Advanced Graph Theory
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Lecture - 21
Line Graphs and Edge-Coloring

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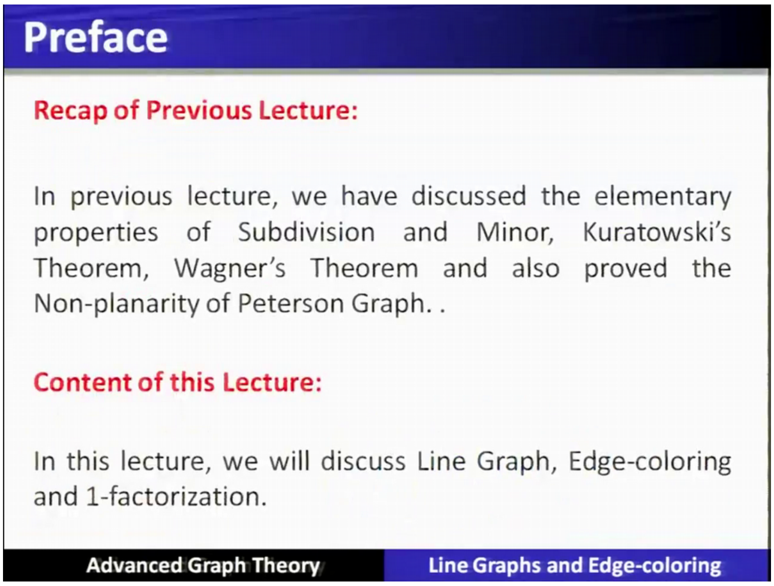

Recap of previous lecture, we have discussed the elementary properties of Sub division and Minor of a graph Kuratowski's Theorem, Wagner's Theorem and also proved Non-planarity of Peterson Graph. Content of this lecture, we will discuss the Line Graphs and how the line graphs are used in Edge-coloring and 1-factorizations of a graph.

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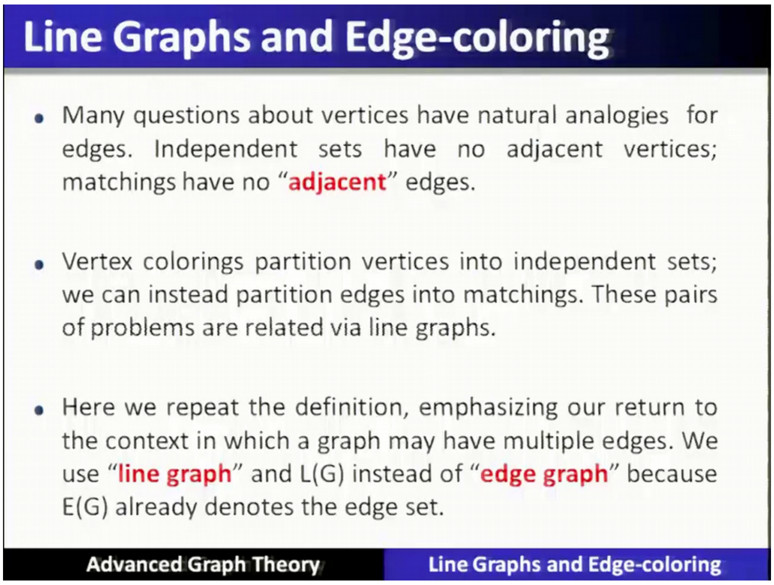

So, let us see the concept of Line Graph and we will use it for the Edge-coloring. So, many questions about the vertices have analogies for the edges. Independent sets have no adjacent vertices and matchings have no adjacent edges, so, analogs.

We have vertices and corresponding problem in the edges are also there with a different name. If no vertices which are adjacent called independent sets; then for the matching no edges are adjacent. Now, vertex coloring partitions the vertices into independent sets; we can illustrate partition the edges also into the matchings. So, these problems we can relate via the line graphs. Here, we repeat the definition emphasizing our return to the context in which the graphs may be may have the multiple edges. We use line graphs because the edge graph is not a proper term.

