Advanced Graph Theory

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Lecture - 20
Characterization of Planar Graphs

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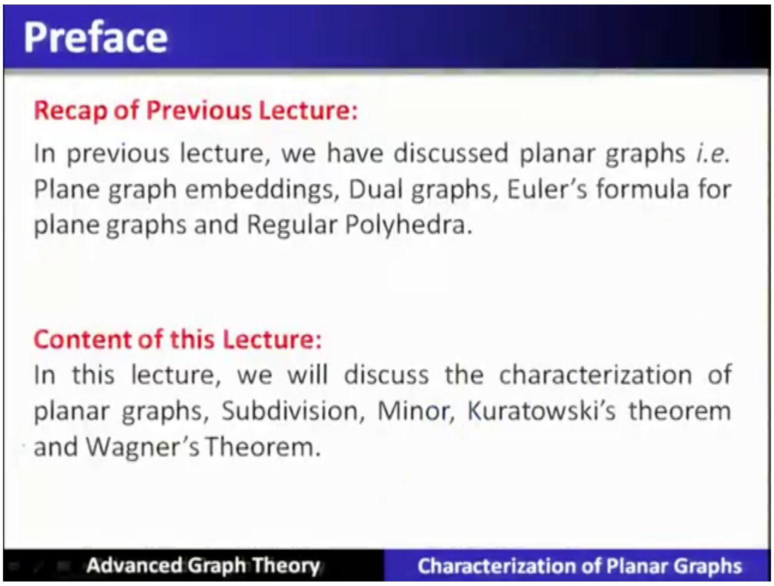

Characterization of Planar Graphs; Recap of Previous Lecture, we have discussed planar graphs that is the plane graph embedding's, Dual graphs, Euler's formula for plane graphs and Regular Polyhedra. Content of this lecture, we will discuss the characterization of planar graphs; that means, at another characterization of planar graphs. Beyond the Euler's formula we will also discuss sub division of a graph minor of a graph Kuratowski's theorem and Wagner's Theorem.

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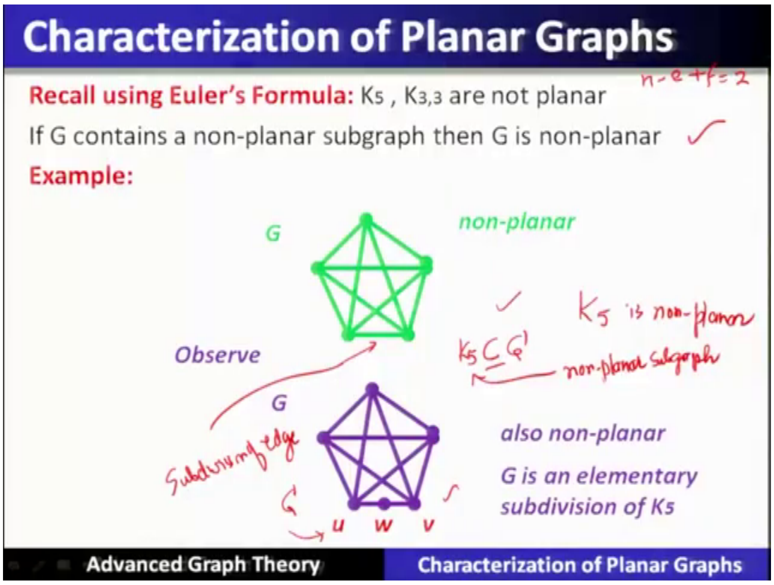

Characterization of planar graphs, if we recall we have used in the previous video the Euler's formula, which is n minus e plus f is equal to 2 and we have seen using this particular Euler's formula that K 5 and K 3 3 are non-planar.

Now, if G contains a non-planar sub graph then G is not a planar that is G is non-planar that, we will see as that another characterization of a planar graph; that means, you know that a planar graph if it has the sub graph of a planar graph is also a planar.

Similarly if the sub graph of a graph is non-planar; obviously, then a graph is not a planar graph or graph is planar graph, another characterization that we will see. For example, we see that this is K 5 K 5 is non-planar.

Now, if we add some vertices and we form another graph, where this K 5 is a sub graph then this entire graph G prime a bigger graph is also non planar why because this K 5 is a sub graph of this particular graph. So, hence any non-planar sub graphs if the graph G contains a non-planar sub graph then G is also non planar this is the necessary condition for the non-planarity, that we will see in for the theorems. Further more if we take G 5 if we take K 5 and we take the sub division of this h.

