Advanced Graph Theory

Dr. Rajiv Misra
Department of Computer Science and Engineering
Indian Institute of Technology, Patna

Lecture - 07
Matchings and Covers

Matching and covers content of this lecture.

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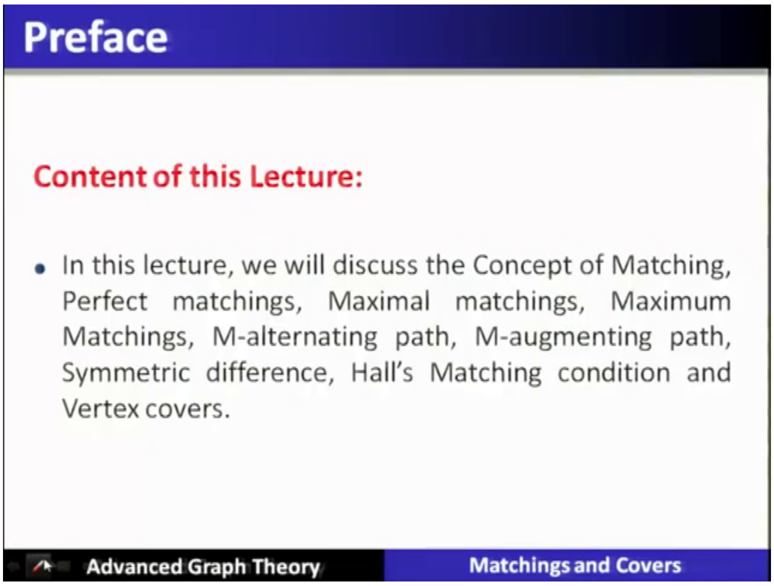

In this lecture, we will discuss the concept of Matching, Perfect matching, Maximal matching, Maximum matching M-alternating path, M- augmenting path, Symmetric difference, halls, matching condition and Vertex covers matching and covers.

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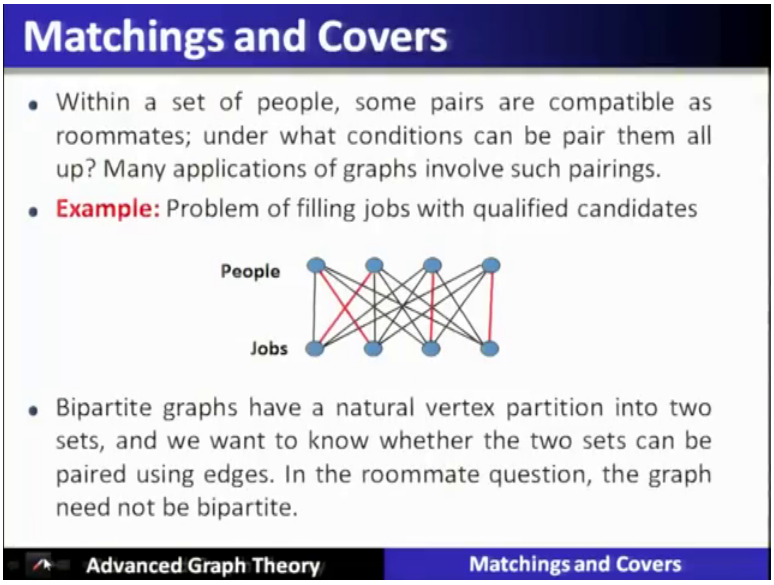

Within a set of people, some pairs are compatible as roommates; under what conditions can be pair them all? So, many such applications of graphs involve such pairing solutions. Take this particular example in which filling up the jobs with the qualified people can be represented. This particular problem can be represented with the form of a Bipartite graph, where set of peoples is a one partite set and the jobs is another partite set.

This particular graph will be found when we can see that there are people, who are not qualified for all the jobs, but for few of the jobs. So, basically the number of people are more than number of jobs, ideal in this particular situation. We have to find out, how these particular jobs are to be field with the qualified people. So, this particular problem setting basically assumes, the problem is to be specified in form of a bipartite graph, where in the set of peoples is one partite set, set of jobs is another partite set. So, given this particular problem setting, we want to find out the pairs.

So, pairs of these two sets are nothing, but they are represented in a form of the edges. So, we can see, here in this particular figure, the edges, which are in the red color. They will form the matching or pairing of people and the jobs. So, jobs are filled by only one person, which is represented with the help of these red lines. These red lines are nothing, but the set of edges. So, matching is nothing, but a subset of edges in a bipartite graph that we are going to explore in this lecture.

Similarly, another problem which is called a roommates the graph need not be a bipartite, yet we have to come up with a matching or a pairing of the roommates. So, those kind of problems; that means, the matching. In a general graph, we will see separately in a different lecture.

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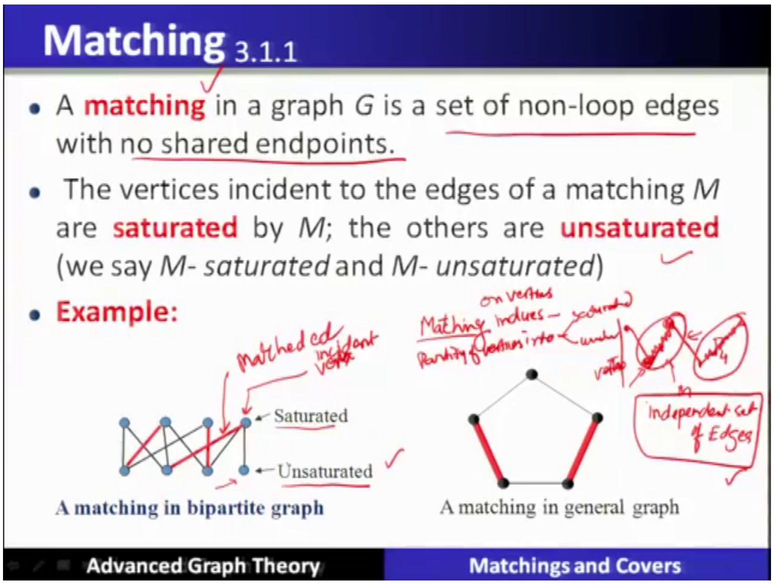

So, matching as I told you that, matching in a graph is a set of non-loop edges with no shared endpoints. Again, I am repeating matching in a graph is a set of edges, with no shared endpoints. So; that means, if a graph is given like this; that means, this particular graph is called P 4 having four vertices. So, one possibility of a matching is this particular edge and this edge will basically, touch these two vertices. This particular edge cannot be included. Why? Because, if you include this edge, they will shared the endpoints of already selected edge in M or a matched edge so, the set of edges with no shared endpoints will form a matching in a particular graph.

So, you can also say that, if a graph is given like this. So, these set of edges, which should not share the endpoints, they will form a independent set of edges. So, it is not a independent set of vertices, but it is a independent set of edges and that is called basically, the matching in a graph. Now, another thing point we have to see that this particular edge which we say that it is a edge in the matching set or it is a matched edge. So, this particular edge will incident on this particular vertex. So, the edges of a matching, when they are incident on this particular vertex that vertex is called a saturated vertex. Take this particular example, red color edge is a matched edge, when it incident on this particular vertex, this vertex is, will be saturated. This vertex will be saturated, the other vertex is on which? This matched edge is not incident, they are called unsaturated vertices.

So, the matching indices, matching on the vertices, indices, the partitioning of vertices into the saturated and unsaturated vertices so, if none of these edges, which are there in the matched edge, if none of these edges are incident, if no matched edge is incident on the vertex, then that vertex is called unsaturated and the vertices, when the matched edge is incident on a particular vertex that vertex is called a saturated vertex.

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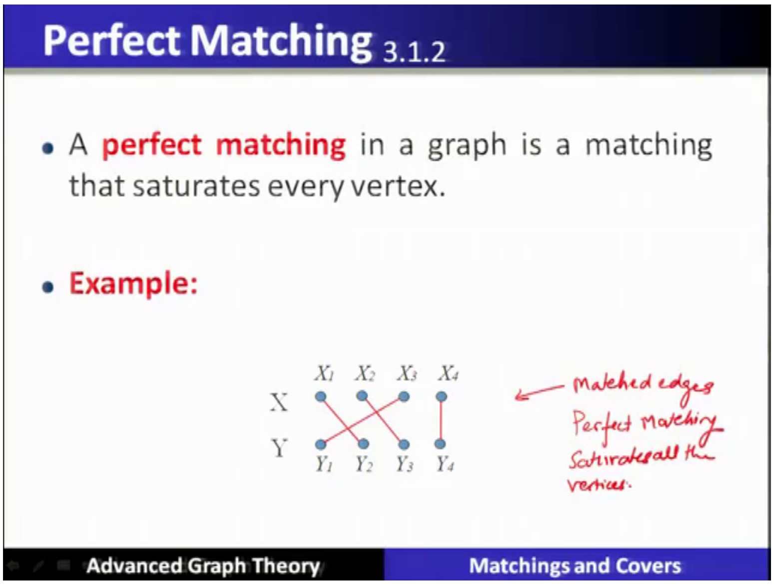

Perfect matching; the perfect matching in a graph is a matching that saturates every vertex, take this particular example graph. Here, you see the red colored edges are basically, the matching edge, matched edges. So, these matched edges are incident on all the vertices. Hence, this is, this kind of situation is called a perfect matching.

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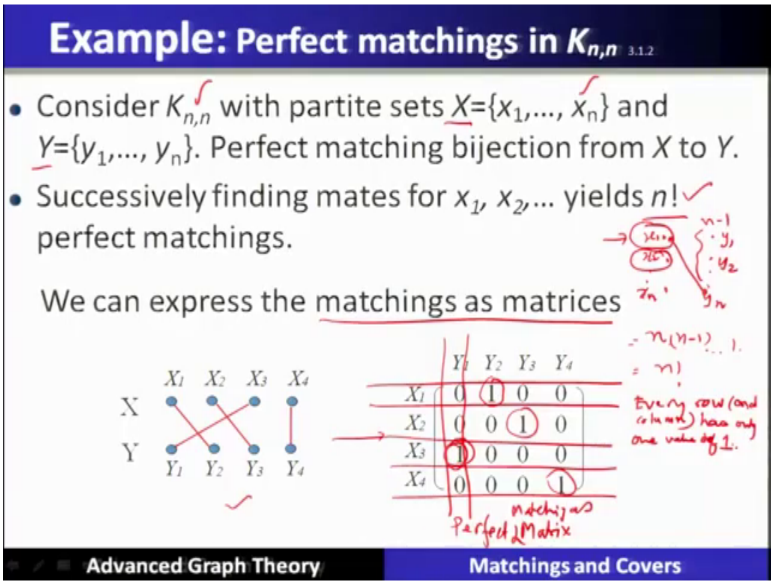

The example of a perfect matching, we can see in a complete bipartite graph K n, n having the partite set as X and Y, the perfect matching, you can see it is nothing, but a bijection from X to Y. Now, successively finding mates for X 1 X 2 and so, on up to X n will yield factorial n perfect different matchings. So, take this particular example. So, let us say it is X 1 X 2 and so on up to X n on the other partite set, you have Y 1 Y 2 and. So, on up to Y n, this X 1 and match any one of these n, different vertices on the other side. So, there is n possibility for X 1 to starts it's matching, having matched with a particular.