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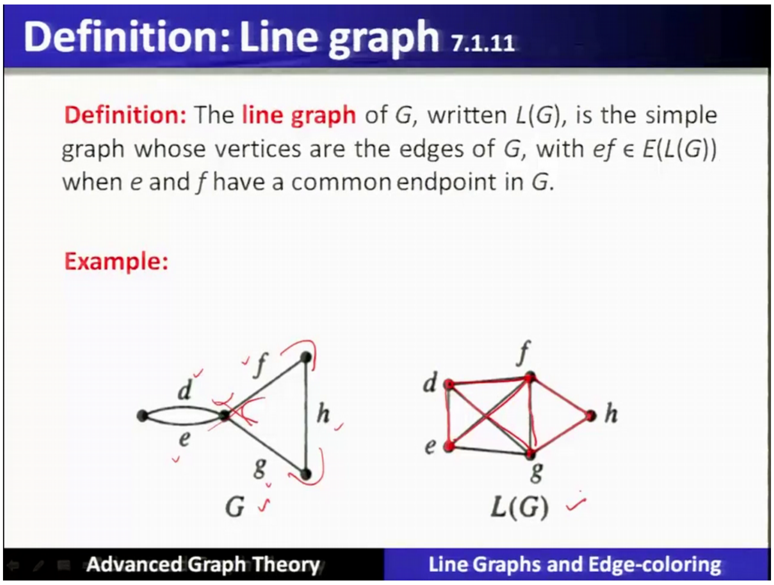

Now, let us see the Line Graph. Line Graph of a graph G is represented as L of G is a simple graph whose vertices are the edges of that graph G with the edges of line graph is nothing but when the edges of G, they are meeting at a common point. Take this particular example, if this is the graph G where we have labeled the edges with their names. So, for every edge there will be a vertex in the Line Graph. Now, as far as a edge in a line graph says that; whenever these 2 edges in the original graph are meeting for example, d and e. So, they meet. So, they will form an edge in the line graph.

Similarly, d and f, they meet, so, d and f as an edge. Similarly, e and f meets, so, e and f as an edge. Similarly, d and g they meet, so, d and g have an edge and f and h they meet. So, f and h they have an edge. So, h and g they meet. So, h and g they have an edge and f and g they also meet. So, f and g they have an edge. So, if we are given a graph g, we can obtain the line graph of that particular graph using this concept.

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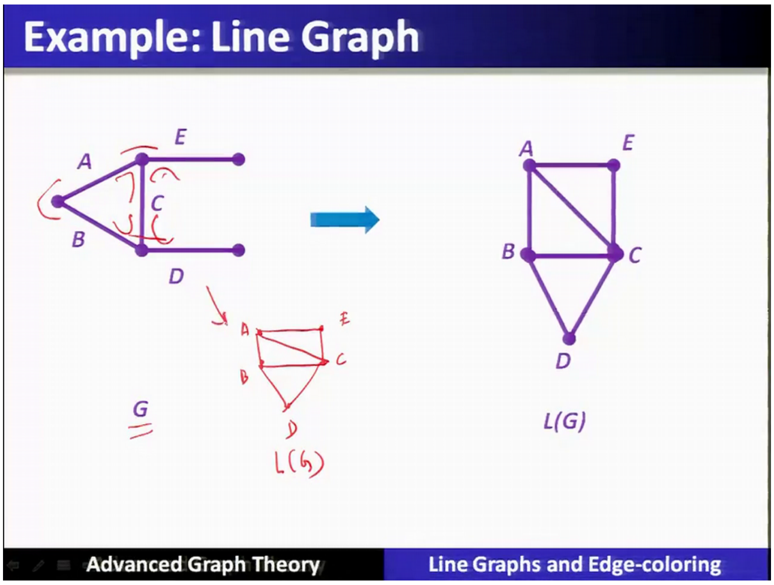

So, again we can take another example. In this example, we are given A graph. So, for a we have an vertex, B we have a vertex, for E we have a vertex, for C we have a vertex and for D we have a vertex. Now, A and B they are meeting, A and B they are meeting, so, we place an edge. A and E they are meeting, we place an edge. A and C they are meeting, we place an edge, then B and D they are meeting, B and C they are meeting, so, we place an edge. B and D they are meeting, we place an edge. Now C and D they are meeting, so, we place an edge.

Similarly, E and C they are meeting, so, we place an edge. So, this is called a Line Graph of this particular graph.