So, sub division of the edge we will obtain another graph and from a non-planar graph if we obtain a sub division of a non-planar graph, then that particular graph also non planar all these things we will discuss with a characterization of the planar graph.

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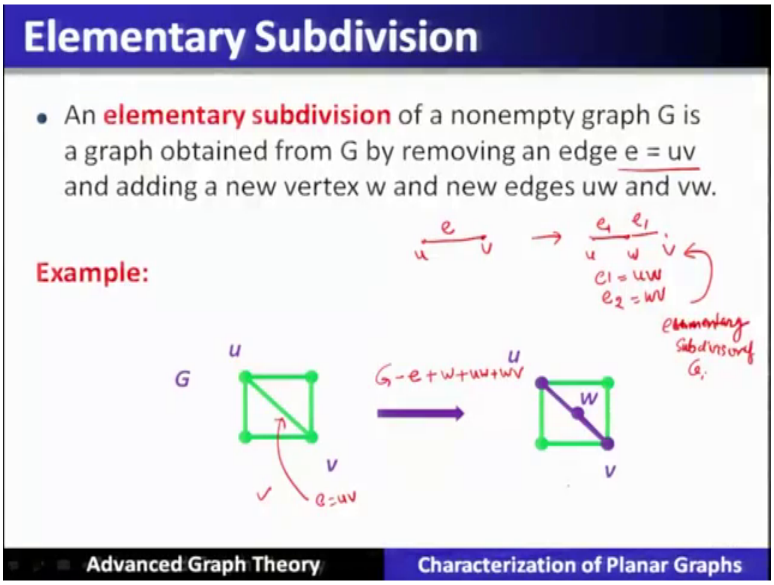

Let us see about the elementary sub division an elementary sub division of a non-empty graph G is a graph obtained from G by removing an edge that is e having 2 end points uv and adding a new vertex w and the new 2 edges that is uw and vw meaning to say that if there is a edge, with the end points uv. So, instead of that what we will do is we will remove this particular edge and we will add a vertex w with the end points u and v and we will add 2 edges, that is e 1 is equal to uw and e 2 is equal to wv this is called the elementary sub division of the graph.

So, for example, if this is the graph now this particular edge we will consider what we will do is we will take this graph, without this particular edge and we will add 2 more edges plus 1 more vertex we will add and we will add 2 more edges that is uw and wv. So, this is called the elementary sub division of the graph g.

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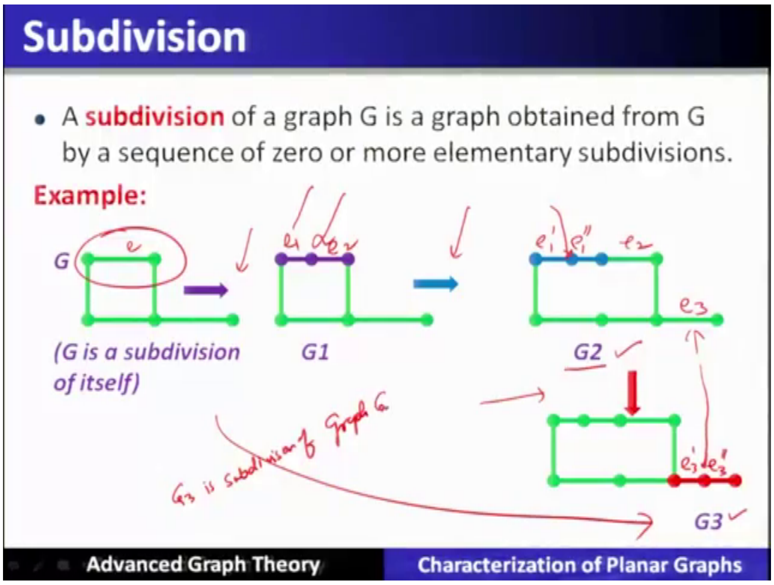

Now, the sub division of the graph G is a graph obtained from that graph by sequence of 0 or more such elementary sub divisions that is called a sub division of a graph G let us take a graph G and obtain a sub division of graph G that is G 3 by applying more than 1, such elementary sub divisions in this particular graph G. Now if we take if we apply the sub division in this particular edge and we will introduce a vertex and this particular edge will be replaced by a 2 different edges.

So, let us say that this is edge e. So, we are replacing by 2 more edges e 1 and e 2 using the elementary sub division. Similarly this particular edge we are further sub dividing it this particular edge even we are further dividing it into e 1 prime and e 1 double prime. This is elementary sub division and we will introduce new vertex into it obtaining another graph that is G 2 and now if you see this particular graph.