Let us say, a vertex on the other side the remaining, the second one X 2 can match the remaining one, which is n minus 1 and so on. So, the last one will require 1 n, this particular equation is n factorial. So, different perfect matchings are possible is n factorial, different perfect matchings are possible, we can also express this matching in a form of a matrix. For example, this is a perfect matching example; this we can represent in a form of a matrix, perfect matching as a matrix, in this particular matrix, if you inspect, what you will find that every row will have only one value of one. Similarly, this row also has one value of n 1.

Similarly, this one and this one in the perfect matching scenario similarly, in the column also you will see every column will have only one value of one. So, in the perfect matching matrix every row or the column and column has only one value of one. So, if a perfect matching is represented in a form of a matrix, then the rows, in the matrix, every row in the matrix will contain only one, one. Similarly, the columns in that particular perfect matching matrix will contain only one place at one.

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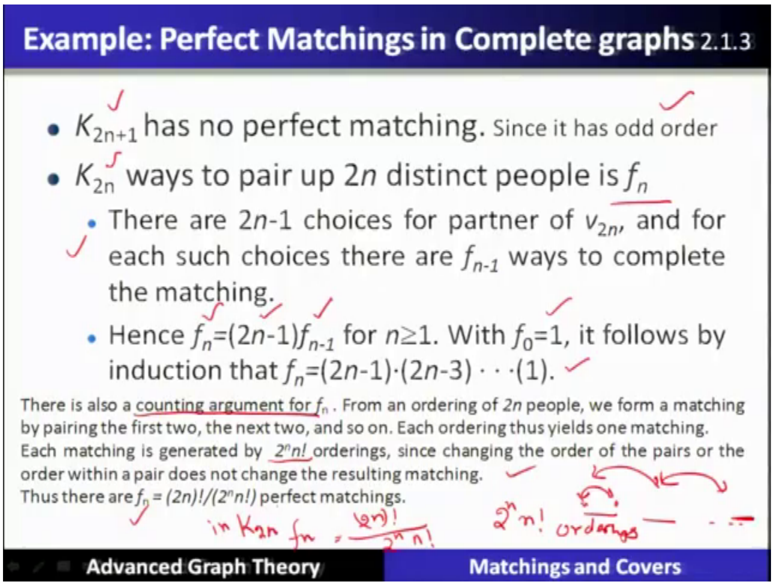

Now, let us go and see more details, if let us say it is a general graph that is K 2, K 2 n plus 1 that is, this particular graph does not have a perfect matching, why? Because it has an odd order. So, there is a possibility to have a perfect matching in a complete graph K 2 n, why? Because in K 2 n, there are different ways to pair up 2 n, distinct people 1 on 2 n distinct people and that is given by a recurrence function f n. So, f n is let us assume that will give you the number of ways to pair up 2 n distinct people.

So, for a particular vertex, there are 2 n minus 1, choices to partner of 2 n and each such choices. There are f n minus 1 ways to complete the matching. So, the first particular vertex will have 2 n minus 1, ways to do the matching and the next one will have the same, recurrence will run on n minus 1 different vertices on one side. So, knowing that f 0 is equal to 2, this particular induction, if we carry out to resolve this particular recurrence, it comes out to be 2 n minus 1 times, 2 n minus 3 and so on, up to 1.

Now, there is a counting arguments for this f n value. So, that says that the ordering of 2 n people, we form a matching by pairing the first two, the next two and so on, that I have already explained. Now, each ordering thus yields one matching, each matching is generated by 2 raise power n times n factorial, different orderings since changing the order of the pairs. So, if there is a pair, you can change the order of the pairs and also if these pairs are identified, then the order of these pairs also you can, you can change. So, that comes out to be 2 raise power n n factorial different orderings. Since, changing the order of the pairs or the order within a pair does not change the resulting matching. So, there are total numbers of perfect matchings. We can see is 2 n factorial divided by 2 raise power n n factorial different perfect matching in K 2 n graph.

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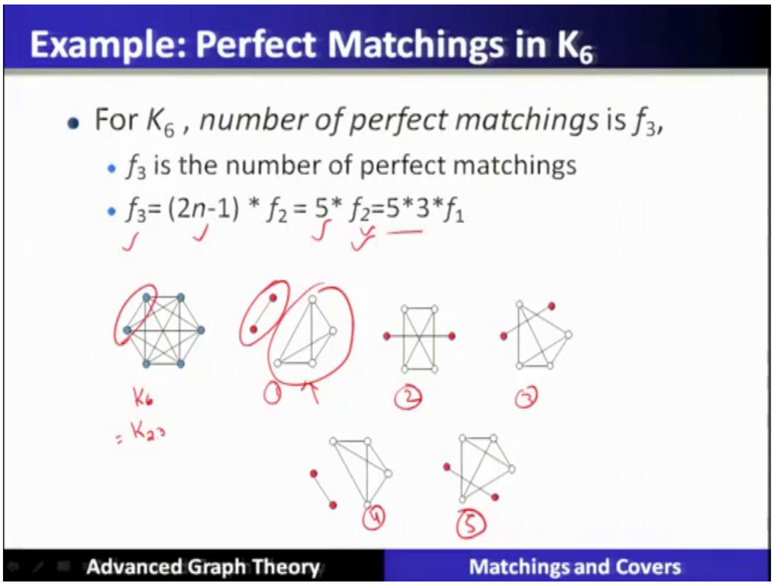

So, this particular example will this particular equation. We have obtain for example, this is K 6 or you can also say, it is K 2 times, K 2 3. So, now, we can see that, if we select this particular edge for example, there are 2 n minus 1 ways, there are five different ways. So, this is 1 way 2 way, you can select this edge 3 way, 4 ways and 5 different, because there are five different pairings possible in case 6.

So, there are five different ways, you can select. So, having selected these one of them, then the second one, we have to find out the perfect matching in the remaining portion of a graph which is nothing, but we have to select f 2; that means, two more pairing of these things and we have to run again on that particular resulting graph. Hence, we are going to get something of this nature 5 times, 3 times f 1 number of different possible perfect matchings.

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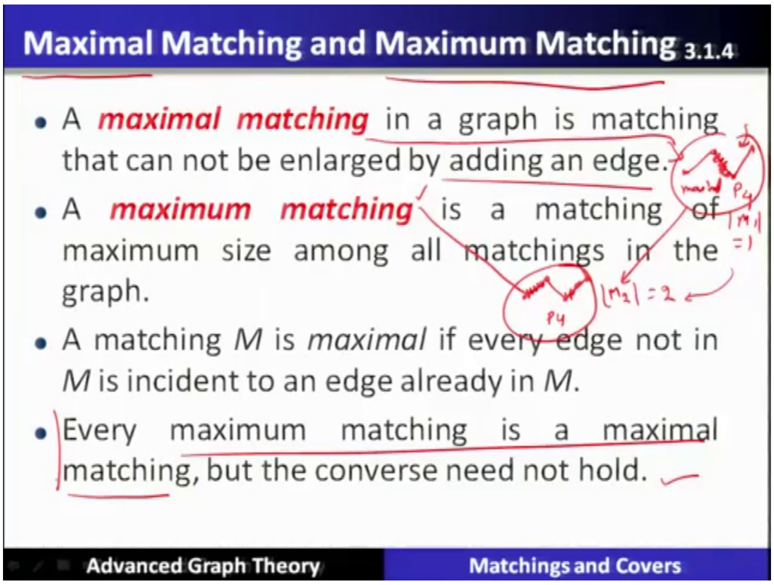

Maximal matching and maximum matching let us understand the contrast between maximum matching and maximal matching. Maximal matching in a graph is a matching that cannot be enlarged by adding an edge. Take this example, this is a P 4 graph and let us say, this edge is a matched edge. We cannot extend, we cannot add this edge, nor we can add the another edge. Why? Because as per as the definition of a matching says that the edges should not share the endpoints so; obviously, there is only one edge. There in the matching and the other two edges cannot be extended, cannot be added.

Hence, this example is a maximal matching. Now, maximum matching is a matching of a maximum size. Size means the number of edges among all possible matchings in a graph. So, take the same example P 4. We have now, this matching edge, this matching these two edges in the matching. So, the size here is two, the size we have seen earlier was one. So, here we have obtained a matching of a bigger size and we cannot extend this particular size in a P 2 P 4 graph. Since, this particular matching will become maximum.

Now, this maximum matching can be the maximal also in some of the situations, but converse need not hold in this particular example, this is a maximal matching is not a maximum in this case, but this particular maximum matching can be a maximal matching that is what is written over here. So, every maximum matching is a maximal matching, but not, but the converse did not hold.

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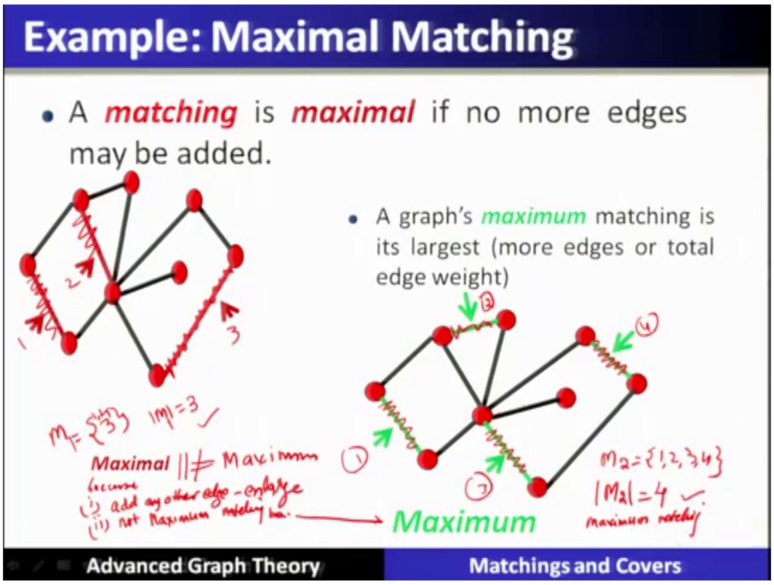

So, we will see through another example, these concepts maximum and maximal matching and their contrast. So, a matching is maximal, if no more edges are added. Take this particular graph. Here, we have this edge, this edge, and this edge. So, the matching size will have 1 2 and 3, 3 edges. So, 1, 2 and 3. So, the size of this matching is 3. In the same graph we can find out another matching. Let us say; this time we are finding out, we are including this edge, this edge, this edge, and this edge. So, we have obtained another matching, which will include four different edges. The size of this matching is 4. The earlier matching, which we have obtained was having the size 3.

