm that is called augmenting path algorithm we are going to cover up.

(Refer Slide Time: 01:13)

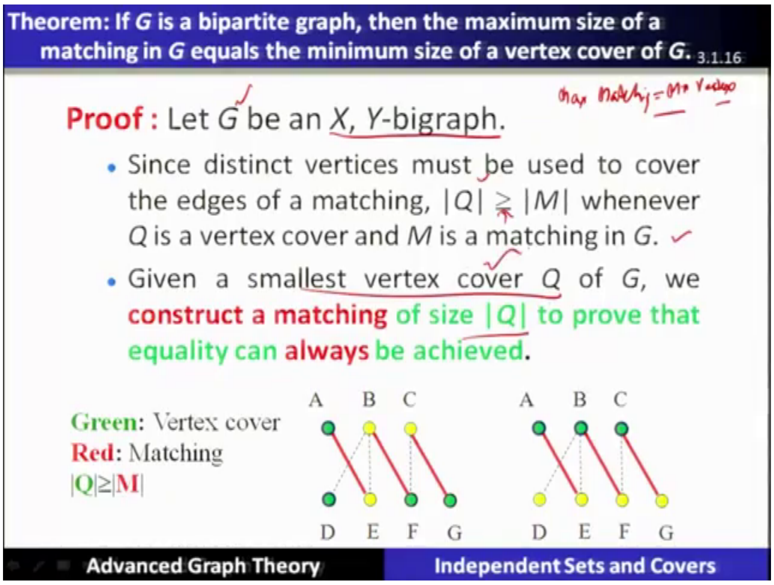


Theorem, Konig Egervary theorem, the statement of theorem says that if G is a bipartite graph then the maximum size of matching in G equals the minimum size of the vertex cover. In the last lecture we have discussed the implication of this particular theorem on a bipartite graph and we have seen that in this particular figure where in the vertex cover is denoted by the green color vertices, here the size of the vertex cover is 1 2 3 4 in this figure and the matching is represented as the red color that is 3. So, vertex cover size is greater than or equal to here maximum matching.

So, matching basically now; so, we are going to see this particular relationship between the vertex cover and the maximum matching in the condition when it reaches an optimum value. So, optimum value means, the maximum size of the matching will equal to the minimum size of the vertex cover so; that means, Q is equal to M and that condition when it arises then the matching which is obtained will be the maximum matching and so, basically if you are given with a minimum set of vertex cover of a graph then we can identify the matching of a maximum size and that is given by the Konig Egervary theorem.

So, this particular theorem is also called min max or a primal dual optimization problem, where maximization is in the terms of matching and minimize in the terms of vertex cover. So, using this particular condition we will see the maximum matching algorithm which will utilize Konigs Egervary theorem or first we will see the proof then we will see the algorithmic proof that is called augmenting path algorithm to find out the maximum matching in a graph. So, this particular theorem has two proofs one proof we are going to give now and after that the algorithm which we will discuss that will be algorithmic proof or this particular theorem.

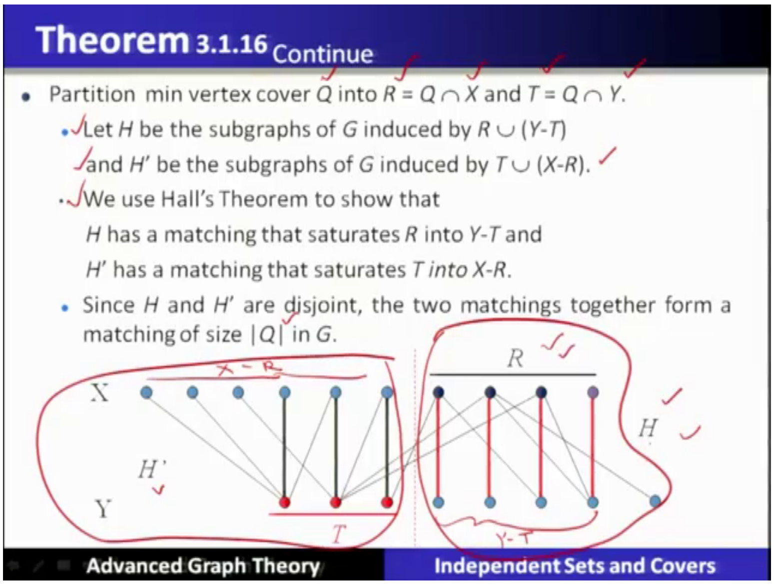
(Refer Slide Time: 03:52)



So, the theorem says that if G is bipartite graph, then maximum size of a matching in G equals the minimum size of the vertex cover. So, maximum size of a matching equals the minimum size of the vertex cover. Let us see the proof of this particular theorem. So, to prove this theorem let us assume G to be a bipartite graph having a X Y bigraph or X Y patrician.

Now since the distinct vertices must be used to cover the edges of the matching therefore, the size of the vertex cover denoted by capital Q cardinality is always greater than or equal to the maximum or equal to the matching whenever Q is basically the vertex cover and M is the matching. Now given the smallest size of the vertex cover; that means, if we reduce the size of the vertex cover; that means, we are going for this particular condition it becomes equal. We construct a matching of size Q to prove the equality always be achieved. So, let us construct the matching if we are given the smallest vertex cover.

(Refer Slide Time: 05:22)

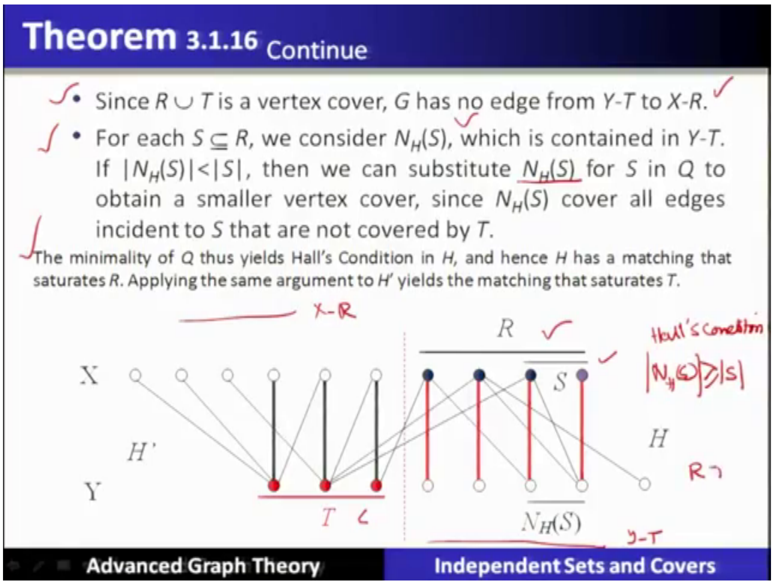


We will partition the vertex cover which is given into two sets R and T. So, R will be in the partition X and T will be in the partition Y let us see in this particular figure.

So, T is in the partition Y and this R will be in the partition X and together T and R they will basically form the total vertex cover. So, these are the vertices which are colored as red in T and blue in R they will from the minimum vertex. Now having partition into R and T, now then let H be the sub graph of G induced by R union Y minus T. So, R union means R and this becomes Y minus T. So, this part of the graph we call it as H, similarly H prime graph will basically also a bipartite or it is a induced subgraph of G that is called as H prime will have T union X minus T union X minus R.