We will obtain and here if you take this particular edge and we will replace this edge e 3 as e 3 prime e 3 double prime and also introducing new vertex. So, we have obtained a G 3. So, G 3 is basically a sub division of a graph G.

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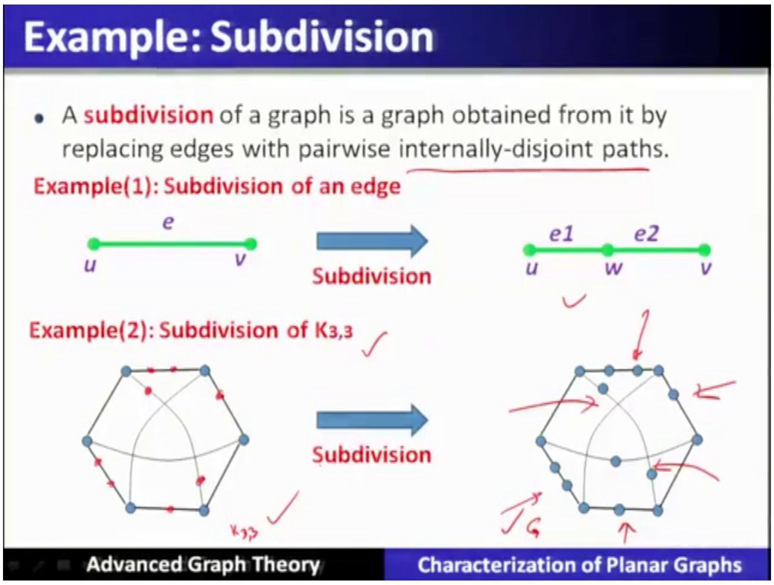

Now, another example of a sub division is sub division of a graph is a graph obtained from it by a replacing the edge with a pair wise internally disjoint paths that is basically here they are also called as a internally pair wise internally disjoint paths. So, if you take the graph K 3 3 and if you apply this particular sub division of this graph we will obtain another graph from K 3 3 by sub dividing these edges, this edge, this edge, this edge, sub division this edge sub division this edge and this edge.

So, let us take this particular graph and apply the sub division. So, from K 3 3 we obtain this particular graph G after doing the sub divisions. Hence this graph G is a is is a sub division of the graph K 3 3.

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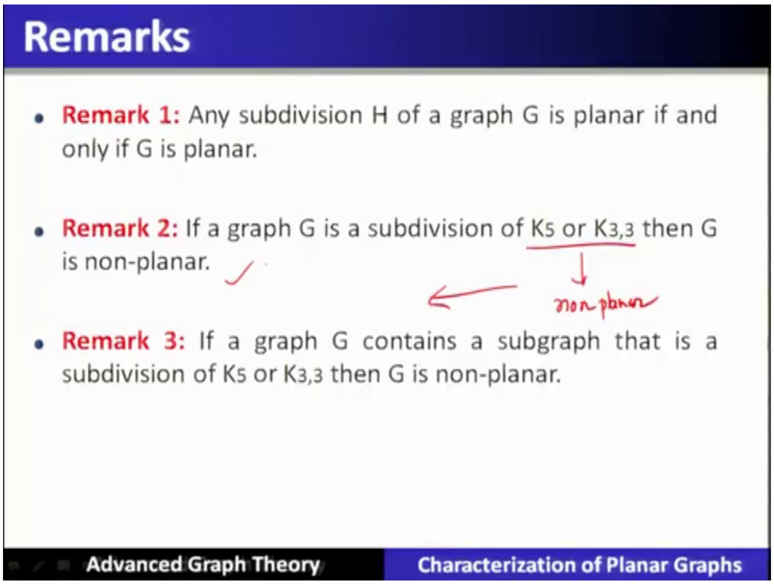

So, any sub division h of a graph G is a planar if and only if G is a planar graph. So, if we take if we obtain a graph let us say H from G using the sub division and that G let us say is K 5 or K 3 3, then any such sub division which we obtain out of these K 3 3 and K 5. Hence basically are non-planar and their sub divisions will basically prove that a given graph edge is also a non-planar.