So, now, we can inspect that in this particular matching, we cannot add any of these edges, which are black colored. So, this is a maximal matching and also we cannot find any other matching of a size bigger than 4. Hence, this becomes a maximum matching. This is a maximal matching, because of two conditions,; one is that, we cannot add any other edge; that means, we cannot enlarge this matching, second thing is, this is a maximal matching, because of you cannot add and this is not the maximum matching because their exist another matching of a bigger size hence this remains only the maximal, but not maximum.

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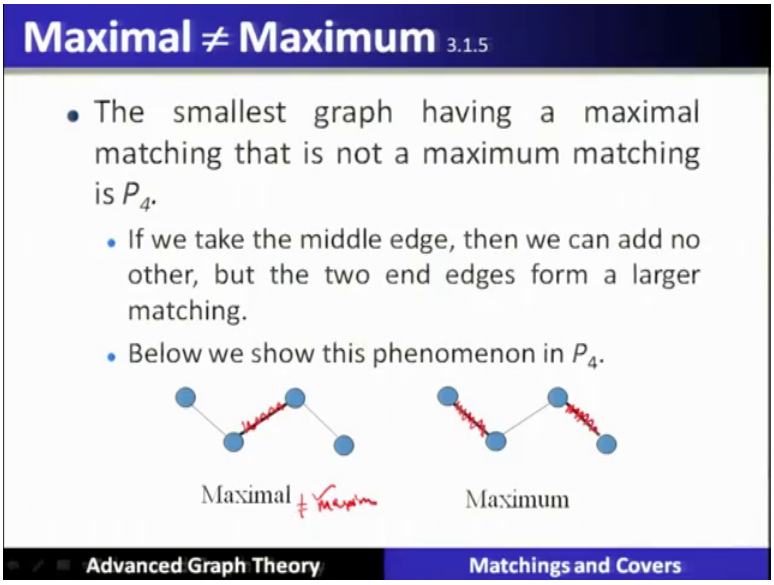

So, we have seen already this particular notion that maximal and maximum is not same. In this particular scenario here, this is a maximal matching having only one edge. This is a maximal why? Because we cannot extend or we cannot enlarge this size, this matching hence, it is a maximal matching is not maximum. Why? Because they are exist another matching, which is having a size more than this particular size. So, hence, this is not a maximum matching therefore, it becomes unequal.

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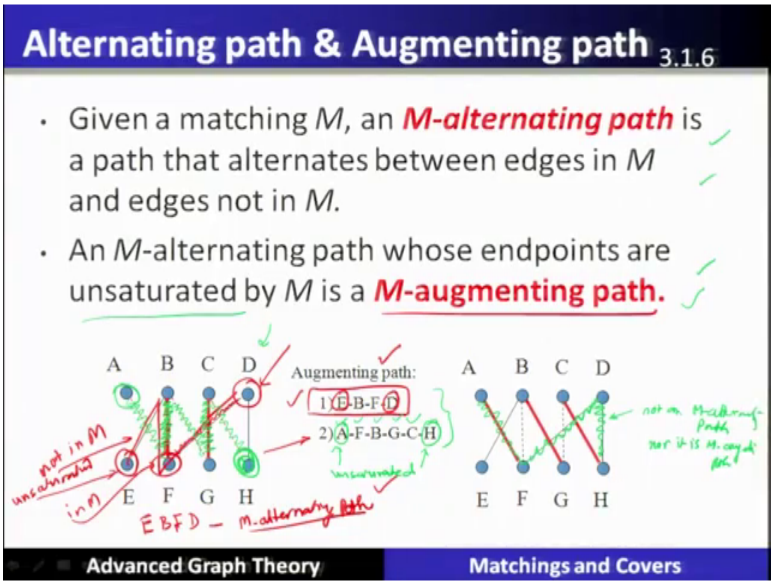

So, now, we are going to see alternating path and augmenting path and their contrast. So, given a matching M an M-alternating path is a path that alternates between the edges in M and the edges not in M. So, take this particular example here, we will take this particular example as M-alternating path. So, M-alternating path, which starts from this particular edge starts from this edge, which is not in a matched edge and then picks another edge, which is in the matched edge, then again picks this edge, which is not in a matched edge and that is all.

So, this becomes 1 2 and 3. So, it is end vertices, this particular path, we can see E B, then F and then D. So, this particular path is called as M alternating path, why? Because it alternates between the edges in M, this is in M and edges not in M, such path is called M alternating path. Now, that alternating path, which has end points unsaturated, it is called augmenting path M augmenting path is un saturated.

For example, this E is unsaturated and D is unsaturated. Hence, this becomes an augmenting path. So, let us see about M- augmenting path an M-alternating path whose endpoints are unsaturated by M is called M-augmenting path. This is one such example, which I have shown you that E B F D is M augmenting path. Why? Because the endpoints that is E and D they are unsaturated let us see another example.

So, another example says that a. So, another example says that a, we have selected then F, we have reached then B, this is a matched edge, then from B, we have to go to G and from G, we are going to see, this is a matched edge and from C we are going to H. So, you see that this particular vertex, H is unsaturated and the star vertex A is also unsaturated. So, both are unsaturated vertices. Hence, by this particular definition that, this is the alternating path, whose endpoints are unsaturated by M so, that becomes an M-augmenting path. So, we have shown you the examples of 2 M-augmenting path in this particular graph.

Now, let me ask you another question, whether A then D and then H, whether this is an M-alternating path or M-augmenting path. So, you see that it starts from a matched edge alternates with another unmatched edge and then it changes to another unmatched edge. So, hence, it is not an M alternating path nor it is M augmenting path. So, we have to be careful, while we define M- alternating path and M-augmenting path. So, it is not that all, the paths in a graph or in this category.

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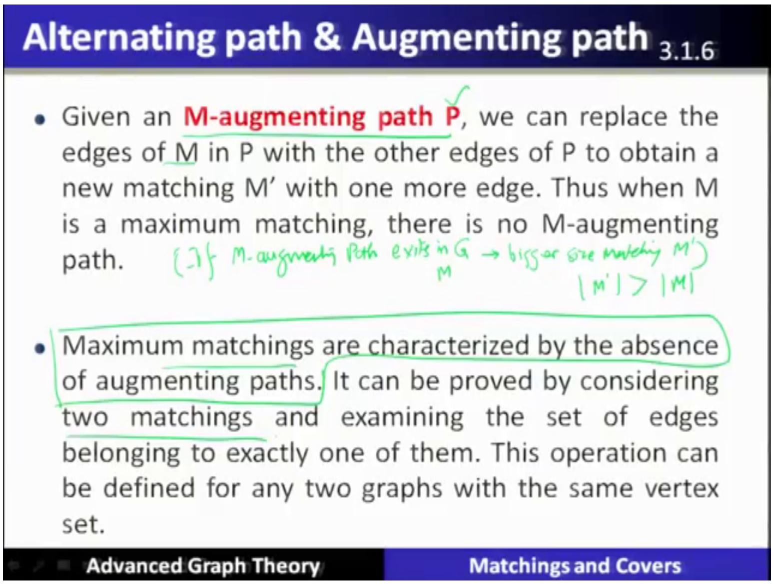

Now, given M-augmenting path, we can replace the edges of M in P with the other edges of P to obtain a new matching M prime with one more edge. Thus when M is maximum matching, there is no M augmenting path, this has told you the utility of M augmenting path; that means, if M augmenting path exists in the graph G, then there is a possibility that we can get a bigger size matching M. So, we can extend the bigger size matching M, if you are given now, M prime is greater than M, if there exist M-augmenting path, we cannot extend the size of the matching, is M augmenting path does not exist in a graph.

Hence, this is a important characterization. So, maximum matchings are characterize by the absence of M- augmenting path, this is very-very important notion. So, that is why I am highlighting it and repeating it again. Maximum matchings are characterized by the absence of augmenting path, again I am repeating it. So, if there is no augmenting path exists and we have obtained a matching. So, that will be the maximum matching.

Now, we can prove it by considering two different matchings and then examining the set of edges, belonging to exactly one of them and this particular operation can be defined for any two graphs, with a same set of vertices that we are going to see now. So, that operation is called a symmetric difference.

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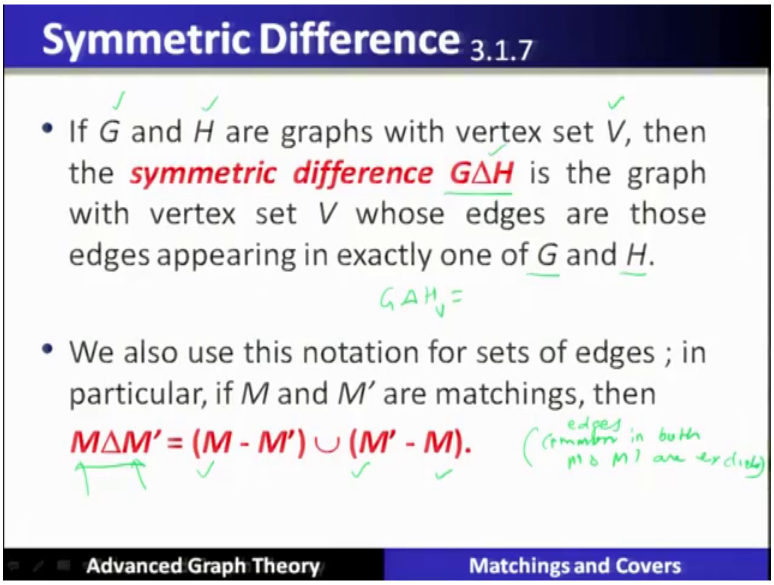

So, let us take two different graph G and H, which are defined on the same vertex set, if we take a symmetric difference is written as G symmetric difference H, which is defined on the same vertex set V, which will include those edges, which are appearing exactly one of these G and H. So, those edges which are common in G and H will be excluded or in another way, we can write down this symmetric difference, if we take two matchings M and N prime. So, matchings is also a graph defined on the same vertex set. So, M symmetric difference, M prime will be M minus M prime; that means the edges in M, but not in M prime union edges in M prime, not in M. So, the edges which are common will not be the common in both M and, M primes are excluded.

So, having defined symmetric difference, let us see how using symmetric difference; we can, we can basically characterize the maximum matching. So, in this particular example, we can understand the example or the working of a symmetric difference.