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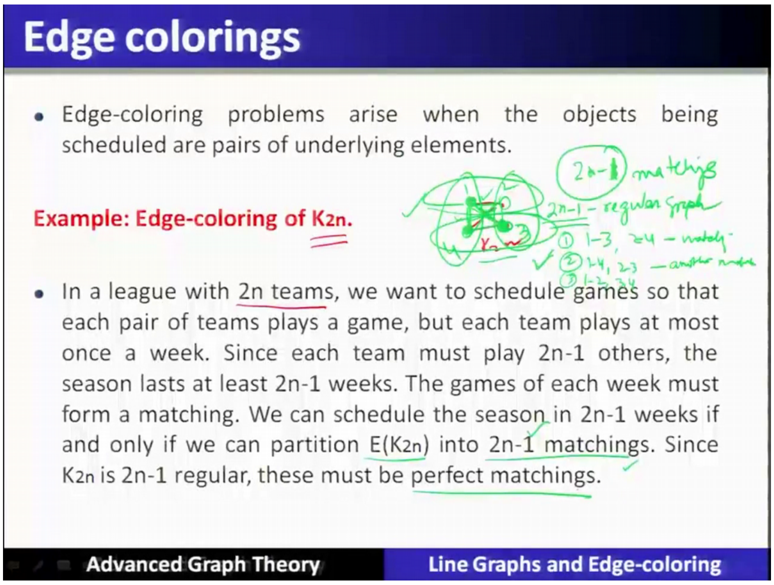

Now, the Edge-coloring problem arises when the objects being scheduled are the pairs of the underlying elements. So, example of edge colorings of let us say k 2n. So, k 2n when n is let us say 2. So, this becomes k 2n graph. Now, re-bond a pair of objects to be considered as per the scheduling of a game is concerned, then it requires an edge coloring not the vertex coloring. So, in the league of 2n teams, you want to schedule the game, so that each pair of teams play a game, but each team plays at most once in a week.

So, since team may must play 2n minus 1, others the season last at 2n minus 1 week. So, the games of each week must form a matching, we can schedule the season in 2n minus weeks if and only if we can partition this edges into 2n minus 1 matching, since k 2n is 2n minus 1 and regular, there must be a perfect matching. So, we can see here that having arranged the bipartite graph instead of that let us add. So, this becomes k 2n graph. So, k 2n graph if we take a particular team. So, how many because it is 2n minus 1 regular graph. So, it can have a pairing with all n minus 1 vertices. So, these edges will pair.

Similarly, the other edges will also have such pairings. So, different pairings are basically possible. So, we can partition these set of edges into 2n minus 1 different matchings and how to obtain this matchings that is the pairings is called an Edge basically coloring. So, when we obtain a pairing, so, there is a rule that if these 2 are selected then the remaining pair; that means, this is one set of match when 1, 2, 3, 4 when 1 and 3 is one such of matching and 2 and 4 is one matching. Second matching we can obtain as 1, 4 and 2, 3 is another matching.

Similarly, we can have another matching like 1, 2 and 3, 4 n. So, this particular graph as 2n minus 3 different matchings and this will require 2n minus 1, so, 2n minus 1 matchings. So, here 2n minus 1 comes out to be 3. So, 3 matchings we have obtained. So, for that, you might have seen that the graph has to be a 2n minus 1 regular and it has a perfect matching then only 2n minus 1 different matchings are possible.

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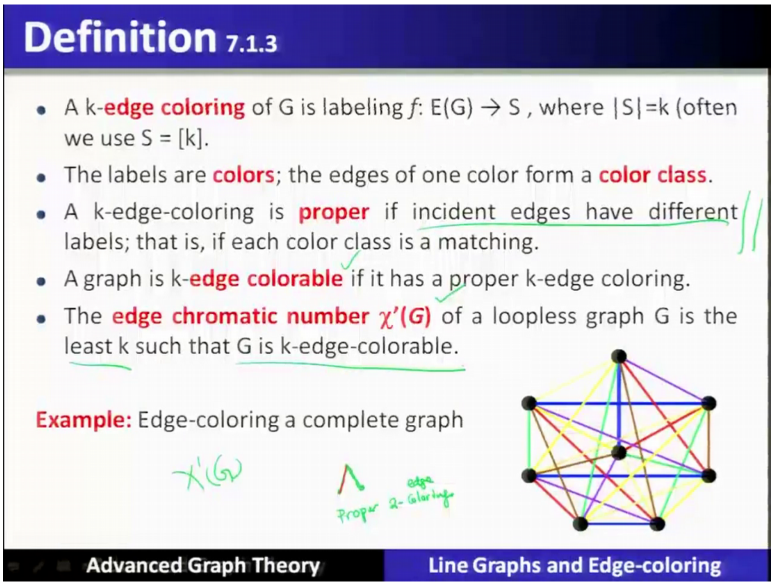