So, this becomes this is R. So, this becomes X minus R this becomes T. So, this becomes. So, we have now divided the graph into two subgraphs, one is called H the other is called H prime and now we are going to use the halls theorem to show that this H has a matching that saturates R into Y minus T and also H prime has a matching that saturates T into X minus R. Now since H and H prime they are disjoint subgraphs, the two matchings together form a matching of size Q in G. So, the matching in H and H prime together if we combine that will be the matching of size Q in the graph, that is how we are going to construct the matching.

(Refer Slide Time: 07:58)

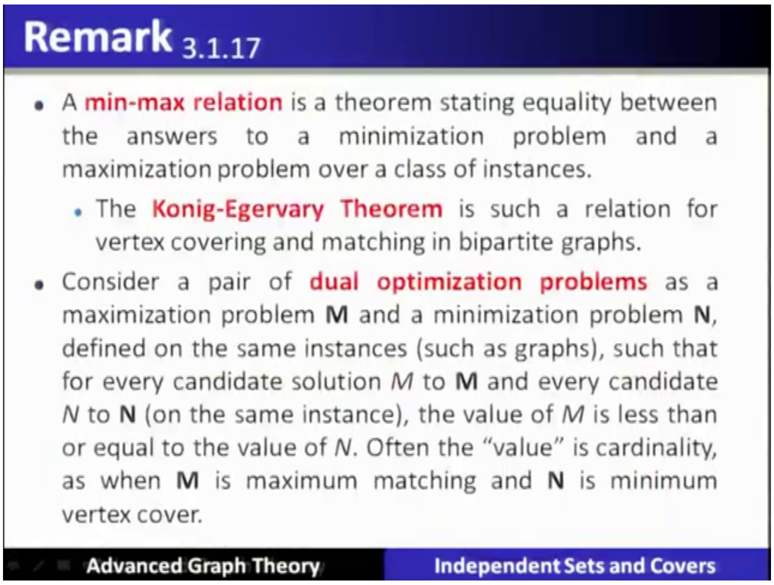


Now, let us see the next step since R union T is the vertex cover. So, G has no edge which is which basically runs from Y minus T; that means, from here and X minus R; that means, here there is no edge which will go from Y minus T to X minus R, that particular condition is clear now, why because R union T is a vertex cover; that means, it will be covering all the vertices. So, no more additional edges are required further to be included in that cover Q why because it is a minimum cover.

Now, for each subset of R, here S is a subset of R, we consider the neighbor of this particular S which is contain in Y minus T that we know. Now if this particular neighborhood relation size is less than S, then what we will do we can replace this particular neighborhood N of S in Q to obtain a smaller vertex cover. So, that is way we are going to construct a smaller or a smallest vertex cover which will satisfy N H of S which is always at least and hence we do this to satisfy the halls condition.

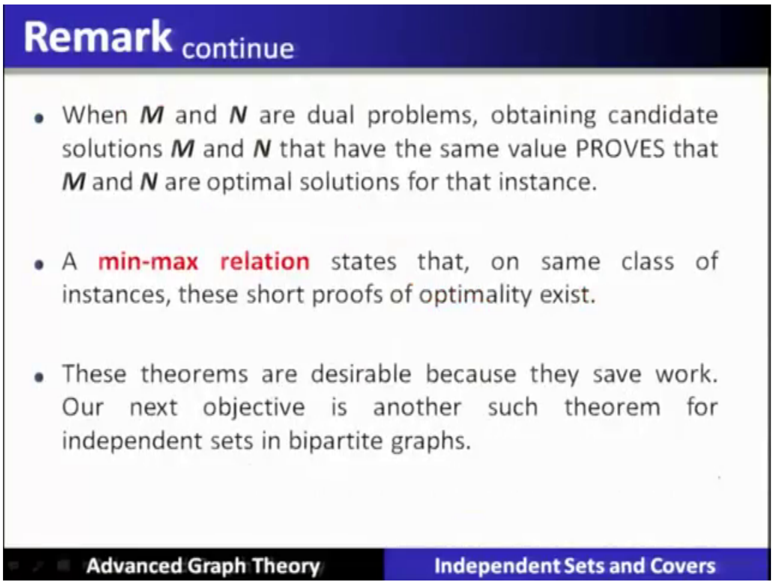
Now, you know that halls condition will saturate this particular R through the matched H. So, the minimality of Q thus yields the halls condition in H here and hence H has a matching that saturates R. So, applying the same arguments in H prime, we will see that they exist a matching which will saturate T. So, R and T they are basically being saturated in this particular construction and hence this basically once we identified this kind of vertex cover of a minimum size then it will generate the maximum matching of that size Q that is shown over here which will saturate R and T.

(Refer Slide Time: 10:49)



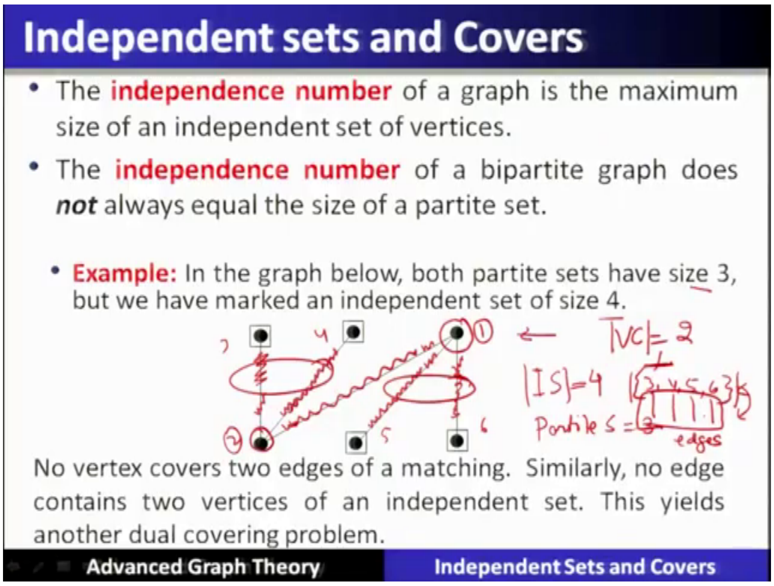
Now, in this theorem we have seen the min max relation. So, minimization of a vertex cover will give you the maximum matching and this called min max relation, is a theorem stating equality between the answers to the minimization problem and the maximization problem over a class of instances. So, Konig Egervary theorem is such a relation or vertex cover and the matching in a bipartite graph. Now consider a dual optimization problem as a maximization problem M and minimization problem N defined on some instances such that for every candidate solution M of to M and every candidate solution n the value of M is less than or equal to the value of N, often the value is cardinality as when M is maximum matching and N is.

(Refer Slide Time: 11:44)



The minimum vertex cover when M and N are dual problems obtaining a candidate solution M and N that have the same values proves that M and N are optimal solution for that instance and that we have seen in Konigs egervary theorem. These theorem are desirable because this save the work. So our next objective is another such theorem for independent sets in the bipartite graph.