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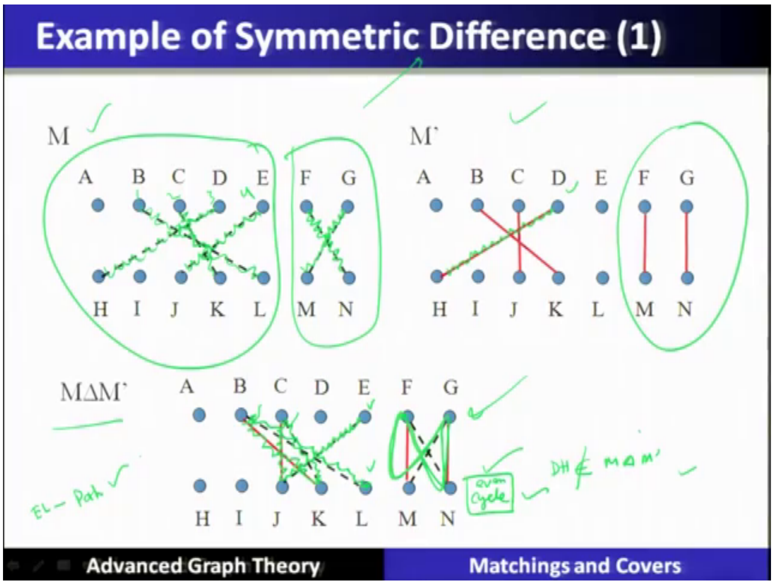

So, let us say that two matchings are given M and M prime. So, M is represented as dotted lines, dotted. I am marking with the, with the green color and the red color is on the right side that is M prime, what we can see here is that. So, basically is that. So, basically in this particular section F and G so, two more edges are there, which was not there. So, basically these both, of these two edges are included. So, which are there in M that is also there, which are in M prime that is also included. So, this becomes the resultant symmetric difference.

Similarly, this particular component will have one, two, three, four, four different edges. So, out of that; so, this edge dotted is present, second edge is also dotted, which is present, third edge is not present. So, third edge is not present, because it must be a common in the other component. Let us see that, yes, it is common. So, this edge is also present in M prime. So, hence a symmetric difference will eliminated. So, that is eliminated; that means, D and H is not in M symmetric difference, M prime or we can also say that symmetric difference.

So, D and H is not in this particular set. So, having computed the symmetric difference of two different matchings in a particular graph, what we will obtain is you can inspect. There are two components; one component is nothing, but; so, if you start from a point reverse, all the edges and come back to the same point is called a cycle. So, it is the cycle and the length of a cycle is the number of edges. So, this particular cycle is having, how many edges? Four edges and this is called even cycle.

The other component, if we start from this point, reverse back all the edges then, we have started from here and finish this particular vertex and not repeating, neither the edges nor the vertices. So, this becomes a path. So, this becomes E L path. So, when we take a symmetric difference of two matchings, it will basically break into or it will obtain either the even cycle or a path, or may be both. Here, in this case, it is appearing both even cycles as well as even path.

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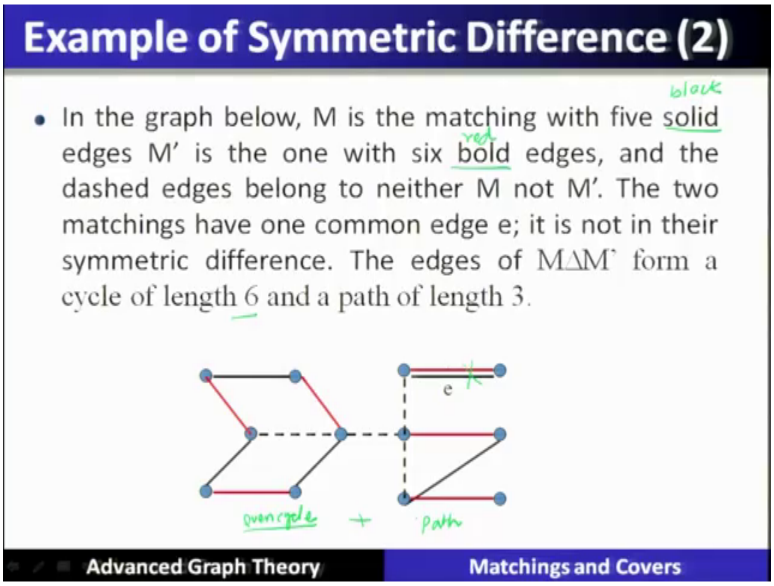

So, this particular example also will illustrate that, that basically M is a matching, with the five solid edges and M prime is one, with the six bold edges. So, bold edges is marked as a red colored edge and solid edge is marked with the black edges, if we take the symmetric difference of these two matchings then we will see, there is a cycle of length 6. So, this is a cycle and length 6 is that is the even cycle and we will get a path of length 3 1 2 and 3. This particular edge is common e is common in the M and red color is common in M prime. So, hence this is eliminated in the symmetric difference. So, it will result into a even cycle and a path that is shown in this example.

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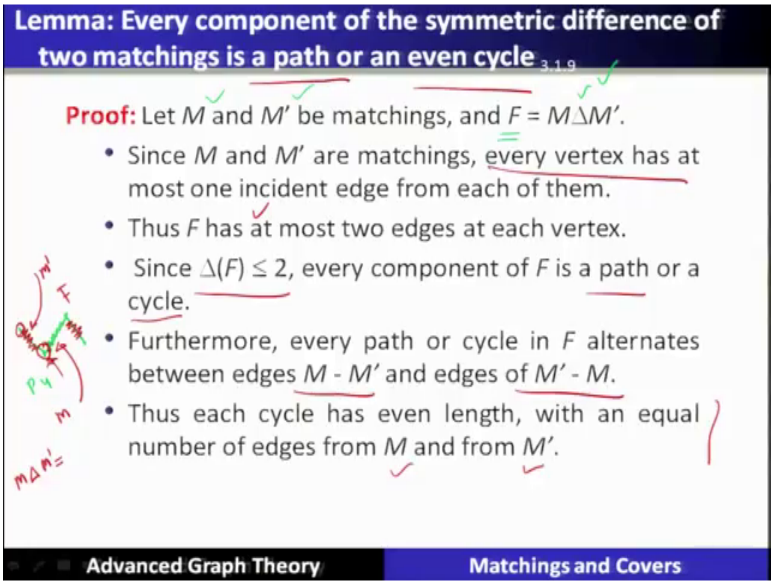

Lemma: Every component of a symmetric difference of two matchings is a path or an even cycle that I have already shown you through an example. Let us see the proof. In the proof, let us assume M and M prime be the matchings and symmetric difference, M symmetric difference, M prime, let us say is F. So, F is a graph, which is a symmetric difference of two matchings in a given graph with a same vertex set. Let M and M prime are the matchings, every vertex has at most one incident edge from each of them, let us understand it.

So, this is a p four graph let us say this is one matching and this red color is another matching. So, red color matching we call it as m prime and the green color matching we call it as m. So, m symmetric difference m prime will form all these edges why because there is no common edge. So, both these edges of the entire P 4 graph will be there in this particular edge. So, every vertex has at most one incident edge from each of them. So, let us say this particular vertex only, this edge from M prime is incident that is less than that is not from M prime.

This particular vertex has an edges, which are incident from M prime and incident from M. So, at most one edge is incident from each of them. Thus, F this is called F has at most two edges at each vertex. So, here there is a two, at most two edges or two edges and here is one edge. So, that also is basically at most two edge. Hence, the degree of the vertices in F is at most 2.

So, if the degree of F is at most 2 then, it will be either the path or a cycle, if all the edges are 2 then, it becomes a cycle, if two edges are basically two vertices or having degree 1, then they will become a path furthermore. Every path or a cycle in F alternates between M minus 1 M minus M prime and edges of M prime minus M that we know by the symmetric difference. Thus, each cycle has an even length with an equal number of edges from M and from M prime. So, hence we have seen that either it is a path or an even cycle. So, if it is a cycle then it is even why? Because the edges have to be equal in number from M and M prime hence, this is called even cycle. Hence, it is proved.

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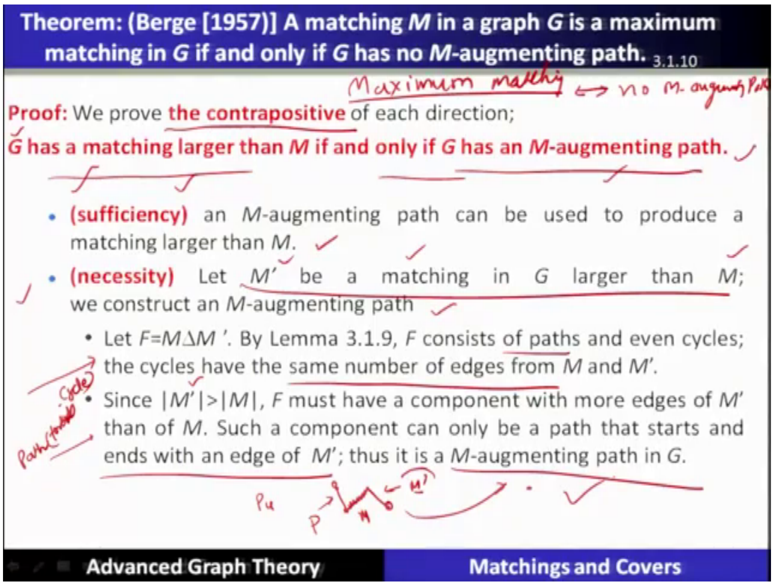

There is a theorem given by Berge in 1957, which states as follows a matching M in a graph G, is maximum matching in G, if and only if G has no M-augmenting path. This statement, we have already seen earlier through an example, but let us see the theorem, this is very important theorem given by Berge. So, again I am repeating this particular theorem, a matching M in a graph G is maximum matching, if and only if it has no M-augmenting path. So, it will characterize a maximum matching, it will characterize a maximum matching.

Maximum matching means it is equivalent to the graph, which is having no M-augmenting path. So, we have to prove in both the directions. Now, proof will use the contrapositive in both the directions so; that means the statement to prove in a contrapositive of both. The directions will become that G has a matching larger than M.

So, this is the negation of this statement. So, G has a matching larger than 1, if and only if G has an M-augmenting path. Let us prove this contrapositive statement; if this is true then the entire statement of the theorem is true. So, let us see the sufficiency condition. Sufficiency condition says that an M-augmenting path. So, M-augmenting path is given and that can be used to produce a matching larger than M prom the previous theorem and hence, this proves this particular statement that G has a matching, which is larger than M.

Now, we have to prove the necessity condition. So, necessity condition says that if G has a matching larger than M then we have to prove that G has M-augmenting path. Let M prime be a matching in G, which is larger than M, because this is given. Now, we have to prove that it has an M-augmenting path. So, we will construct an M-augmenting path, given this M prime matching that is larger than M. There are two matchings. Now, we are given M and M prime. So, immediately we will take a symmetric difference of these two matchings that becomes a graph F and by Lemma 3.19. We know that F consist of paths and even size cycles. Now, since cycles or have are having the same edges from M and M prime. Hence, for cycle both M, both the, both the edges from M and M prime is equals.