Therefore, let us see the definition of k edge coloring of a graph is the labeling of the edges with the color set S, where the cardinality of axis k so; that means, applying k different colors, we are placing the colors on these edges.

And k edge coloring is nothing but labeling or a function. So, the labels are the colors. The edges of one color will form the color class, so k edge coloring is proper if the incidence edges have different labels that is if each color class is a matching. So, let us understand what is the proper coloring the incident edges have different labels, so, take this particular graph. So, the incident edge on this particular vertex if this is having label let us say red. So, the other edge will have another label why because, the incident edges should have the different labels for a proper. So, it is a proper 2 colorings 2 edge colorings.

So, a graph is k-edge-colorable if it has proper k edge colorings. The edge chromatic that is, chi prime G of a loop less graph is the least value of k such that the graph is k-edge-coloring. Example of edge-coloring a complete graph is basically shown over here.

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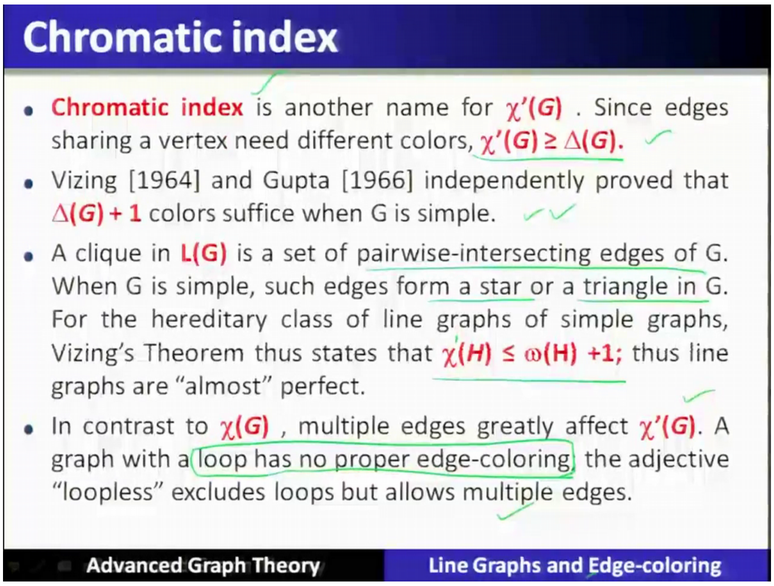

Now, we call the edge chromatic number also as the chromatic index. So, chi prime G is called chromatic index, since the edges sharing a vertex need different colors. Therefore, this bound on this edge chromatic number that is chi prime G is at least the maximum degree of a particular graph. Vizing and Gupta independently proved that delta G plus 1 colors suffice when the graph is simple.

So, we will see that what are the conditions when the edge chromatic number of the graph that is the chi prime G is at least delta and whatever the conditions as per as Vizing and Gupta Theorem that this particular chromatic index will become delta G plus 1. Now, a clique in the line graph in the set of pair wise intersecting edges of G; because the vertices are representing the edges of graph G. So, there is a clique in line graph; this means, there is a set of pair wise intersecting edges of a graph. When G is simple, such edges form the star or a triangle in particular G for the hereditary class of line graph of a simple graph Vizing Theorem states that chi prime edge is less than or equal to omega h plus 1. Thus, the line graphs are almost perfect.

In contrast to chi G, multiple edges is greatly affect the chi prime G that is chromatic index. So, graph with a loop as no proper edge-coloring. So, that is important point to be noted.