(Refer Slide Time: 12:12)

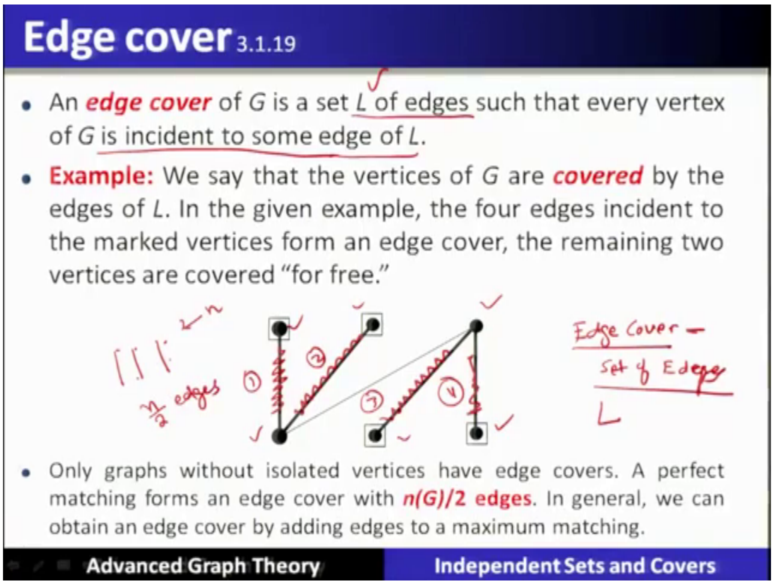


Independent sets and covers. So, independence number of a graph is the maximum size of independent set of the vertices and independence number of a bipartite graph does not always equal to the size of the partite set. In this example here the both the partite sets are of size 3, but the independent set if you see are the 4 vertices. So, independent set size is 4 whereas, the partite set is of size 3. Now here we will see that what is the vertex cover. So, if we include this particular vertex and this particular vertex, it will cover all the edges of this particular graph. Hence the vertex cover is vertex number 1 and vertex number 2 that is of size.

So, we will see that no vertex cover covers two edges of a matching, similarly no edge contains 2 vertices of an independent set for example, this particular edge will touch only 1 independent set, this particular independent set will be touched by another edge this independent set is basically touched by another independent set. So, set of independent sets 3 4, 5 m and 6 they are touched by the different edges, why because if there is an edge which also touches another set another nod in the set independent set, then it will not be independent set why because there will be an edge which is connecting it.

So; that means, the number of nods which are there in the independent set and that many number of edges are required to cover the size of the independent set. What about other edges? So, the other edges here in this particular example this particular edge is not basically touched by is not considered here which is touching the independence set, but this particular edge will be touched by one of these edges which are there present to touch the independent set. So, similarly no edge. So, this will yield another dual covering problem. So, these set of edges which will basically touch all the independent sets, they will form an cover which is called an edge cover.

(Refer Slide Time: 15:11)

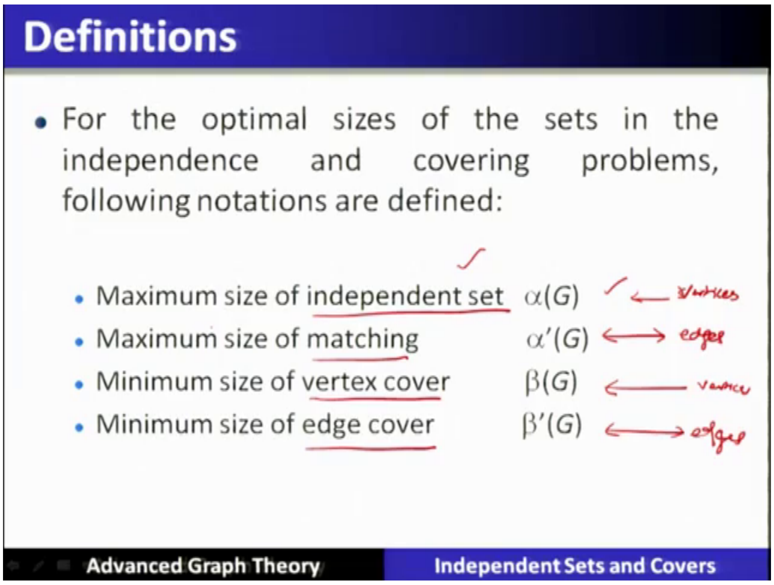


So, edge cover of a graph G is a set L of edges such that every vertex of G incident to some edge of G some edge of L. So, in this particular example let us say that this edge cover includes this edge, this edge, this edge and this edge. So, when we include all these 4 edges, what are they going to over they are going to touch all the vertices of a graph. So, let us see that this vertex all the independence vertices are basically touched by the 4 edges.

What about these 2 edges? They will be automatically be touched upon why because automatically be touched upon. Now, we will see why it is going to happen. So, these this is called the edge cover it is opposite to the vertex cover. So, the vertex cover is the set of vertices which will touch all the edges and here edge cover means the set of edges which will touch all the vertices called edge cover. So, edge cover is the set of edges in the graph and is represented by capital L.

So, when there is a perfect matching, perfect matching means all the vertices on both the sides they are being saturated. Then if let us say the graph is having n vertices. So, how many different what is the size of the edge cover will be having n by two different edges, which will give a perfect matching.

(Refer Slide Time: 17:13)

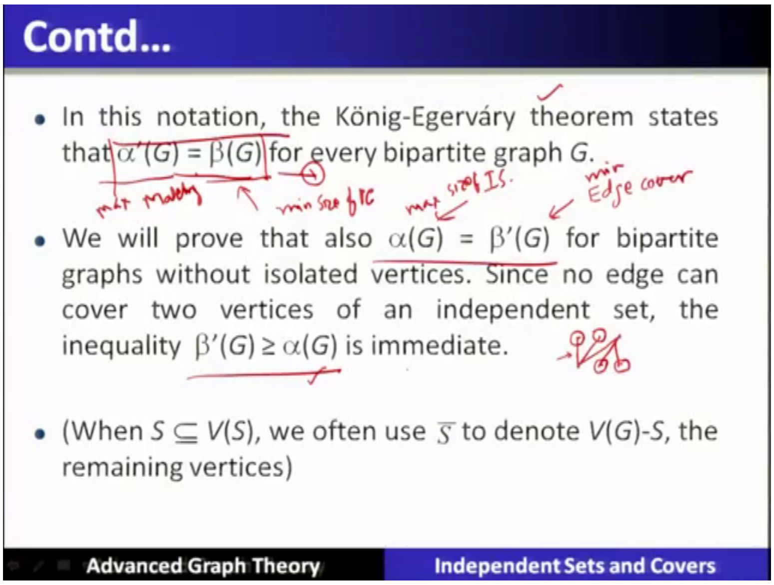


So, with this we are going to give some definitions and some terms also which we are going to use in the for the theorem.

So, for optimal sizes of the set in the independent sets and covering problem the following are following terms are defined. So, alpha G is nothing but the maximum size of independent set. So, this is the set of vertices. Then, minimum size of the vertex cover as I told you is vertex cover is a set of vertices which will touch all the edges which is also a vertices.

Then, the maximum size of the matching here this is nothing, but a set of edges and a minimum size of the edge cover this is also the set of edges. So, the edge cover problem is to minimize this particular set called minimum size of the edge cover, vertex cover we also going to minimize that is the minimum size of the vertex cover is important, matching we always maximize called maximum size of matching, independent set also we are trying to maximize called maximum size independent set. These will become the optimization problems and let us see their solutions.