So, for planarity condition so, any sub division h of a graph G is planar if and only G is planar, this is will characterized yet by another way the planar graphs. So, remarks 2 says that if graph G is a sub division of K 5 or K 3 3, then G is non-planar why because we have earlier prove that K 5 and K 3 3 they are non-planar and any sub graph, which we obtain of the as per the subdivision of these graph will also be a non-planner. Now remark 3 says that if a graph G contains a sub graph that is a sub division of K 5 or K 3 3 then G is non-planar again I am repeating if a graph G contains a sub graph that is a sub division of K 5 or K 3 3 then G is a non-planar.

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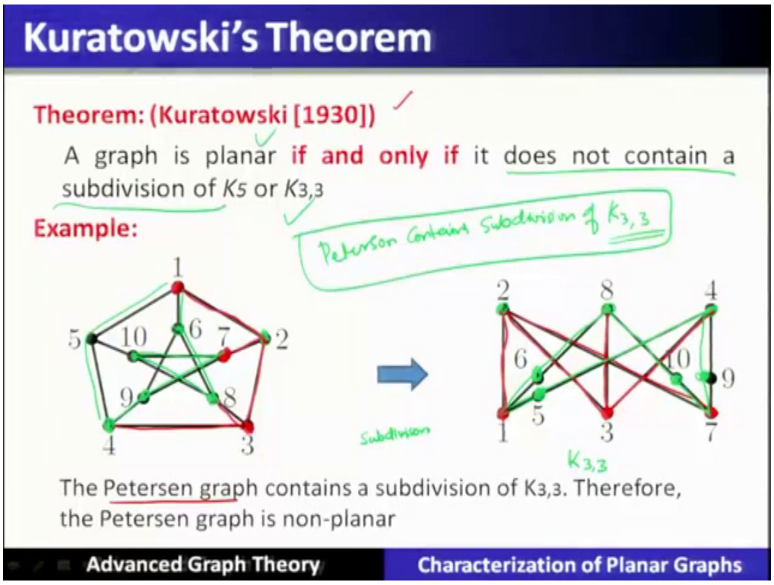

So, with this let us see the famous theorem for the characterization of planar graph that is given by Kuratowski's theorem Kuratowski has given a theorem in 1930, the theorem is stated as the graph is planar if and only if it does not contains the sub division of K 5 or K 3 3.

So, let us take the example of the statement of this particular theorem. So, in the example let us take a famous graph is called a Peterson graph. Now we will obtain. So, we will show that this Petersons graph contains the sub division of K 3 3 and as per the statement of the theorem if this particular Peterson graph contains a sub division of K 3 3, which is non-planar also hence that graph will also be non-planar. Now the sub division is if the sub division is planar then only the Peterson graph can be planar, but we will show that the sub division of Peterson graph is K 3 3 which is non-planar hence the Peterson graph is non-planar.

So, to show that Peterson graph is non-planar let us find out let us see the bipartite graph 1 3 and 7. Let us take these vertices on 1 side of a Peterson graph and the other side of a Peterson graph let us choose 2 then 8 and then 4.

So, you can see they are independent sets 2 4 8 they are independent sets let us draw the bipartite K 3 3 graph. So, as per as 2 8 4 is concerned we will draw an vertices and the other side we will draw 1, then 3 and then 7 let us draw the edge between them 1 and 2 is an edge. So, that edge we have taken up then from 1 2 and 3 there is an edge 2 and 3 there is an edge. So, we added the edge then 3 and 4 are having an edge. So, this edge we have added then 3 and 8 we have an edge.

So, this edge we are going to add then 2 and 7 we have an edge that we will place an edge here in our bipartite graph. Now as per as 1 and 4 is concerned 1 and 4 they do not have an edge, but we will take the sub division of 1 4 with 5 to be included as a sub division. So, 1 and 4 they have an edge with a sub division at 5. So, we will place an edge between 1 and 4 with 5 as a as it is sub division.

Similarly, 1 and 8 will have an edge with the sub division at 6. So, 1 and 8 is having an edge with a sub division at 6. Now we will take 7 and 8 now 7 and 8 is having an edge with 10 as it is sub division 7 and 8 is an having an edge with 10 as it is sub division point.

Now 7 and 4 will also have an edge with 9 as it is sub division so, we have included we have obtained K 3 3 as a sub division of. So, this particular graph contains this particular graph Peterson contains sub division of K 3 3. Now we know that this particular Kuratowskis theorem says that graph is planar if and only if it does not contains a subdivision of either K 5 or K 3 3 since it contains a sub division of K 3 3 hence it is not a planar graph.