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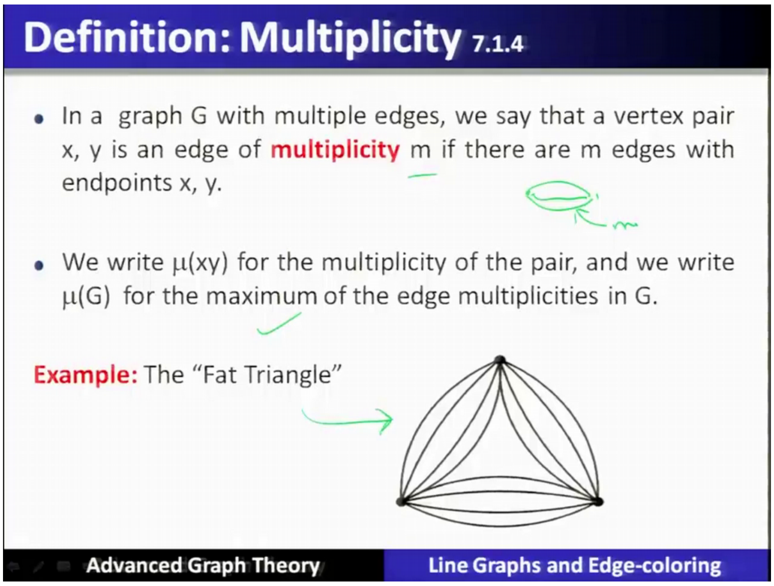

Hence, the loop less will be excluded here, but multiple edges are allowed in the edge-coloring. So, multiplicity multiple edges mean between 2 vertices multiple edges can be there and how many edges are there that is called multiplicity. So, in a graph G with multiple edges, we say a vertex pair x, y is an edge of a multiplicity m if there are m different edges with the end points x and y.

So, here let us say there are m. So, hence the multiplicity of m will be there. Now, we write of mu of x y for multiplicity of a pair and we write down mu G for the maximum of the edge multiplicities in the graph. So, this kind of structure where multiplicities of where multiplicity of a graph or the multiplicity of the pair are allowed and this is called a Fat Triangle.

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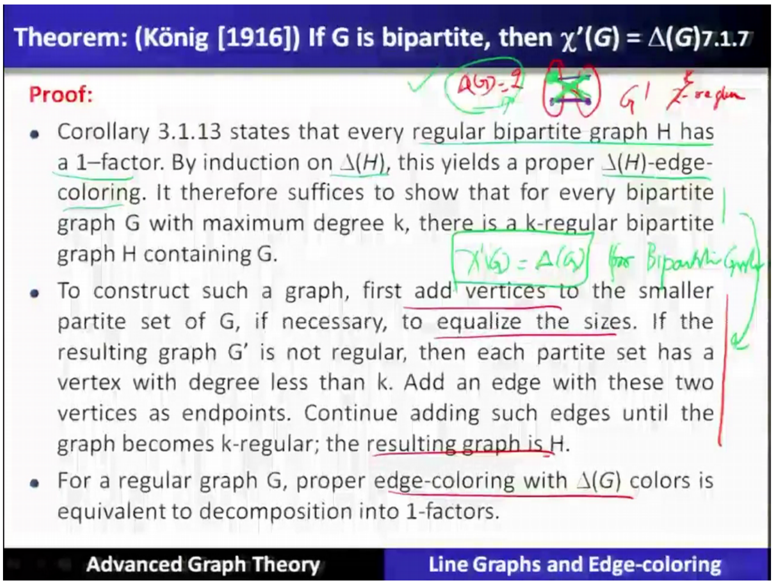

Now, there is a theorem which is given by Konig, which states that if the graph G is bipartite, then chi prime G is equal to delta G.

So, let us see the proof corollary which is stated in the previous videos, 3.1.13 states that, that is the Halls Theorem states that, every regular bipartite graph H has one factor. So, it is a it is a Halls Theorem about matching. So, one factor is nothing but a matching, so, every regular bipartite graph H as one factor. So, by induction on big delta of H, this yields a proper delta H edge colorings if therefore, suffices to show every bipartite graph G with a maximum degree k, there is a regular k bipartite graph containing G. To construct such a graph, we add the vertices to a smaller partite set of part G if necessary to equalize the sizes.