(Refer Slide Time: 18:37)

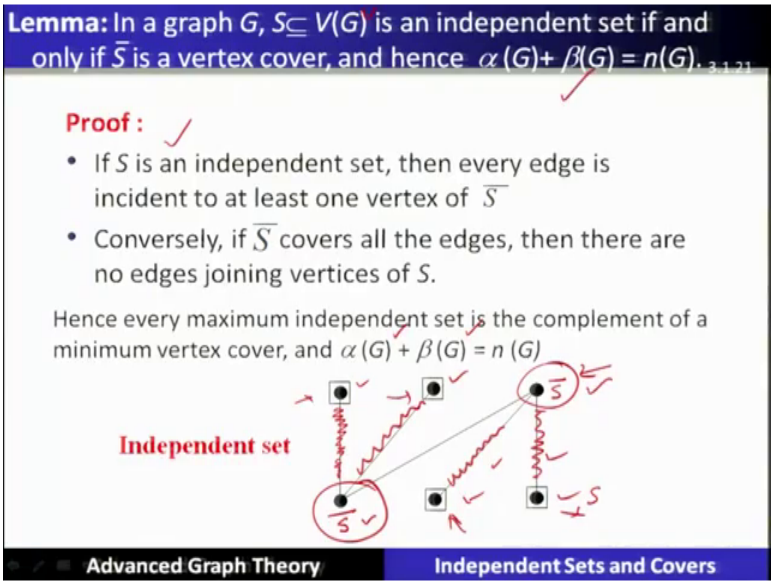


Now, having stated these notations and definitions let us again restate the Konig Egervary theorem. So, Konig Egervary theorem is the size of the maximum matching equal to the size of the minimum size of the vertex cover. So, beta indicates the minimum size of vertex cover and alpha prime G denotes the maximum matching. So now, Konigs egervary theorem we can write also in this particular manner, let us say that this is equation number 1.

Now, we will prove also that alpha G is equal to beta prime G. So, beta prime G is nothing, but the edge cover and this is the minimum size of the edge cover and here alpha G is an independent set, the maximum size of independent set. Now here we can see that no edge can cover 2 vertices of a independence set. So, take the example this is 1 independent set, another independent set, this is another independent set and one.

So, for every independent set, for every nod in the independent set there will be an edge. So, how many edges will be there in the edge cover at least that many number of edges will be there that is beta prime G has at least alpha G number of edges in it. So, hence this particular equality basically is established.

(Refer Slide Time: 20:40)

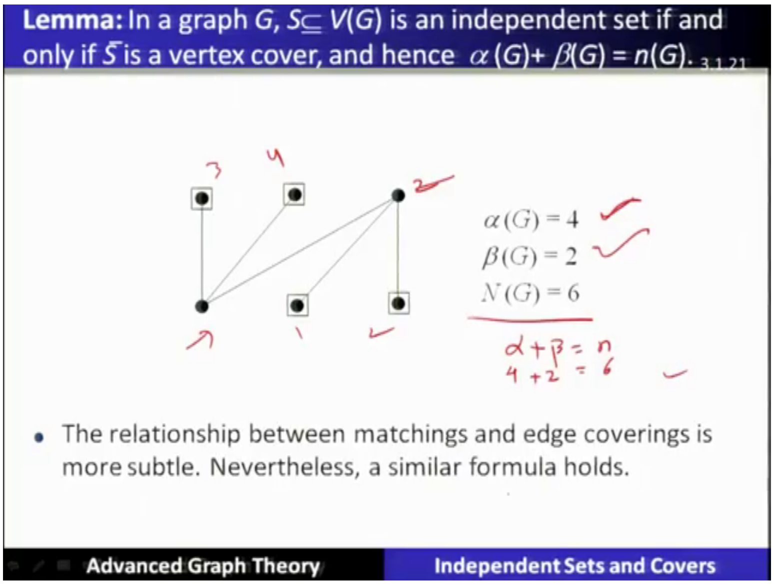


Now, there is a lemma which says that in a graph G, the subset of the vertices is an independent set if and only if the compliment of that particular subset S is a vertex cover and hence alpha G plus beta G is equal to the total nods in a graph. So, proof, let us see, now if S is an independent set which is a subset of vertices then every edge is incident to at least one vertex of S prime. So, let us see that this, this, this, this, these 4 are basically the independent set and this vertex belongs to the compliment of that independent set.

So, it says that if S is an independent set then every edge incident to at least one of S prime. So, you see that this particular edge which basically touches the independent set, it also touches S prime one of these similarly here also the same thing is happening. This edge is basically touching the independent set, it also touches the compliment, also touches the compliment, this also touches the compliment.

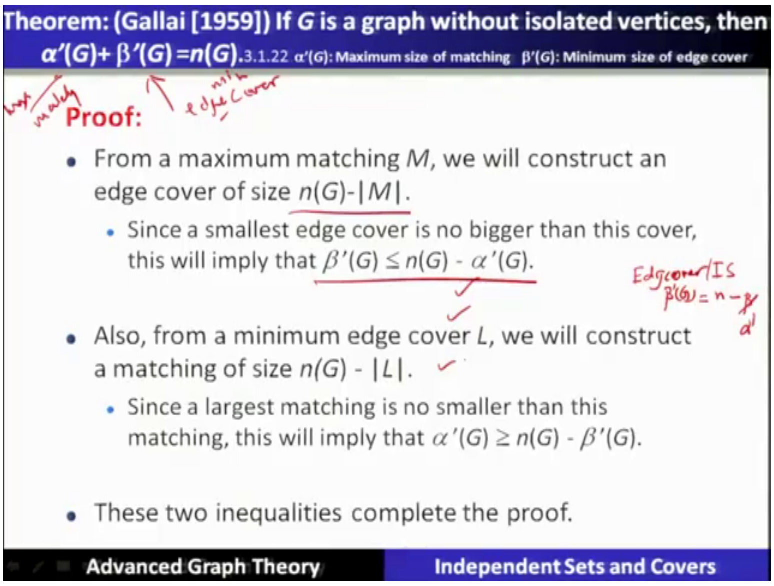
So, conversely if S prime covers all the edges S prime these basically set of vertices which covers all the edges, then no edges joining the vertices of S, hence they will becomes the independent set, Hence every maximum independent set is the compliment of the minimum vertex cover. So, here the vertex cover we have to see that alpha G is basically the independent set and beta G is basically the. So, this is basically the independent set plus this is the vertex cover. So, that becomes the total number of vertices in the graph and that is what we are going to prove by this particular process.

(Refer Slide Time: 23:11)



Now, in a graph G, the same things which we can see through the example. In this particular lemma, we can see that independent set size if 4, 1, 2, 3, 4 and the vertex cover size is 1 and 2. So, total if we sum them, alpha plus beta that becomes N and that is nothing, but a 6. Hence the relationship between the matching in the edge cover is more subtle now nevertheless the same formula holds.

(Refer Slide Time: 23:50)



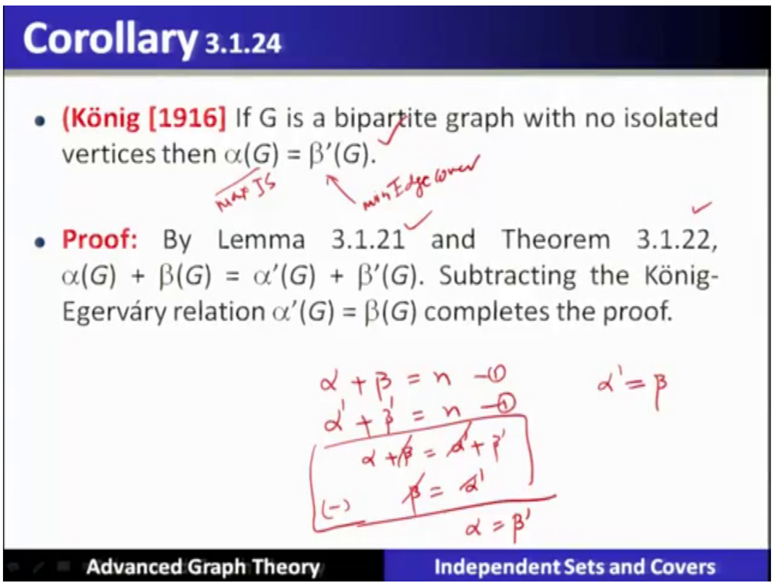
Now, we are going to see another theorem, we says that if G is the graph without isolated vertices, then alpha prime G plus beta prime G is basically the total vertices of a graph.