So, hence we have to see another component that is called a path. Since, M prime is greater than M, we have assumed it. So, F must have a component more than more edges of M then M prime, and that is a path, and this will forbids the cycle. So, is it, is a path component, then we have to see the path. So, such a component can only be path that is starts and ends with an edge of M prime. Why? Because M prime is greater. So, it has to start and end from M prime.

Thus, there exist and M-augmenting path. So, take this particular example; P 4 example again we are going to take. So, this is an M and these edges will become M prime. So, this particular M prime; that means, the augmenting path will start from a vertex from M prime and then alternates with the edge of M and then again the edge of M prime. So, this will become an M-augmenting path. So, M-augmenting path exist and with the help of the edges, which are not in M, we can invert it, which are there in path, but not in M. We can invert it and we can obtain a matching of a bigger size and hence, M-augmenting path exist.

So, we have proved the previous theorem, which is given by Berge that will characterize the maximum matchings. So, maximum matching in a graph is equivalent to saying that the graph has no M-augmenting path.

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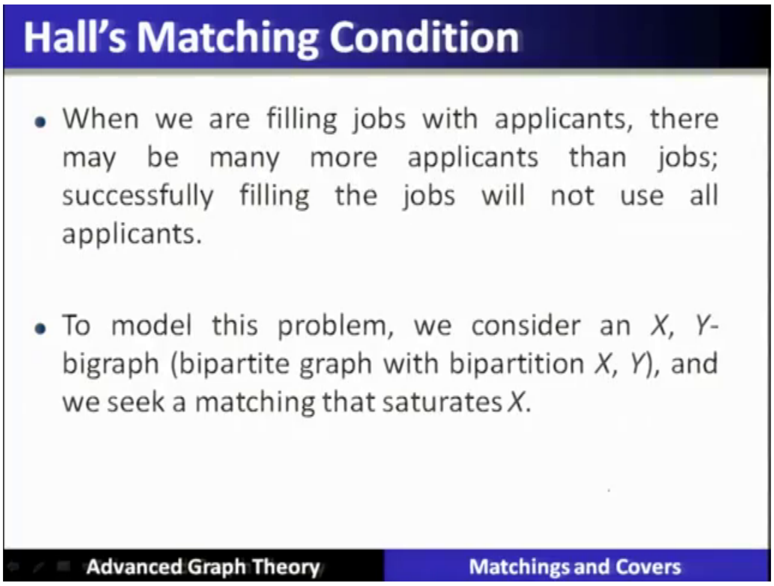

Now, another theorem, which is called a Hall's matching condition or a Hall's theorem. It is called; we are going to see that. First, we are going to see the use and then we will be proving it, when we are filling up the jobs with the applicants. We may be seeing that there are many more applicants than the jobs and successfully filling up the jobs will not use all the applicants.

So, one of these two partite sets, which basically are representing the jobs, will be all used up while the other partite sets, which are representing the jobs are not going to be utilized are going to be saturated by this pairings. This is an important notion and we have to see this particular condition. So, to model this particular problem, we take X Y bigraph, and we seek a maximum matching and we seek a matching that will saturate all the vertices of X and that is a solution of this particular problem of job filling, with the applicants.

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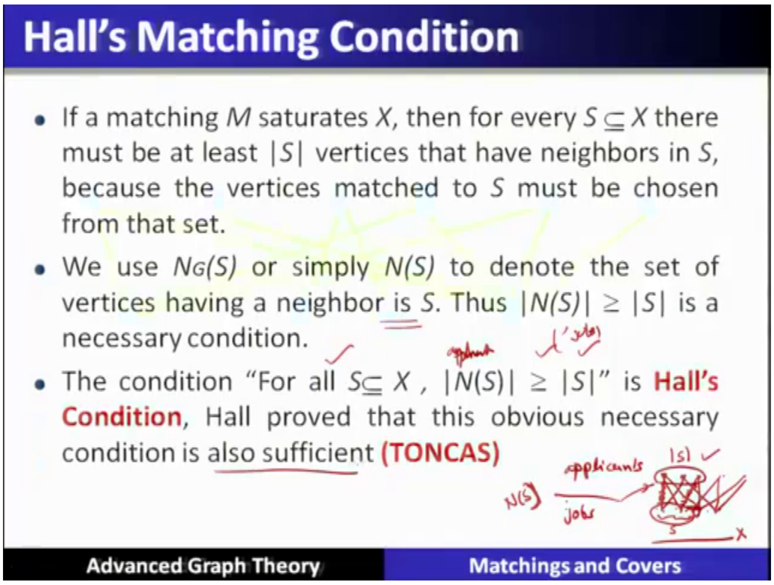

Now, if you matching M will saturate X; that means, all the jobs, then there is a condition that for very subset of X, that is every subset of jobs, there must be at least the number of that set of vertices that have neighbors in S. Why? Because neighbors in S meaning to say that the jobs must have mod S or that subset size of the jobs at least that many number of people or applicants. Let us use the notations to see that. So, neighborhood of S S is a subset of X, which is going to be saturated by the matching. So, it is simply represented as N of S. Here, denote the set of vertices having the neighbor in S. So, again in the problem of the applicants verses jobs. So, here there are many applicants and there are few jobs.

So, so the applicant, some are the applying to these many number of jobs, others are applying to these many number of jobs and so on. So, what we are seen that, if this is a subset S out of capital X, then every subset S out of X will have that many number of S in the neighborhood of S and that is called a Hall's conditions. So, we have to define the neighborhood of S. So, this is the neighborhood S.

So, these set of vertices, which are being matched by these edges. They are basically, matched on the neighborhood of S, that will be on the other side, that is in the applicants. So, this particular condition is called a Hall's conditions. So, to saturate all the jobs by the matching or the pairing of the applicants, the condition is called a Hall's condition, which says that for all subset of X, which is represented as S, which says that the neighborhood size of S, which will match S or a jobs to the applicants.

So, neighborhood will be in the applicants, is always at least the number of jobs. So, that is quite obvious condition that the jobs, which are going to be filled up all set of jobs subset, if you take, should be having more applicants, then only, every job will be saturated and that obvious condition is basically, the necessary condition and halls has also proved that this obvious condition, obvious necessary condition is also sufficient. So, that we are going to see in the theorem.

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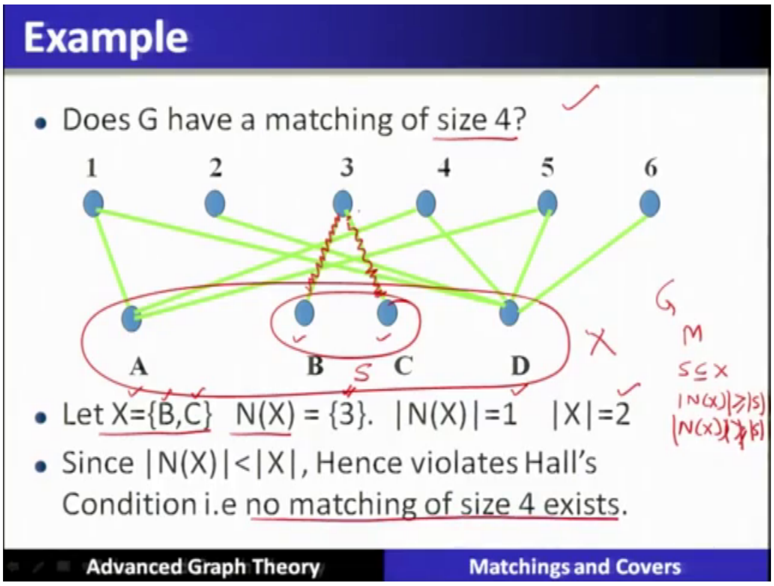

The implication of this particular theorem, we can first understand through some examples and then we will understand or we will get the motivation of this usage, of this important, important theorem, that is called a Hall's theorem. This is the first example; is basically in this particular graph G. Does this graph has a matching of size 4? So, what is 4? If you see this set is 4.

So, he is asking whether, they are exist a matching, which will saturate all the vertices of X, if it is then basically S subset of X that particular condition holds, which says that this many number of; that means, always on the other side, the neighborhood side, the size of the neighborhood is more than any subset of that particular set of X, which we are going to say, which we are going to check to find out a subset. Here, in X, where this particular condition is violated to show that there does not exist a matching of size 4. Let us assume that these two vertices form a S. So, that is B and C. So, the size of this X becomes 2, why? Because there are two elements B and C.

Now, the neighbor of B is 3 and the neighbor of C is there, that is all. So, the neighborhood of X; that means, neighbor of B is 3 and the neighbor of C also 3. So, the neighborhood of X, if we see the size is basically, it is 1. Hence, this particular N of X is not satisfying, this halls condition hence, they are does not exist a matching of size 4, in this particular graph.

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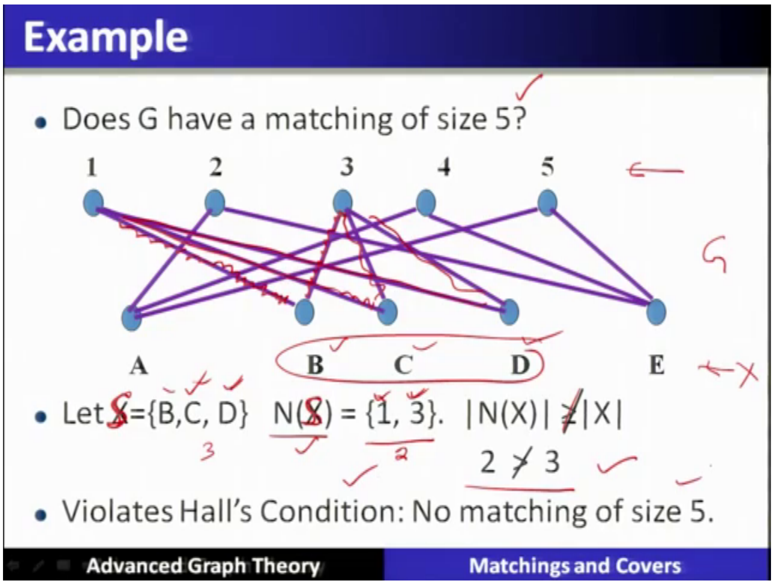

Let us take another example, this example says that does. This graph G has a matching of size 5. So, here you see that this part of the graph is also having the five elements on the other side, it is five elements, both the partite sets are of the same size. Let us t, let us see the conditions, let us find out a subset, assume that this is X. So, subset, we call it as S, which consist of B C and D. Let us find out the neighborhood set of this S.