If the resulting graph G prime is not regular, then each partite set vertices less then that degree k at that edge and thus make it s k-regular. For a regular graph G, the proper edge-coloring with big delta G colors is equivalent to decomposition into one factor let us see through an example. So, let us say that this is the graph which is given a bipartite. Now, here one vertex is missing, we can add that vertex to a smaller partite set this is a smaller partite set, we have added one vertex to equalize the sizes. Now, if this particular G prime is not regular, then we will place an edge these edges are added extra and hence these becomes 2 regular or let us say it is a k regular graph.

Now, having obtained; that means, given any such partite graph, we can obtain a k regular graph by these sort of operations by adding vertices and by adding edges having done that? Now, you know that the big delta G of this particular graph is 2. So, we can obtain an edge coloring with that most big delta G colors. So, let us say that if these edge in color, then we cannot color the other edge which are incident on these vertices with the same color. So, the other 2 vertices which need to be colored the other 2 vertices will require the same color, which are not same.

Now, the remaining edges require another color. So, hence we are using a green color. So, the remaining edges we can apply the green color. So, every vertex which is incidence having the edges of different colors. How many colors? That is big delta G the colors will be required. Let us read again. So, corollary 3.1.13 it states that, every regular bipartite graph as one factor. So, that is basically the perfect matching if the graph is regular bipartite graph. Now, by the induction on big delta of this particular graph this yields proper delta edge edge-colorings. So, we have shown that and we have also shown through these particular steps that, if the graph is not k regular, we can obtain it by adding the vertices or adding the edges.

So, hence this chronic, we have seen the theorem that if graph G is bipartite, then chi prime G is equal to big delta G that is for bipartite graphs. So, for bipartite graph, this particular bound holds according to the Chronic Theorem.

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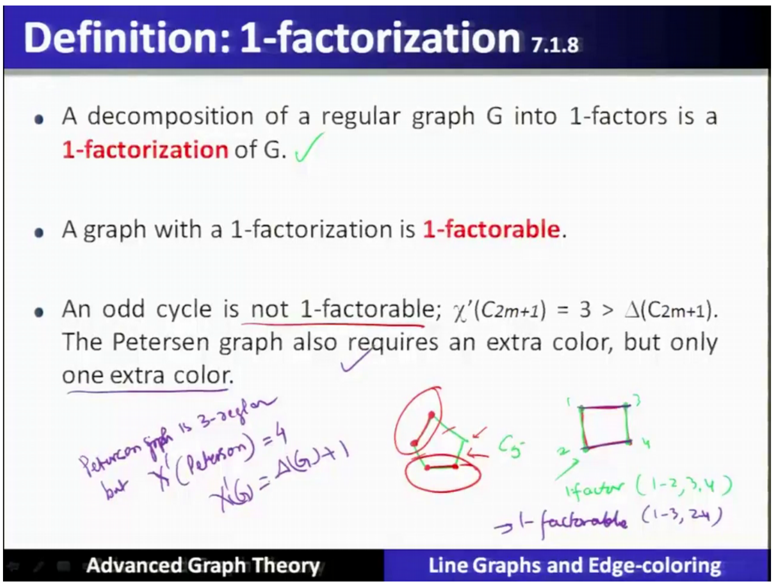

Let us see 1-factorization is nothing but the matchings the decomposition of a regular graph into the 1-factor is the 1-factorizations of a graph G. So, graph with 1-factorization is called 1-factorable. So, an odd cycle is not 1-factorable. So, take an odd cycle, this is C 5 it is a odd cycle it is not 1-factorable; why because, if you can select this edge, then this edge cannot be selected this edge if it is selected this vertex will be unsaturated.

Hence this is not the perfect matching. Hence, it is not 1-factorable. Similarly, if an even cycle if we can take, so we can factorize, we can obtain the 1-factor in the following way. So, this is these are all called 1 factors of this particular graph 1, 2, 3, 4. There is another one factor, you can obtain between 1, 3 and 2, 4. So, this is 1 factorable. So, the Petersen graph also requires an extra color, but only one extra color that is important point to note. So, that means, the Petersen graph is 3-regular, but it requires how many colors chi prime of Petersen is equal to 4; that means, it requires big delta G is equal to 3 plus 1 more. So, Petersen graph is an example where chi prime G becomes plus 1.