Alpha prime G is the maximum matching and beta prime G is the edge cover this is the minimum size, this is the maximum matching.

And now we are going to see the proof of this particular equation or a theorem. So, from a maximum matching n, we will construct an edge cover of size n minus M. Now since the smallest edge cover is no bigger than this particular cover. So, this will imply that beta prime G is at most n minus alpha prime G. So, again you can see that this particular inequality you have derived from the previous results, in the sense we have seen the edge cover that is, let us say that its beta prime G and we have also seen that this edge cover is equal to the independent set. we have also seen that independent set plus vertex cover becomes the total number of vertices so; that means, the edge cover equal to n minus the set of vertex cover and this vertex cover basically we can denote by beta.

Now, this vertex cover is equal to the maximum size of the matching. So, that becomes alpha G. So, hence beta G is less than alpha prime G and hence this particular inequality is established. Now, we can also see from minimum vertex cover we can construct a matching and that is quite obvious that is, if the edge covers are given then we can construct a matching of size n G minus one since the largest matching is no smaller than this particular matching. Hence this will imply that alpha prime G is at least n G minus beta G beta prime G where beta prime is basically the edge cover.

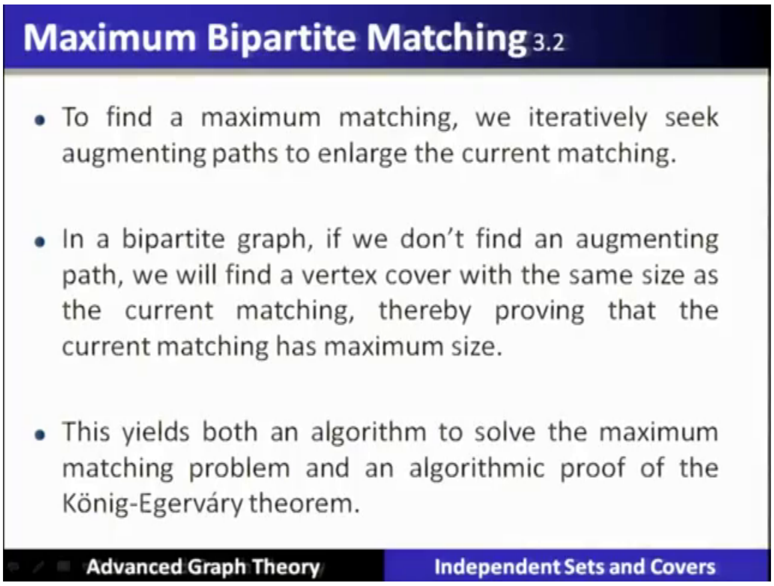
(Refer Slide Time: 26:37)



Now, corollary which says that if G is a bipartite graph having no isolated vertices then alpha G is equal to the beta prime G alpha G is basically the maximum size of the independent set and beta prime G is the minimum size of the edge cover. So, by previous lemma, we have seen that alpha plus beta is equal to n, also we have seen alpha plus alpha prime beta prime also is equal to n.

So, from this lemma and from this two theorems we have got this. So, we can write down alpha plus beta is equal to alpha prime plus beta prime, Now Konigs egervary theorem in previous we have seen that alpha prime is equal to beta, then what we are going to do is we can subtract alpha prime and beta from this particular equation. So, when we subtract we will obtain alpha is equal to beta prime and this basically is a corollary which will complete this proof.

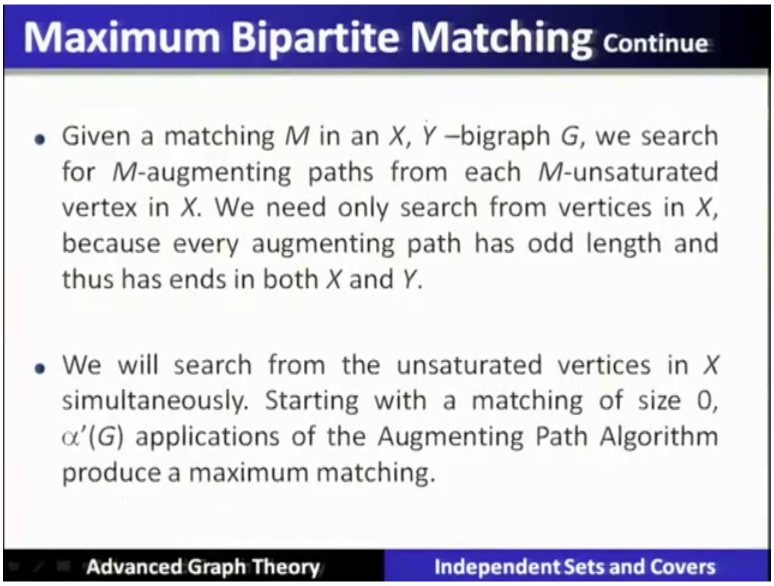
(Refer Slide Time: 28:01)



Now, maximum bipartite matching, maximum size of bipartite matching; that means, the theorem Konig Egervary theorem states that maximum matching is achieved when there is a minimum size of the vertex cover also is achieved, hence its min max theorem we are going to use over here as a algorithmic proof this algorithm we will see. So, to find the maximum matching we iteratively seek for augmenting paths and if we found the augmenting path we can enlarge the current matching.

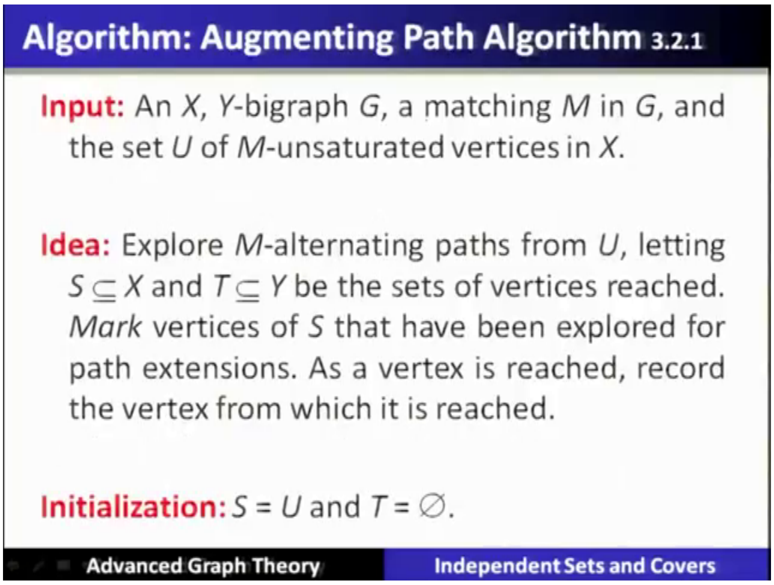
So, in a bipartite graph if we do not find augmenting path then we stop at that point and we will conclude that we have found the vertex cover with the same size as the matching and thereby proving the Konig Egervary theorem that the matching has of the maximum size. So, this yields both the algorithm to solve maximum matching problem and algorithmic proof or Konig Egervary theorem.

(Refer Slide Time: 29:11)



So, we will see this is the first algorithmic proof in this course. now given a matching M in X Y bigraph we search for m augmenting path from n unsaturated vertex in X we need only search from the vertices which are unsaturated in X because every augmenting path has an odd length and thus has ends in both X and Y. We will search from unsaturated vertex in X simultaneously starting with a matching of size zero alpha prime G applications of augmenting path algorithm will produce the maximum matching.