So, the neighborhood of B is 3 and 1. So, 1 and 3 are present. So, neighborhood of C, it is also 1 and it also 3. So, nothing change neighborhood of D is also 1 and also 3 nothing change. So, the neighborhood of S will become 1 and 3 size is basically 2 and here, S is of size 3. So, the neighborhood set ID smaller than the subset of S. Hence, this particular condition is violated. Hence, there is violation of a Hall's condition, their does not exist a matching of size 5.

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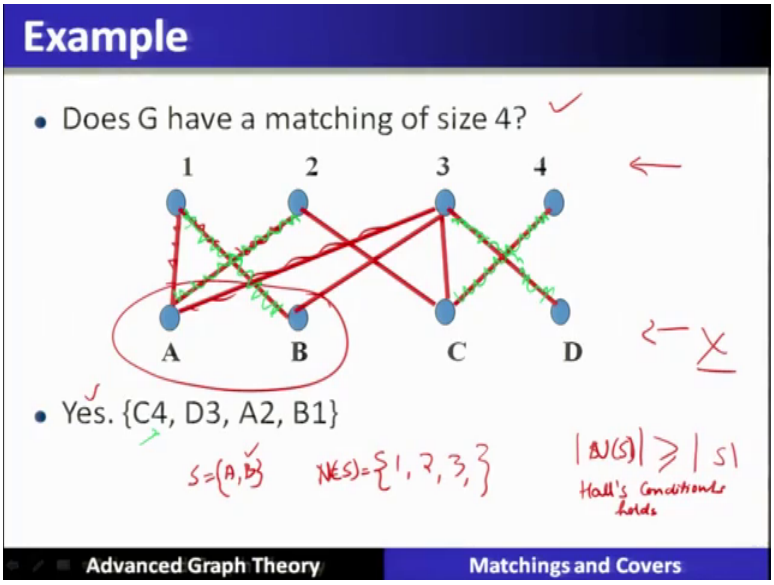

In another example, we can see that, this particular graph whether, it has a matching of size 4, both the sets are having of size 4. So, here the conditions are satisfied; that means, if you take any subset. Let us take this particular subset of X. So, subset. So, neighborhood of let us say S is equal to A B quickly, we have to rush through this particular and if we found the neighborhood of this S.

So, A is having a neighbor 1, another neighbor 2, and another neighbor 3, about B, it has neighbor 1, nothing change. It has neighbor 3, nothing change. So, the neighborhood of S, if you take the cardinality is; obviously, more than the cardinality of S. Hence, the Hall's condition, holds for every subset of X. Hence, this particular case, it basically has a matching of size 4, if it is ask then you have to produce. This matching of size 4, let me show you the matching of size 4, C 4, this is the matching D 3, A 2, and B 1. So, these are the four different set of edges and which is required whether the matching of size 4 exist using halls condition, we have seen.

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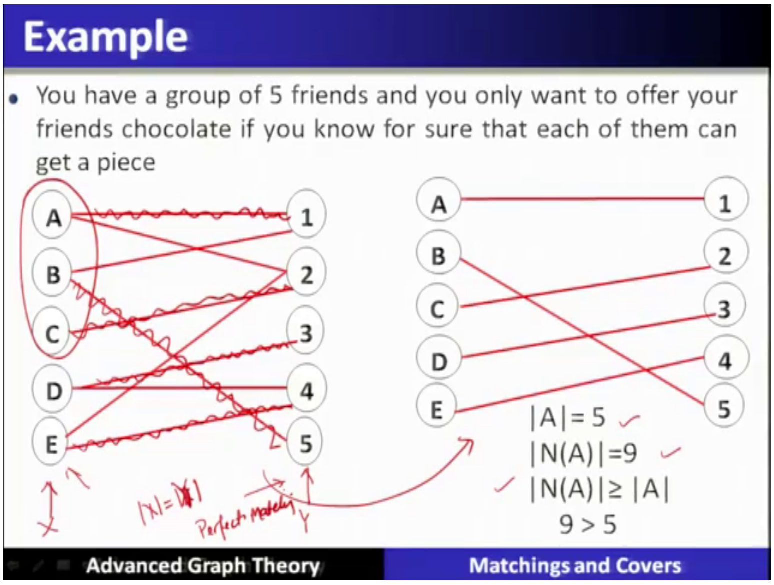

Let us take another example of a Hall's theorem in this example, you have a group of five friends and you have another five different type of chocolates, which you want to offer to your friends and their preferences are also shown on the ah, left side of the graph in shown mentioned, in the red color. Now, we have to find out a matching; that means, if everyone basically, is guarantee to get a chocolate, then only we are going to distribute it, that is a, that is the condition. Let us see whether, you are going to saturate the, all the friends with a pairing of chocolate.

So, let us see that the number of the size of A is 5 and the neighborhood, you have basically, if we count the neighborhood, how many different edges are there preferences then basically, it is comes out to be 9. So, we can see that whether, we can evolve or we can come out with a matching. So, there is a possibility of a matching that A can be matched here, with 1, his own preference then B can be matched with the preference 5. C can be matched with the preference of 2, D can be matched with the preference of 3, and E can be matched with the left over 5.

So, there is a possibility of a matching so; that means, this particular Hall's condition exist for any subset of this particular set that is called X. So, X is saturated and here simultaneously, Y is also saturated so; that means, and the size of the X and Y. They are same. So, this kind of matching is called a Perfect matching, where in all the vertices are saturated; that means, every person is going to get a different kind of chocolates.

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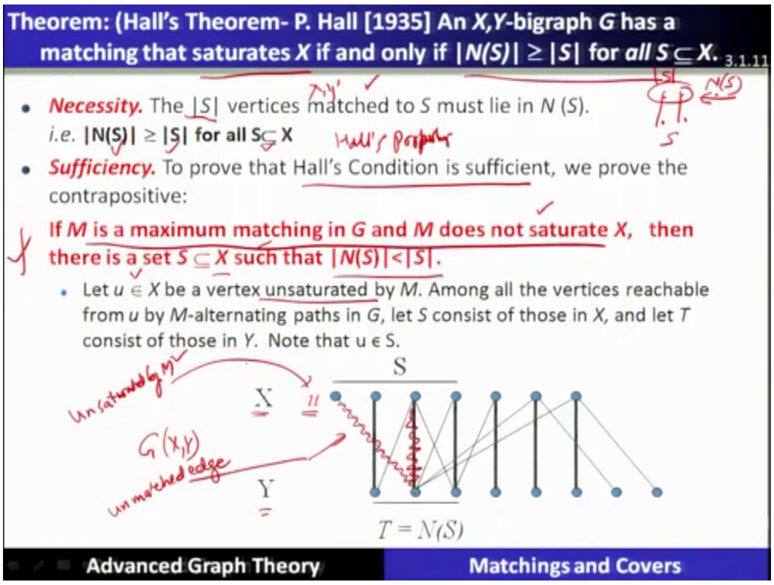

So, having seen the motivation of use of this important theorem, that is called a Hall's theorem, which will give you the condition, whether you are able to saturate one set of partite set either, it is the jobs or it is the people, who are going to see different chocolates. So, if that is the case then this condition, which is called a Hall's condition must be valid. So, we are now going to see the proof of this Hall's conditions an X Y bigraph, G has a matching that saturates X, if and only, if the neighborhood of S is at least that size of S for all S, which is a subset of X.

So, that condition, we have already seen. Now, let us see the proof. So, first we have to see the necessity condition. So, in the necessity condition, we are given that there is a X Y bigraph, has a matching that saturates X. So, if there is a matching that saturates X; that means, these S number of vertices, which are matched to S meaning to say that, if this is the set of vertices on the neighborhood, if it is match to S. How they are matched, they are matched, through the edges.

So, these particular edges will lie on the neighborhood of S. So, this is basically lies on the neighborhood of S. So, this neighborhood of S has at least, S number of elements present S number of elements present for all S, which is a subset of X. Hence, this particular property that is called Hall's property is satisfied that is necessity condition is followed.

Now, let us see the sufficiency condition in sufficiency condition. We are given this Hall's condition; let us say so Hall's condition. Now, we have to prove by contrapositive, let us see the statement of a contrapositive.

So, if M is a Maximum matching in G and M does not saturate X. Here, it saturate X over basically negation is there in contrapositive then, they are exist a subset S, which is subset of X, such that there is a violation that is N S is less than S, let us see this particular condition, this particular contrapositive is proved, holds then basically the halls condition, we have it is sufficiency condition.

So, for this let us assume a vertex u, which belongs to X. So, this is the bigraph X and Y bigraph and they are exist a vertex u and this is unsaturated vertex, by matching M. So, that is the condition that is given. So, we have to assume that this is the vertex, which is unsaturated by the matching M. Now, among all the vertices reachable from u by M by M alternating path in G; let us see that u we can reach to T via unmatched edge, why? Because it is a unsaturated, this is unmatched, we can reach to a nod on Y and let us call it is a part of T or a neighborhood of S using a matched edge. It will take you back to the set X and we it that this particular vertex, which is matched by the set of or a nod of Y is present in S. Let S consist of those in X and let T consist of those in Y.

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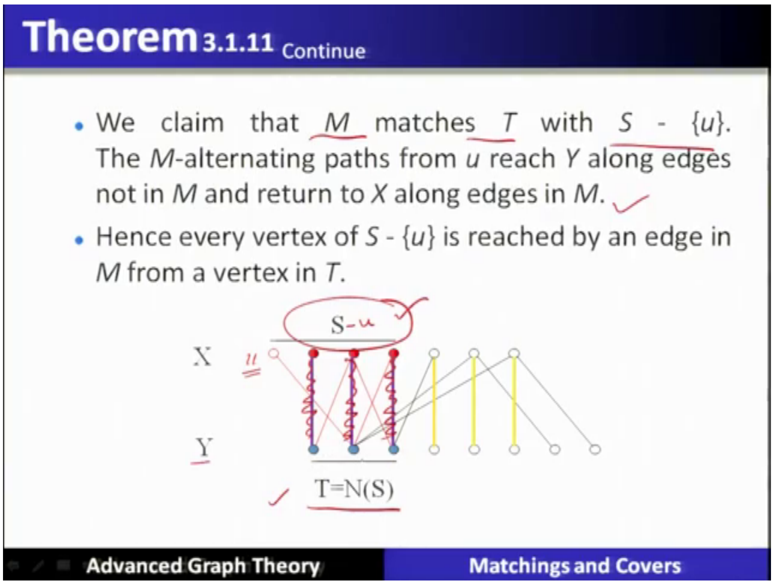

Now, we claim that this matching M will match T with all the elements of S minus u, because this is unsaturated. So, S minus u, if there is a matching between S minus u and between T. So, we have to see those edges. So, the M alternating path from u will reach y along the edges, which are not there in M and then return X along the edges, which are there in M hence, every vertex of S minus u is reached by an edge, which is present in M from a vertex T. So, that we can see in the purple color, which is shown; that means, these set of vertices, which is there in Y, we call it as a T, all these particular vertices basically, will have a matched edge and that will take you back to S minus u, which is there in X.