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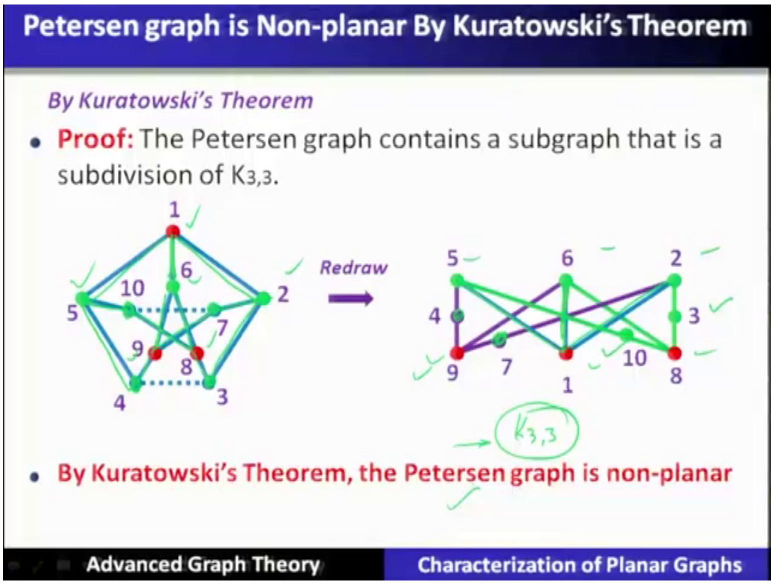

So, Peterson graph is non-planar by Kuratowskis theorem we have again shown this in this particular figure we will take another example to basically obtain a sub division of K 3 3. So, if you let us see it again. So, let us take red particular dots as basically 1 side of part of it is sets that is K 3 3 and all the green points as the other sides of the part of it is sets.

Now let us add an edge between 1 and 6 there is a direct edge between 1 and 2 there is a direct edge between 1 and 5 there is direct edge that we have included, now between 8 and 2 between 8 and 2 8 and 2 there is a edge with 3 has it is sub division similarly 8 and 6 is a direct edge, then 8 and 5, 8 and 5 is having an edge with 10 has it is sub division. Now you will see the 9 9 and 6 is having a direct edge then 9 and 2 is having an edge with 7 has it is sub division.

Similarly, 9 and 5 is having an edge with 4 has it is sub division. Hence we have obtained we have shown that this Petersons graph contains a sub division of K 3 3 according to the Kuratowskis theorem this particular K 3 3, which is a non-planar it contains a sub division as a non-planar graph hence it is not a planar graph as per as the theorem is concerned.

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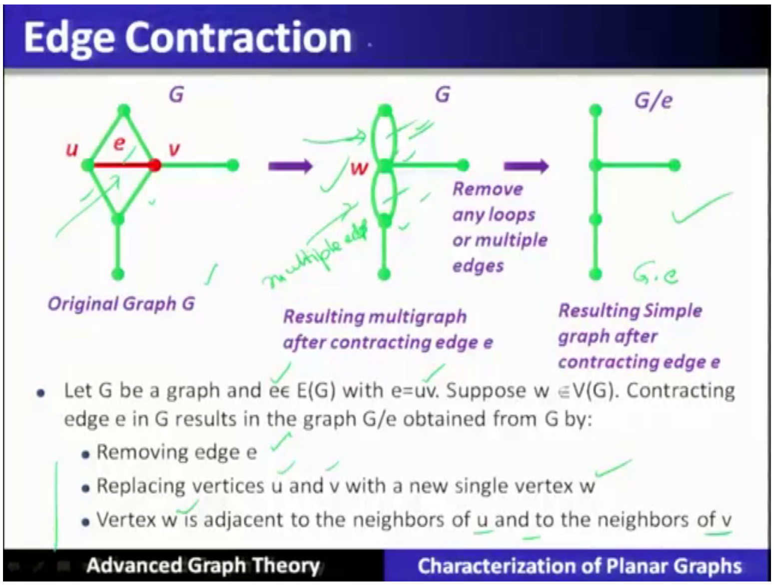

Now, let us see another way of characterization of planar graphs by the way of edge contraction. Let us revisit the concept of edge contraction which we have earlier shown in the video lecture.

So, if this is the original graph and let us choose a edge e by contraction means we will remove this edge and join these particular vertex u and v they will become 1 vertex that is called edge contraction and all the other incident edges will remain interact. So, this becomes basically the loops these becomes the multiple edges between these 2 end points.

If you remove them the extra edges we will obtain a graph without multiple edges that is a simple graph, which we obtain out of the edge contraction edge contraction. Sometimes we will write down using dot also or using another such notations.