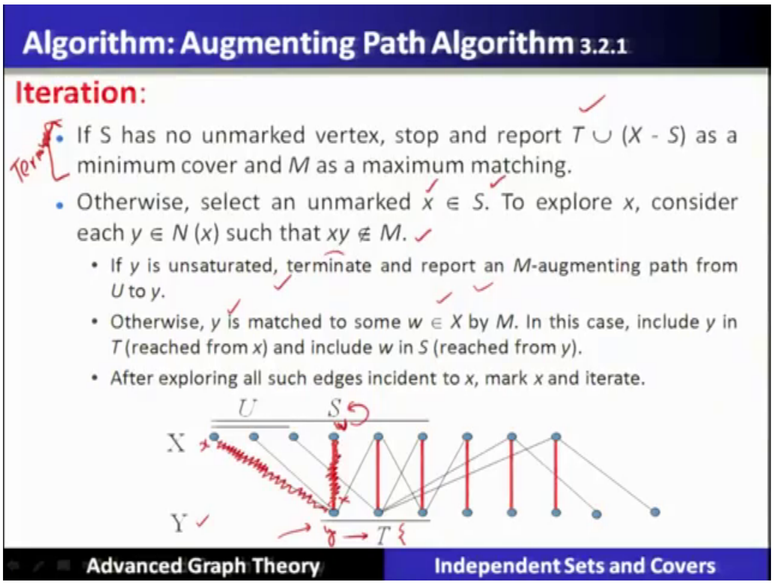
(Refer Slide Time: 30:02)



So, augmenting path algorithm input is X Y bigraph, also input is the matching M in the graph is given and also the set u of M unsaturated vertices. The idea of this augmenting path algorithm is to explore M alternating path starting from unsaturated vertex, U letting S which is a subset of the vertices X and T which is a subset if the vertices from Y be the set of vertices which are reached by this M alternating path starting from U.

Mark the vertices of S that how been explored for the path extension as the vertices is reached record the vertices from where it is reached and let us see the initialization where S initialize to u; that means, initially all the vertices are unsaturated and T becomes empty why because we have not visited any vertices.

(Refer Slide Time: 31:13)



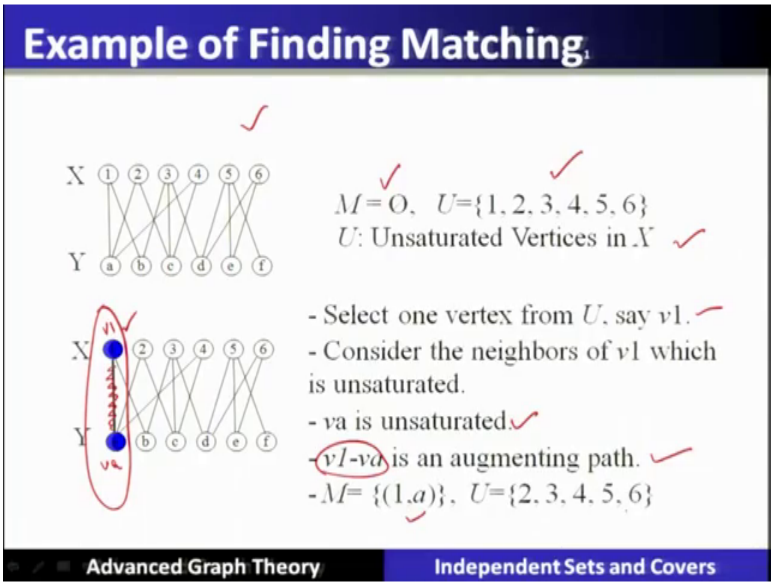
Now, the iteration of this algorithm if S has no unmarked vertex, then we will stop and report T union X minus S as the minimum size of the vertex cover and M as the maximum matching. So, this is the termination condition of the algorithm otherwise we have to select an unmarked vertex let us say X which belongs to S to explore X consider each Y which is a neighbor of X such that X Y is not in matching.

So, it is nothing, but if you recall it Is basically a to explore augmenting path starting from unsaturated nod X. Now if Y is unsaturated then terminate and report an augmenting path from u to Y otherwise Y is matched to some X by M in this case include Y in T reached from and X and include W in S reached from Y after exploring all such edges incident to X mark X and iterate. Again I am explaining let us see that this is the set of nods out of S which are unsaturated.

So, let us say that X belongs to U and which is unmarked. Now, we will take this is unmatched edge we will take this edge and reach some of the nod in Y and let us call this Y which is a neighbor of X here and this particular edge X Y should not be in the matched edge. So, the matching which is shown as the red color it is not one of them. So, we can reach from X to Y via unmatched edge.

Now, if Y is unsaturated, now here it is saturated, Y is matched to some W which is there in X, let us say this is W and it will reach through a matched edge to a W, this is not a possibility why because this is not a matched edge. So, in this case include Y in T. So, T will be including these particular nods which is reached from X and include w also into S and so on we will explore. Now, if we reach to a unsaturated vertex here; that means, if you start in the beginning it will be unsaturated then Y will be, then we have identified the augmenting path and we terminate this particular augmenting path algorithm. So, we have to basically call this augmenting path algorithm how many times that we are going to see in this particular example.

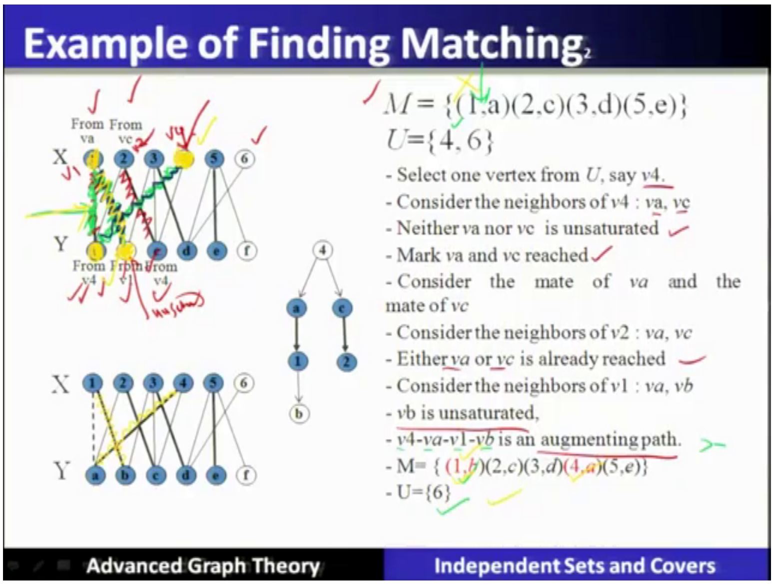
(Refer Slide Time: 34:13)



So, if let us say that in the beginning we are given X Y bigraph. X contains the nods which are labelled as 1, 2.3, 4, 5, 6 Y contains a b c d e f. Now initially there is no matching. So, M is initialized to M T and all the vertices of X are in U, that is all are unmarked unsaturated. Now, we will select 1 vertex from U. Let us say it is vertex v1. Now consider the neighbor of v1 which is basically unsaturated when we reach to a vertex a this also is unsaturated.

So, hence this particular v1, va is an augmenting path and this algorithm will be terminate and then again we will include this particular edge in the matching and also we will modify U, why because 1 will be removed from the current U. So, it will be now 2, 3, 4, 5, 6, So, we will continue doing this.