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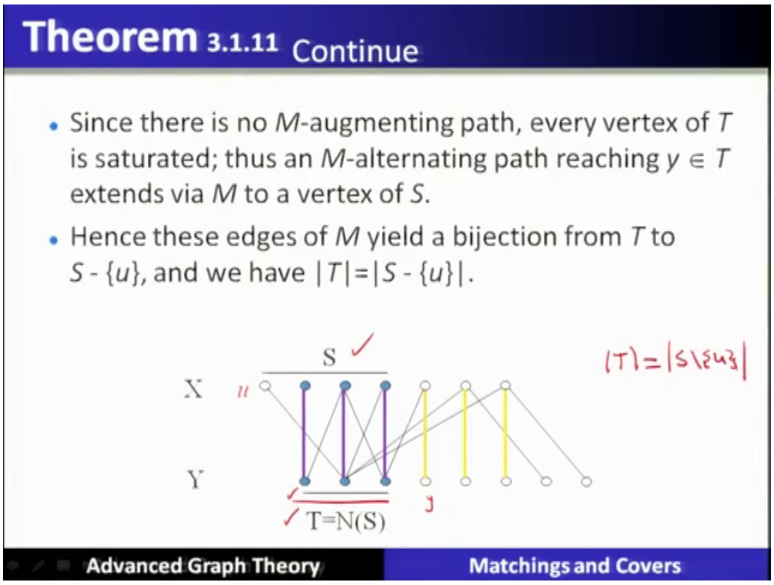

Let us go ahead. So, there is no M-augmenting path since, every T, every vertex of T is saturated. Thus, M-alternating path reaching Y here, extends M extends via M to the vertex of S. Hence, these edges of M yields a bijection from T to S minus u and we have the size of the T is same as S minus u. So, this particular matched the vertices, which are saturated is T is same as the set S minus u both are same.

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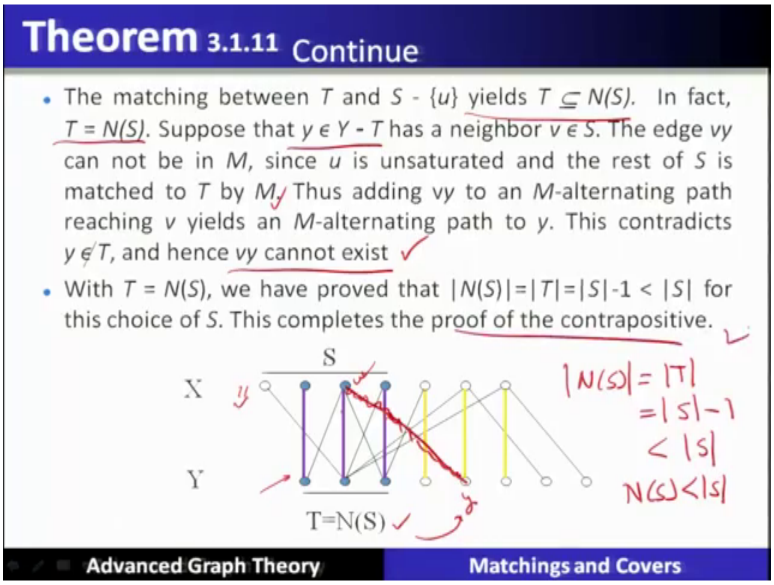

Furthermore. So, the matching between these two sets; T and S minus 2 will yield that if you take any subset in T that is basically, the subset of the neighborhood rather, we can see that T is equal to neighborhood of S and suppose, there exist a Y, which is not in T and has a neighborhood v in S. Let us say that this is v, it has a neighborhood. So, the edge v Y cannot be in M.

So, this cannot be in the M; that means, it cannot be in the purple color Y since, u is unsaturated and rest of S is matched by match to T by M and adding an edge v Y to an M-alternating path reaching v yields an M-alternating path to v and this contradicts, the condition that Y is not an element of T ok. It is not in T, this completes the prove of contrapositive. Hence, this v Y cannot exist you can also see that simultaneously, this edge is a matched edge and if you add another edge, also both cannot be in the matched edge. Why?

Because they would be sharing a vertex over here hence, such a v, such a v Y edge cannot exist in a set of matched edge. Hence, v Y cannot exist, because all set of edges, which are in match, we have already covered it. Now, here this T is the neighborhood of S and we have proved that neighborhood of size, of S is same as the T and here the T; that means all the vertices of T, which are matched to S except the u, which is unsaturated. So, S minus 1 and if we basically can see that this is less than S, hence N S is less than S and hence, this will prove, he contrapositive.

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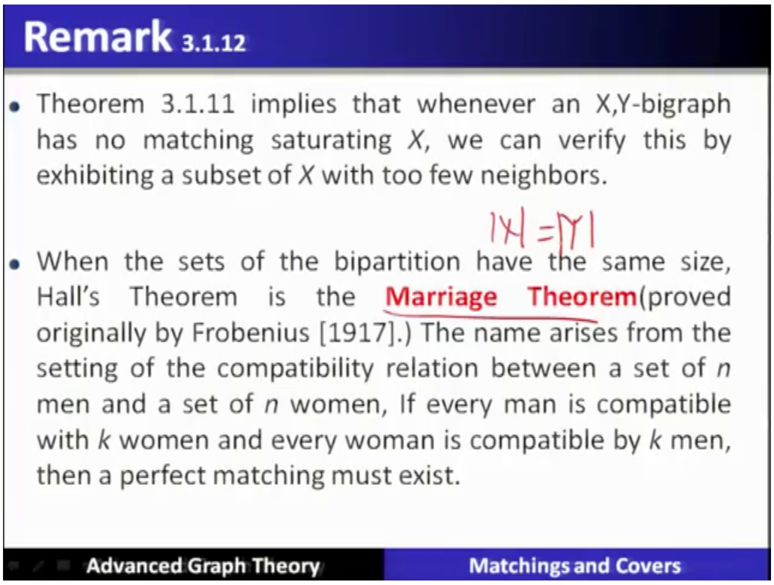

Remark; the Theorem implies that whenever X, Y- bigarph has no matching saturating X, we can verify this by exhibiting a subset X. The Theorem implies that whenever X Y bigraphhas no matching saturating X, we can verify this by exhibiting subset X with two few neighbors. On the other side; that means, violation of a Hall's condition is basically good enough to prove that it is not saturating X.

So, when the set of bipartition have the same size, this Hall's Theorem is called a Marriage Theorem, so; that means, when X is equal to Y then this particular Theorem is called a match, is called a Marriage Theorem. The name arises from the setting of the compatibility relation between a set of n mens and the set of k womens, if every men is compatible with k women and every women is compatible by k men then a perfect matching must exist.

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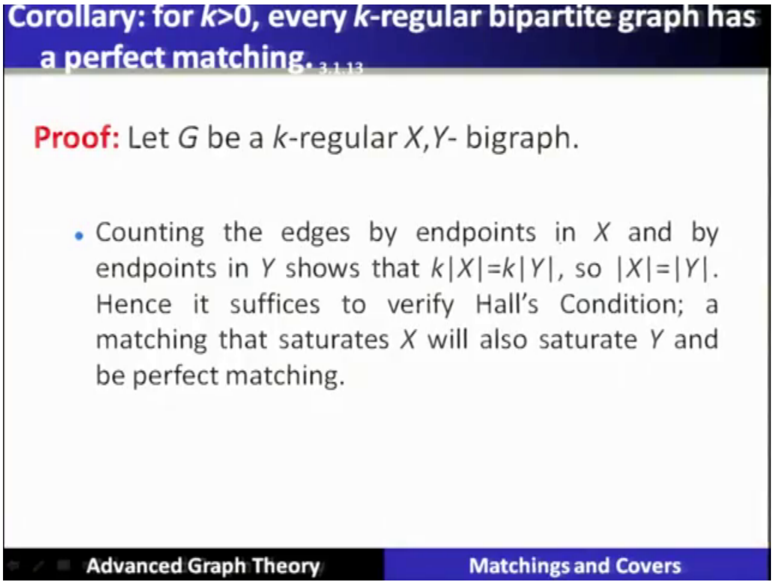

This gives a corollary for k is greater than 0, every k regular bipartite graph has a perfect matching proof is quite simple. Let G be a k regular, X Y bigraph counting the edges by the endpoints in X and by the endpoints in Y. We can see that k times, the cardinality of X is equal to the k times cardinality of Y. So, X is why? Hence, it is suffice to verify the Hall's condition is valid a matching that saturates X will also saturate Y and that is the perfect matching.

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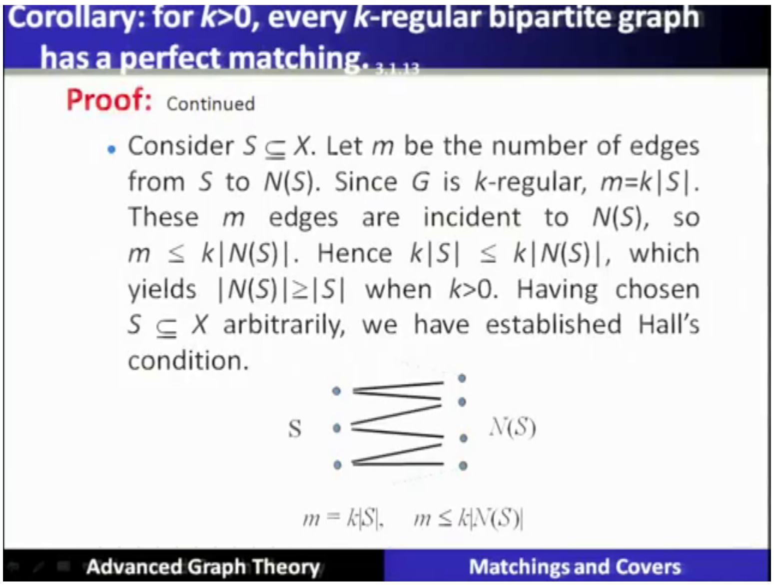

Now, consider S is a subset of X, m is the number of edges from S to the neighborhood of S. Since, G is k regular, m is equal to k times S. These m edges are incident to the neighborhood of S. So, m is less than k times neighborhood of S, that is the validity of the Hall's condition.