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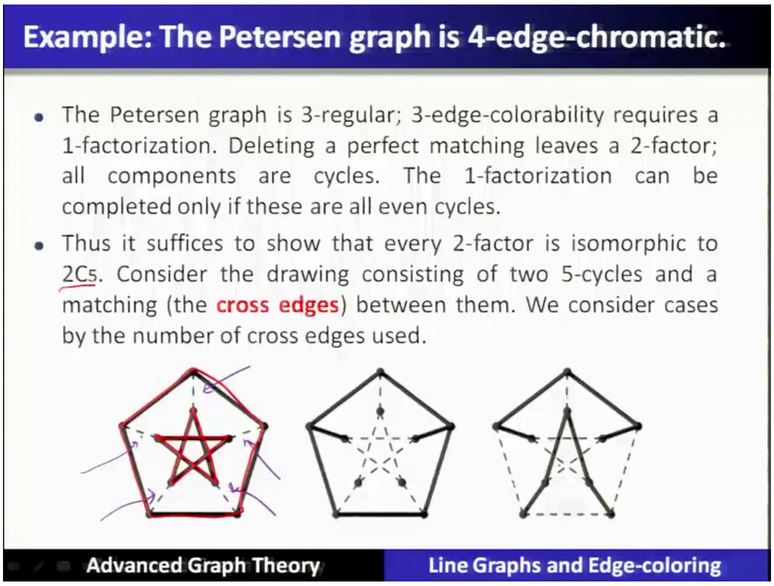

So, Petersen graph is 4-edge-chromatic. So, Petersen graph is 3-regular 3-edge-color ability requires 1-factorizations deleting a perfect matching leaves 2 factors, all components are cycle. Here, we can see if you delete it, this is a perfect matching. So, we will obtain 2 different cycles, which I will show you through this is 1-cycles C 5 and this is another cycle. So, 2 cycles 2C 5 cycles, we will obtain. All components cycles the 1-factorizations can be completed only if these are all even cycles. Thus, it suffices to show every 2-factor is isomorphic to 2C 5. So, here we have seen that consider the drawings of 2C 5 and a matching.

The cross edges between them we consider the cases by the number of cross edges are used.

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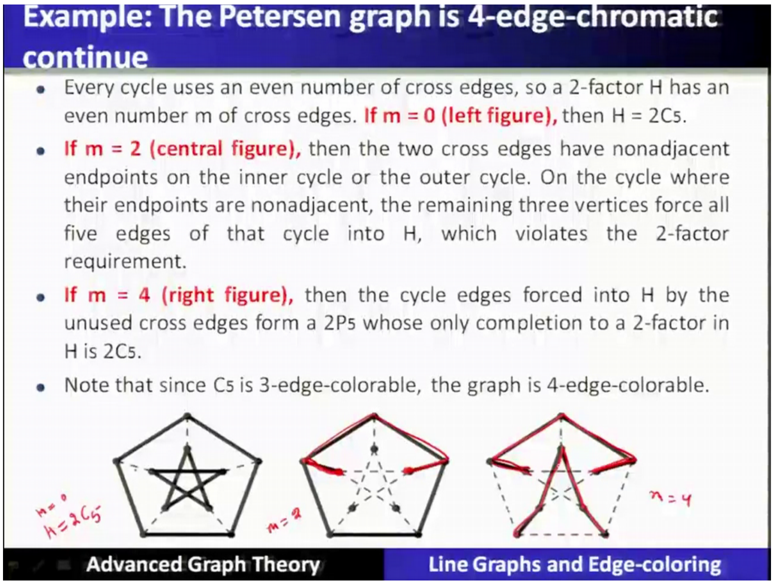

So, every cycle uses an even number of cross edges. So, a 2-factor-edge has an even number m of cross edges. If m is equal to 0, then H is equal to then H is equal to 2C 5. Now, when m is equal to 2, so, here we can see this is 1 and 2 then the 2-cross-edges have non-adjacent end points on the inner circle or the outer circle. On the cycle where their end points are non-adjacent, the remaining 3 vertices 4 or 5-edges of that cycle into edge which violates the 2-factor requirements. So, when m is equal to 4 1, 2, 3, 4, then the cycle edges forced into H by unused cross edge form a 2ps, this is one p and this is another p whose only completion to 2 factor in H is 2cs. Note that, since c 5 has 3-edge-colorable the graph is 4-edge-colorable.