So, let G be a graph and e be an edge in a graph with it is end points as uv suppose w is an vertex in v now contracting an edge e in G results in that h contracted graph obtained from the graph by removing that edge replacing these vertices u and v with this with a single vertex w, that we have done then the vertices then with the vertex w is the adjacent to the neighbors of u and neighbors of v. So, all the adjacent relation of u and v are preserved by w hence these particular edges are added. So, that is called edge contraction steps.

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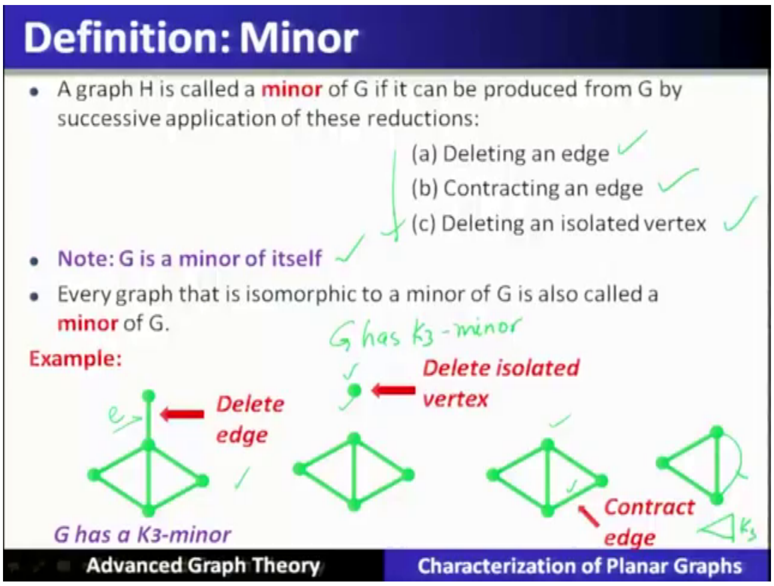

So, having understood the edge contraction let us define the minor of a graph let H is called the minor of a graph G if it can be produced from G, by successive application of the edge contraction or by deleting an edge or by deleting an isolated vertex.

So, with these 3 reductions if a graph which is obtained out of graph G, which is that graph is called a minor of that graph. So, graph G is a minor of itself if none of these operations are performed now every graph that is isomorphic to a minor of graph G also called a minor of graph G.

Now, let us see non trivial cases. Now here we will show that this particular graph G has K 3 as a minor. So, to show that to obtain a minor of this particular graph let us perform 1 of these steps. So, let us choose this particular edge E and delete this edge. So, this particular vertex will remain in isolates vertex.

Now perform another operation called deleting an isolated vertex we will obtain this particular graph in this graph we will select this edge and perform the edge contraction. So, when we perform the edge contraction the multiple edge will come and let us delete this particular edge so, that we will obtain a K 3 as it is minor. So, the graph G we have shown as K 3 as the minor.

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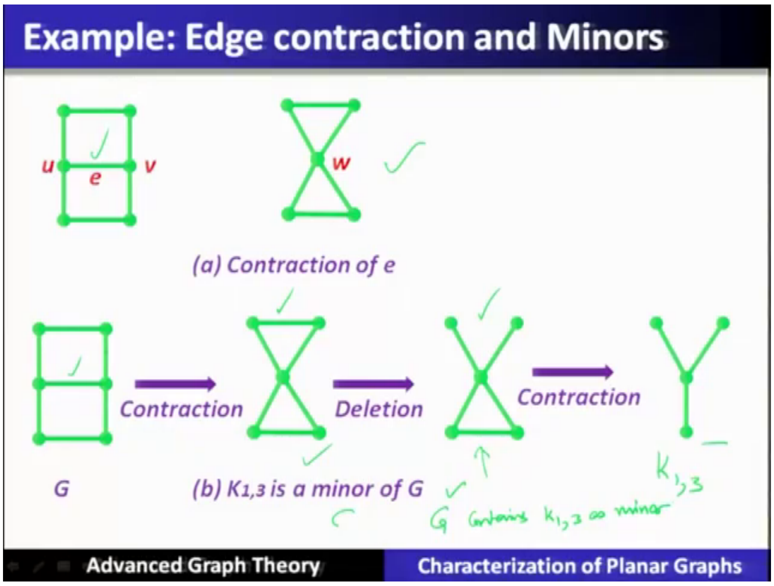

Now, in this particular picture we will take this edge contraction example here if you contract this particular graph around along this particular edge e we will obtain the edge contraction of that graph on the edge e that is called edge contraction. So, let us see that from a graph G if you perform the edge contraction of around this particular edge we will obtain this graph and again we will perform the deletion of this particular edge, we will obtain this graph and then perform another contraction this edge contraction we will obtain this particular graph.