(Refer Slide Time: 35:23)



Let us see that we reach to a stage where the matching has reached 4 different edges, 1, 2, 3, 4 which are shown as the bold and let us say U is having 4 and 6 which is unsaturated. Now, we will select one vertex from U, let us assume we have selected 4 and we call it as v4.

Now, if we see the neighbors of 4, there are only 2 neighbors, one is a, the other is c. So, these a and c, they are saturated, they are not saturated. So, neither of them are unsaturated. So, we mark va and vc they are reached. Then consider the mate of va and the mate of vc; that means, via a matched edge we can reach to 1, similarly from c using matched edge we can reach to 2 and we have to write down from where we are reaching to these vertices of X and that will be.

Now, consider the neighbors of v2 this is v2 which are va and vc. So, either vc or va they are already reached. So, nothing new will be explored further from this v2. Now, we will see the neighbors of v1. So, the neighbors of v1 is a we have already explored and b where b is unsaturated. Hence, we have identified starting from unsaturated vertex and we have reached to a unsaturated vertex and that is called a augmenting path. .

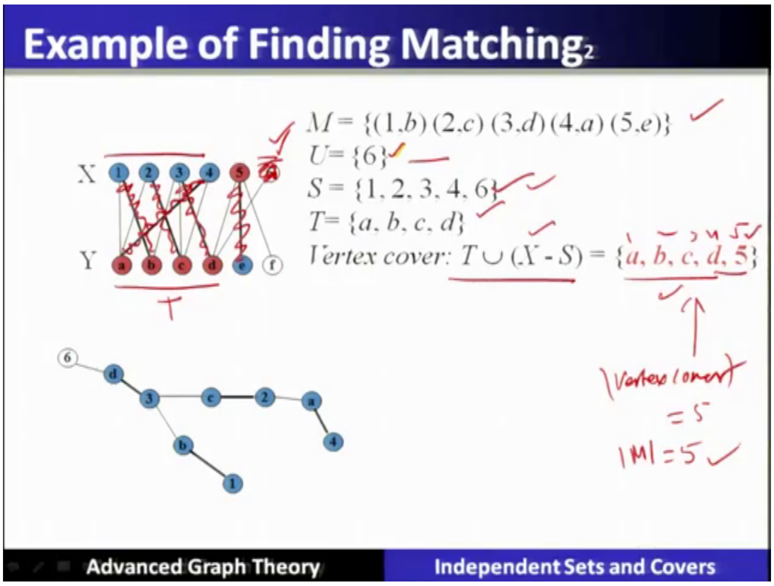
.So, if you trace this augmenting path, which we have identified like this we started from v4, then we have reached to va and from va we have reached to v1 and from v1 we have reached to. So, this is an augmenting path which is shown as the green color and once the augmenting path is identified, then the algorithm terminates and then as per as the matching is concerned we have to modify the matching. So, that the next iteration will began.

So, now we will mark 4 as the nod which is now is saturated. So, it will be out of this box. So, only the unsaturated nod remains is 6. As per as the matching is concerned. So, now, we can see that we have we have to modify this particular matching paths so; that means, the earlier was 1 and a that particular path will be removed, why because this will no longer be in a matching and a new particular edge 1b, this particular edge will be included, similarly another edge which will be included is 4a this edge.

So, again if you recall. So, 1bis this edge and four a this edge is included and what are the other edges which are there is 2c which were there in earlier. So, now, in this particular new figure we can see that 1b is added and 4a is added in the matching and this particular path which is starts from v4; that means, this particular nod is saturated, why because it has a matched edge it will go to a that also is the saturated nod, then it will go to 1 via unsaturated. So, it was there earlier.

So, now, it has to be removed from M. So, it will become unsaturated edge then again we have to include this edge, why because this will be a matched edge that is from 1 to 1 to b that we have included. So, we starts from unsaturated nod here and we have reached to unsaturated nod here that is fully explained here in this particular example.

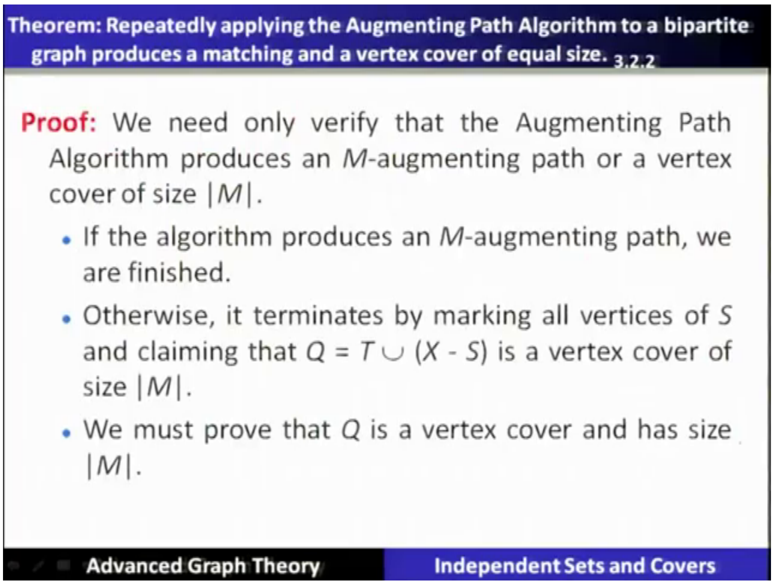
(Refer Slide Time: 40:06)



Now, in the last when we have only the remaining unsaturated nod that is nod number 6 and this is the matching which is given and shown here in the dark color and S is basically these set of vertices which is saturated by these particular edges. So, since 6 cannot be included, why because we cannot find an augmenting path. So, here we terminate, when we terminate the vertex cover will be T union X minus S. So, what is T, this is T and what is S, S is 1, 2, 3, 4 and 6, 6 also will be there why because there will be a path which will be identify.

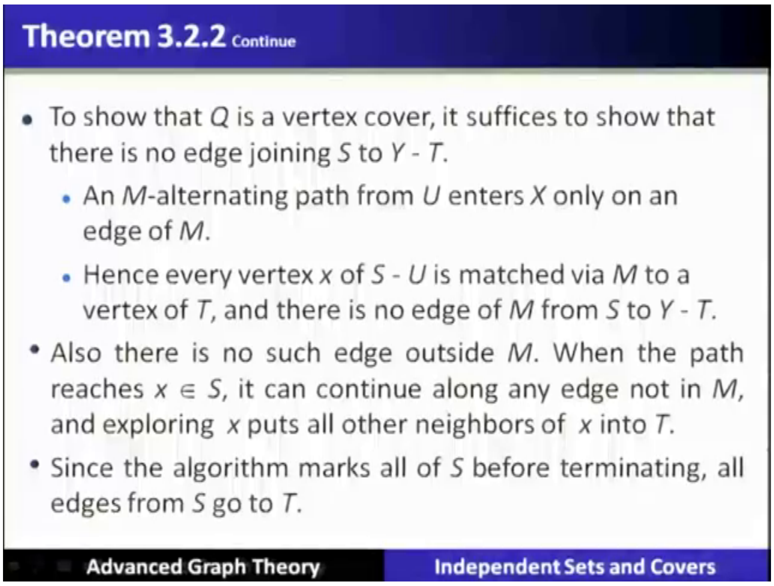
So, we will terminate then; that means, all the nods which are there in T that will be included and then X minus S; that means, the nods which are there in 1, 2, 3, 4, 6, 1, 2, 3, 4, 6 is also saturated will be included X minus S will become 5. So, 5 will be included. So, that will become the vertex cover. So, vertex cover is 1, 2, 3, 4, 5 different size of the vertex cover is 5. So, we have to find out we have also identified a matching of size 5. So, this is 1, 2, 3 4, 5. So, when the algorithm terminates we have already identified the 5 different matched edges.