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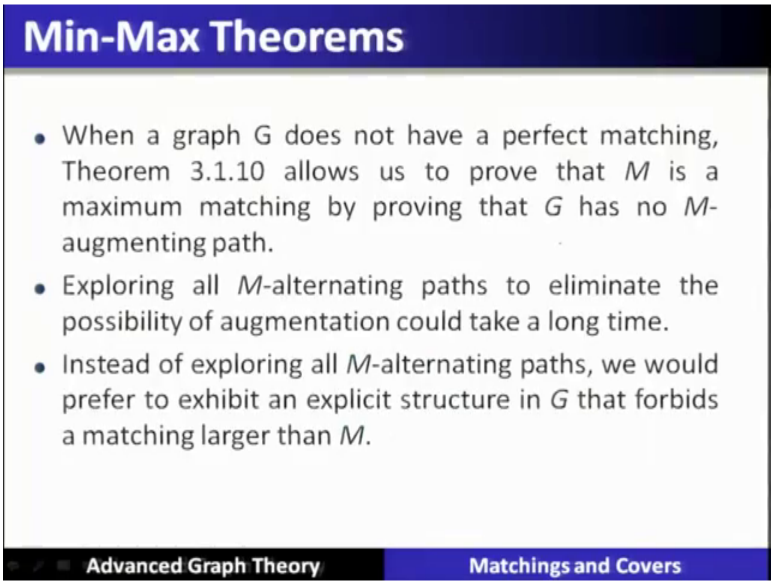

Now, we have to go ahead and see another important thing, when a graph G does not have a perfect matching, Theorem 3.1.10 allows us to prove that M is a maximum matching by proving that G has no M-augmenting path. All these things, we have already explained. Now, exploring all M-alternating path to eliminate the possibility of augmenting could take the long time instead of exploring all M-alternating path, we could prefer to exhibit an explicit structure in G that forbids a matching larger than M.

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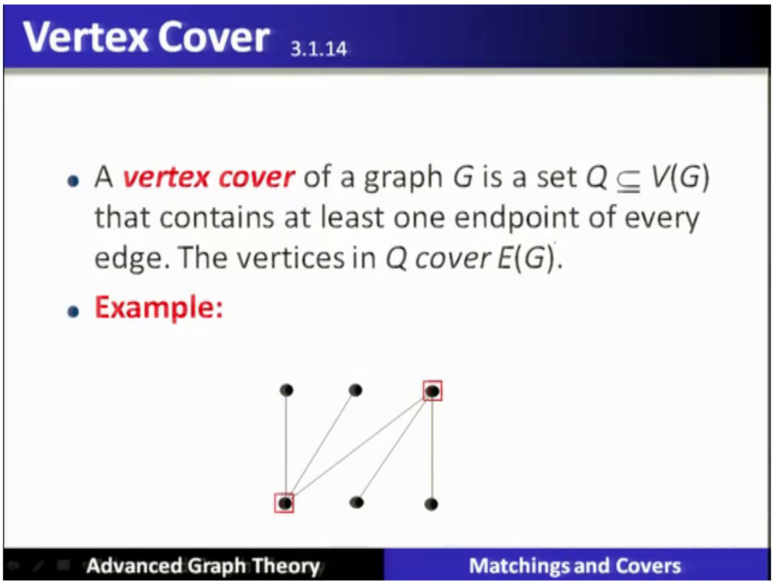

So, we are going to see, what is that condition, which will forbid for getting a bigger size matching than the maximum matching; a vertex cover of a graph is a set of vertices that contains at least one endpoints of every vertex so; that means, a vertex cover, covers all the set of edges, shown in this particular example as squared symbol.

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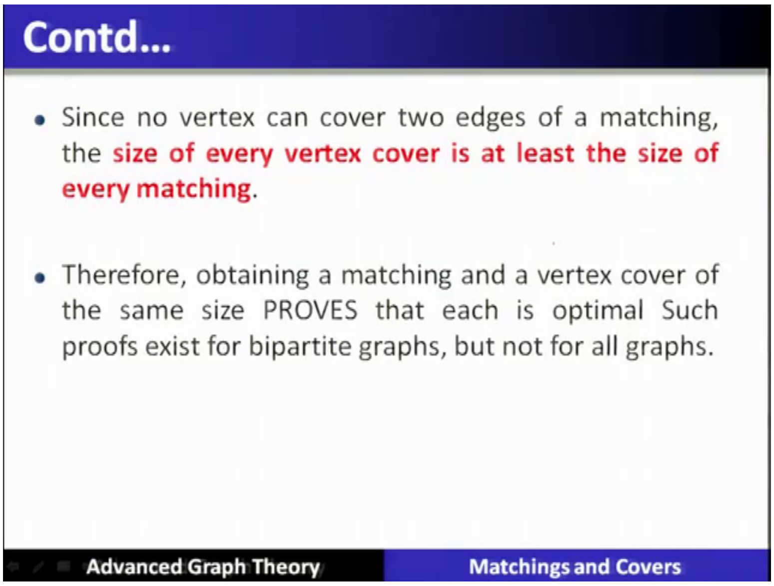

Now, since no vertex can cover 2 edges of a matching. Therefore, the size of every vertex cover is at least the size of every matching. Therefore, obtaining a matching and a vertex cover of the same size will prove that each is optimal, such proofs exist for bipartite graph, but it is not valid for all the graphs.

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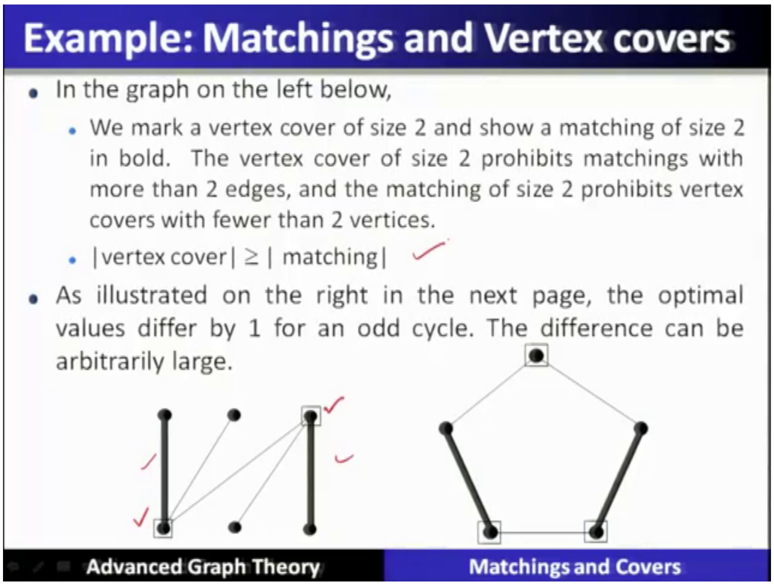

In the graph shown on the left, we mark the vertex cover of size 2, this is 1 vertex cover, this is another these 2 vertices will cover all the edges; that means, all the edges are touching these. So, the vertex cover of size 2 prohibits the matching, with a more than 2 edges and the matching of size 2 will prohibit the vertex cover with the fewer than 2 vertices. Hence, vertex cover is at least the size of the matching and this is the theorem, which is given, which is known as Konigs Egervary Theorem.

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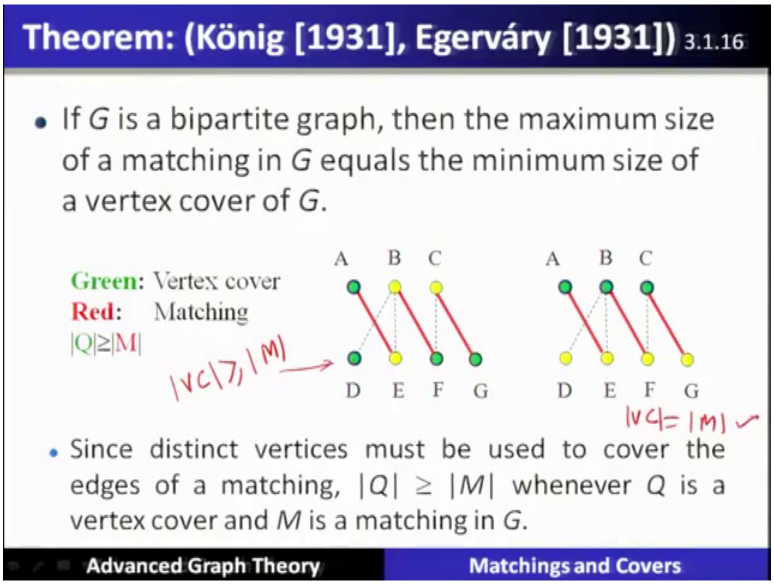

So, let me state the theorem, if G is a bipartite graph, then the maximum size of the matching in G equals the minimum size of a vertex cover. So, here in these particular figures, we can see that the vertex cover is shown as the green colors. So, here in this particular graph, the size of the vertex cover is 1 2 3 4 and the size of the matching is 1 2 3, so; that means, size of the vertex cover is at least the size of the matching.

Similarly, here we can see that, that the size of the vertex cover is 3 and the maximum matching is also 3. So, we have reached the condition, when vertex cover is equal to the size of matching and this is the condition of a maximum matching, and a minimum vertex cover; that means, we cannot basically get a matching bigger than M. Similarly, we cannot get a vertex cover less than this particular size.

Hence, this is a condition, which is given by the Konigs Egervary Theorem.

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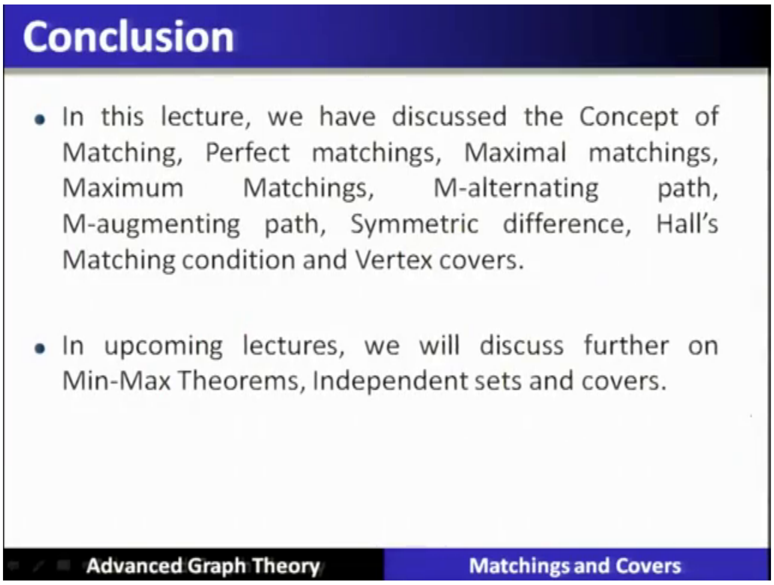

Conclusion in this lecture, we have discussed the concept of Matching, Perfect matching, Maximum matching, Maximum matching, Maximal matching, M-alternating path, M-augmenting path, Symmetric difference, Hall's condition and Konig Egervarys theorem, and Vertex covers. In the upcoming lectures, we will discuss more on the Max, Min-Max Theorem, and independent sets and covers.

Thank you.