So, this graph is called K 1 3 hence this graph G contains a minor or we can also state that K 1 3 is a minor of graph G or G contains K 1 3 as the graph minor.

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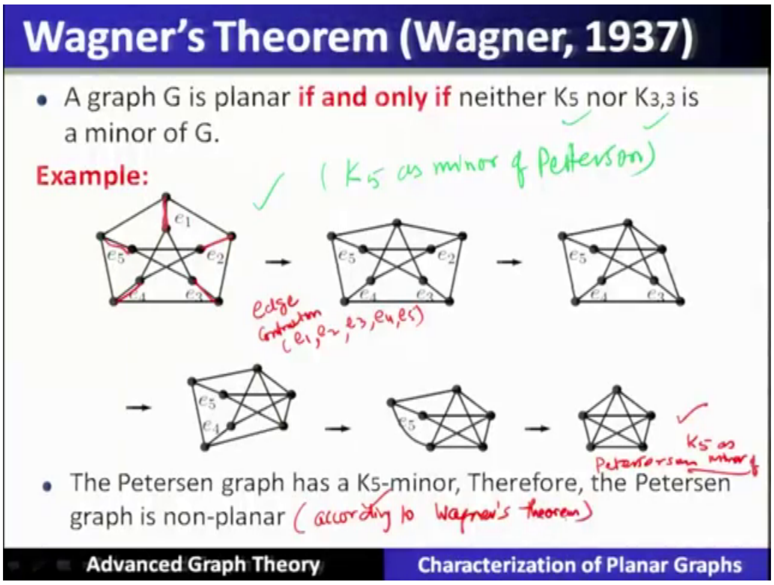

With this let us state the Wagner's theorem given in 1937 a graph is planar if and only if neither K 5 nor K 3 3 is a minor of that graph G. Let us take the examples of a Peterson graph that is well known graph and try to obtain the minor of K it is minorized K 5 or K 3 3.

Now, for Petersons graph it is easy to obtain a minor K 5. So, we will choose to obtain K 5 as the minor of Peterson graph, because this particular graph looks like a K 5 or K 5 as it is minor. So, we will choose this K 5 and let us perform let us show that this Petersons graph is are K 5 is the minor of this Petersons graph. So, let us perform these steps first we will choose we will choose the edges these edges, where we perform the edge contraction on Peterson graph following edges we have chosen e 1, then e 2, then e 3, e 4 then e 5; 5 different edges when we contract h when we do the when we perform the edge contraction we will obtain this graph that is nothing, but a Peterson graph.

So, this is K 5 as a minor of Peterson graph since Peterson graph as K 5 minor therefore, Peterson graph is non-planar according to Wagner's theorem.

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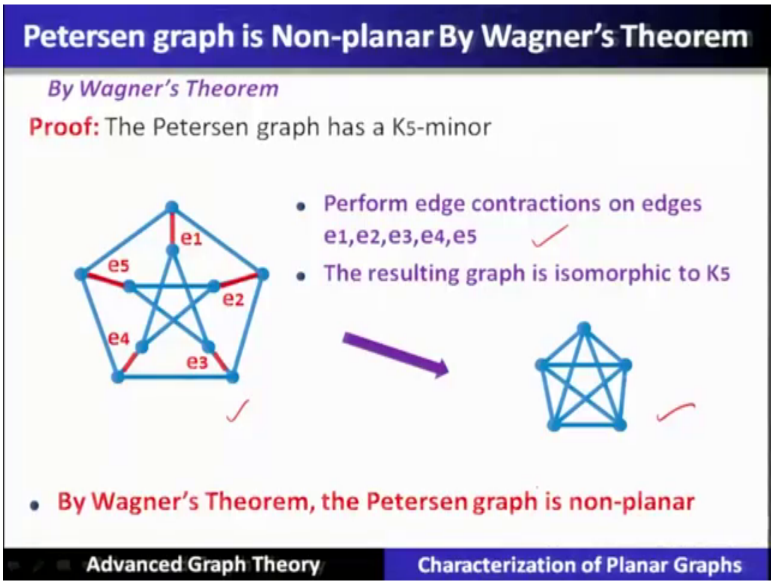

So, same thing again we are representing it let us say that this is the Peterson graph we will perform the edge contractions on different edges shown in the red color the resulting graph is isomorphic to K 5. So, Wagner theorem as prove that this Petersons graph is non-planar. So, we have seen that Petersons graph is non-planar by 2 different ways that is by Kuratowskis theorem and Wagner's theorem.