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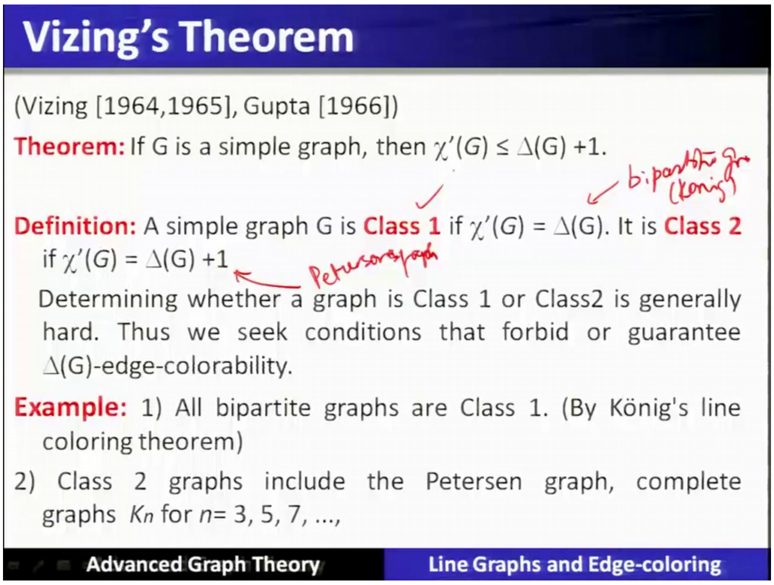

Vizing's theorem, Vizing and Gupta has given a theorem which says that, if G is a simple graph, then chi prime G is less than or equal to big delta G plus 1. Big delta G means, the max degree of particular graph is 1. Now, if we correlate with the previous discussion that we have seen that bipartite graph is having chi prime G equal to big delta that also is included in to it. So, with this, we include the definition; a simple graph G is class 1, if chi prime G is equal to big delta. It is called class 2, if chi prime G is equal to big delta G plus 1. So, the example here is a bipartite graph by Konig's theorem, we have seen this and this becomes valid for a Petersen graph.

Hence, the bipartite graph is class 1 graph and Petersen's graph is class 2 graph. Determining whether a graph is class 1 or class 2 is generally hard. Thus, we seek the condition that orbit or guarantees big delta G-edge-colorability.

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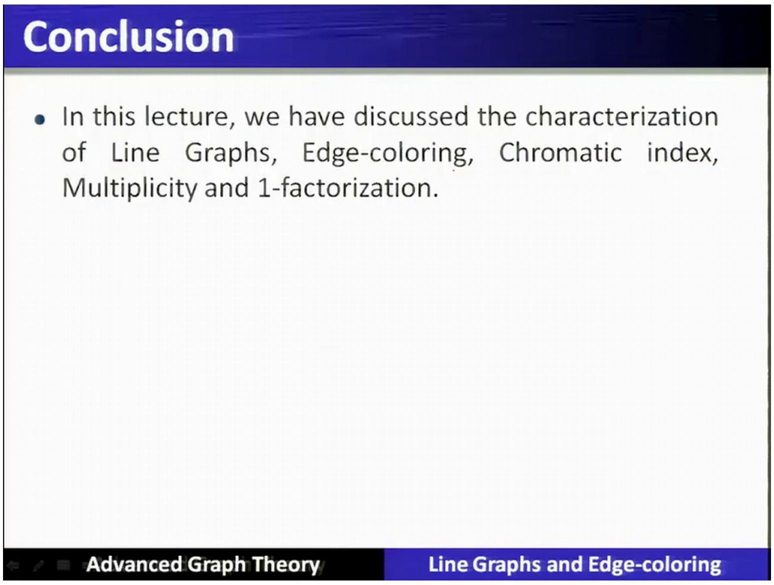

So, conclusion; in this lecture, we have discussed the characterization of a Line Graphs, Edge-colorings, Chromatic Index, Multiplicity and 1-factorizations.

Thank you.