(Refer Slide Time: 42:01)



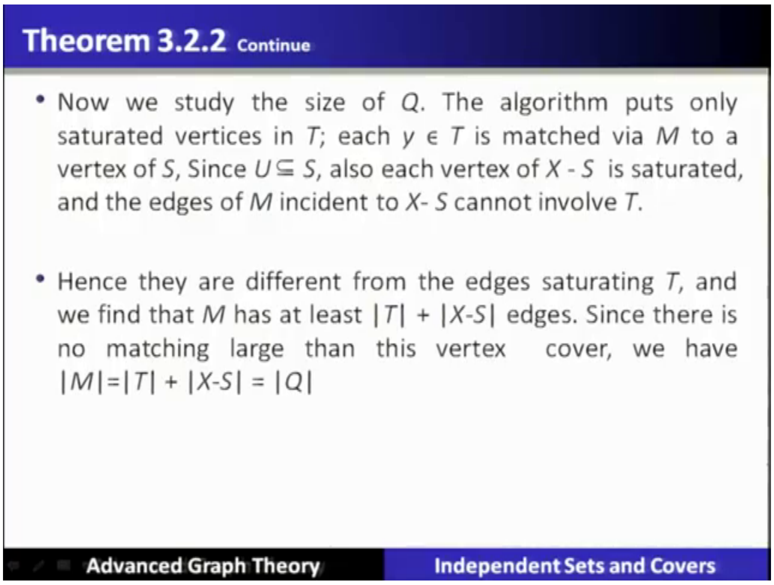
So, let us see the proof of this particular theorem, theorem says that repeatedly applying the augmenting path algorithm to a bipartite graph will produce a matching and a vertex cover of equal size let us see the proof. So, we need only to verify that the augmenting path algorithm produces an M, augmenting path or a vertex cover of size M because every iteration is doing this. So, if the algorithm produces M augmenting path then we are finished otherwise it terminates by marking all the vertices of S and claiming that Q is a vertex cover is T union X minus S is the vertex cover of size m. So, we must prove that Q is a vertex cover and has size M to show that the Q is a vertex cover, it suffices to show that there is no edge, which is joining S and Y minus T that we have already seen figures.

(Refer Slide Time: 42:51)



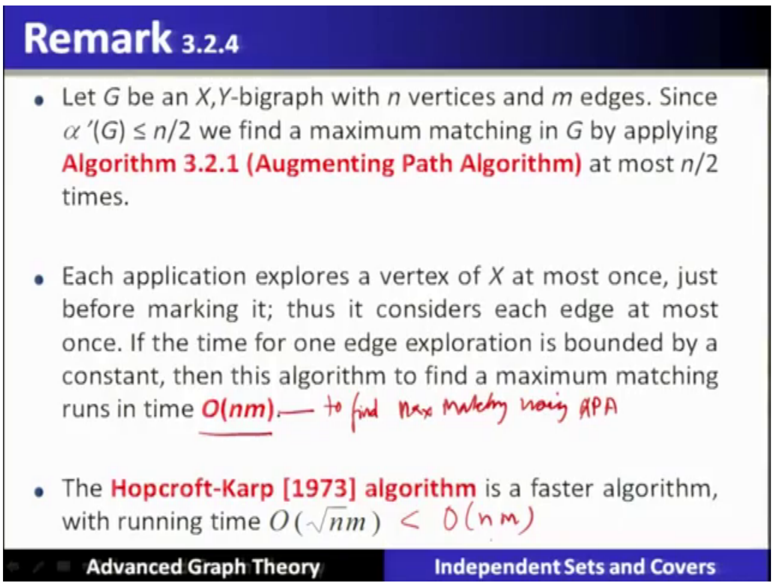
And now alternating path from u enters X, only on an edge in the matched edge. Hence every vertex of X of S minus U is matched via M to a vertex of T and there is no edge of M from S to Y minus T. Also there is no such edge outside m when a path reaches X it can continue along any edge not in M and exploring X puts all other neighbors of X into T ; that means, we are tracing back the augmenting path. So, whatever nods we are reaching we basically will form S and T those set of vertices as the vertex cover and also we are forming the edges which are matched.

(Refer Slide Time: 43:50)



So, now we are now we study the size of Q. So, algorithm puts only the saturated vertices in t. So, each Y which is there in T is matched via M to a vertex of S, hence U is a subset of S also each vertex of S minus X is saturated and the edges of M is incident to X minus S cannot involve T hence they are different from the edges saturating T and we find M has at least T, T plus X minus S edges. Since there is no matching large than this vertex cover. So, we have we have concluded that the cardinality of M is equal to cardinality of T plus cardinality of X minus S that is nothing, but a Q.

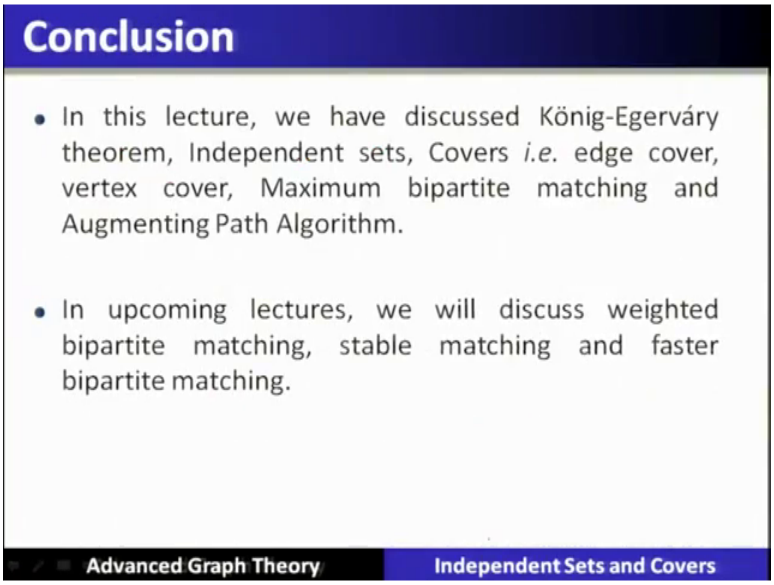
(Refer Slide Time: 44:40)



So, here we have seen that if a bigraph is given with a n vertices and m edges. Now, since alpha prime G that is the maximum size of the matching is always less than n by 2, why because if it is a perfect matching we have earlier seen that the size of perfect matching will be n by 2, if n different vertices are there. So, we find a maximum matching in G by applying this augmenting path algorithm, how many times, n by 2 times. So, each application explores a vertex of S at most once just before marking it. Thus it considers each edge at most once.

So, if the time for 1 edge exploration is bounded by a constant time then the algorithm to find out the maximum matching by applying, by running, by calling augmenting path algorithm n by 2 times total time will be order of n times m to find out maximum matching using augmenting path algorithms. There is another algorithm given by Hop croft and Karp algorithm this is the faster version of the algorithm which runs in a quicker time that is of the order root n of m which, which is the value which is less than this particular time.

(Refer Slide Time: 46:23)



So, we will see this algorithm in later classes conclusion, in this lecture we have discussed Konig Egervary theorem, independent sets covers, that is edge cover, vertex cover, maximum bipartite matching and augmenting path algorithm. The upcoming lectures we will discuss weighted bipartite matching stable matching and a faster bipartite matching algorithms.

Thank you.