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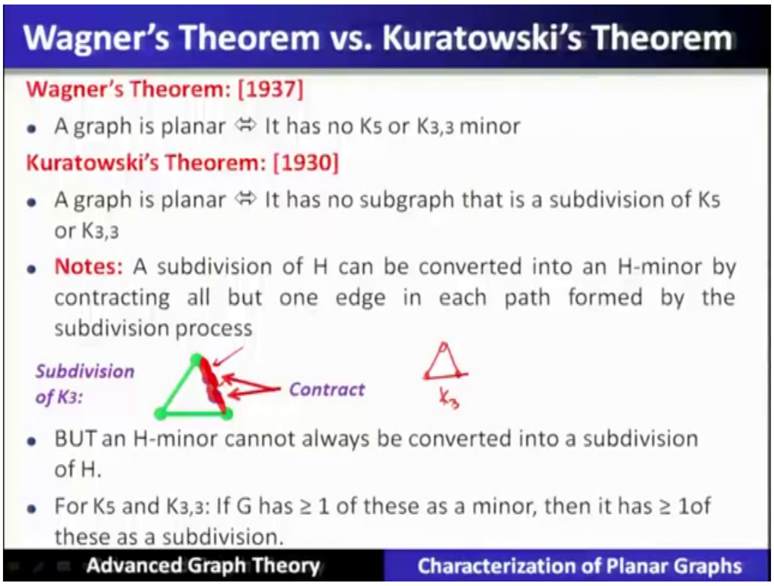

Now, let us see the difference between these 2 theorems that is given by Wagner and Kuratowskis Wagner's theorem is stated in 1937 much later than Kuratowskis theorem, which says that a graph is planar that we will characterize that it has no K 5 or K 3 3 as it is minor Kuratowskis 1930 he stated that a graph is planar. That is equivalent to saying that it has no sub graph that is the sub division of K 5 or K 3 3.

Now this sub division of H can be converted into H minor by contracting all, but 1 edge in each path form by the sub division process, let us see this particular example. So, if this is the sub division we can convert into H minor by performing the edge contraction. So, when we perform the edge contraction this will become one particular vertex then again we will perform the edge contraction this edge contraction.

Then this will become 1 and then 1 more edge contraction. So, this will become 1 edge between and hence we obtain the K 3 3 as it is minor, but an H minor cannot be converted into the sub division of H. So, for K 5 and K 3 3 if G has more than 1 of these as minor then it has more than 1 of these as the sub division.

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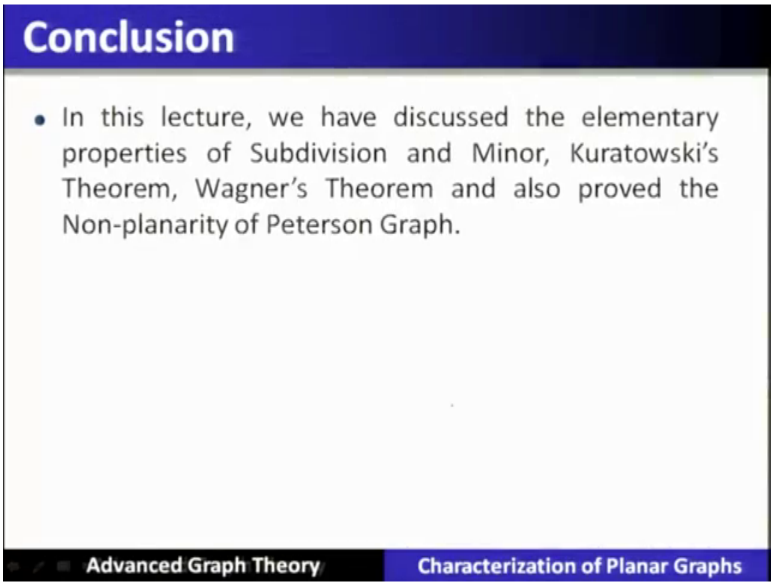

So, conclusion in this lecture we have discussed the elementary properties of Sub division and Minor, Kuratowski's theorem Wagner's theorem and also prove the non-planarity of Peterson graph.

Thank you.