Advanced Graph Theory
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Lecture – 02
Paths, Cycles, and Trails

Lecture 2; path, cycles, trails and walks. Recap of previous lecture.

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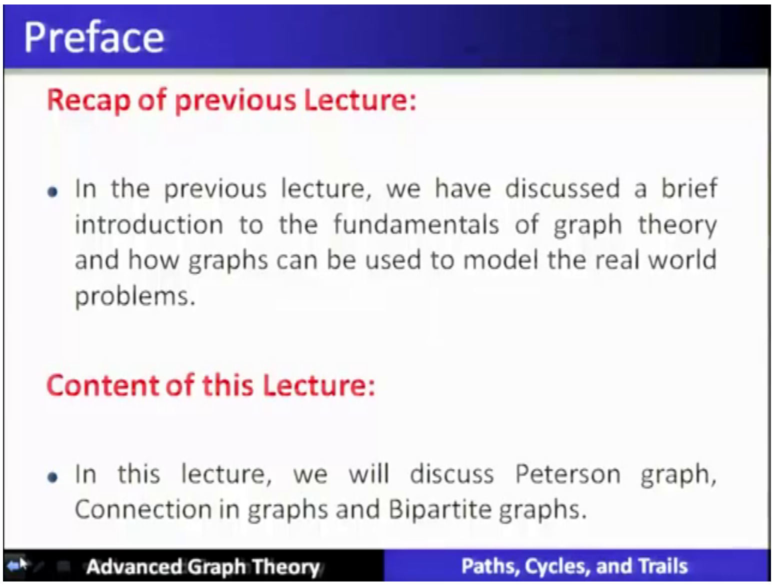

In previous lecture, we have discussed brief introduction to the fundamentals of graph theory, and how the graphs can be used to model the real word problems. So, in the previous lecture we have seen what is a graph.

Content of this lecture; in this lecture we will discuss the petersen graph, the connections in the graph, bipartite graph. There are 3 things with primarily which we are going to cover in this particular lecture, and how these 3 things are going to be used; that is, the walk trail path and a cycle.

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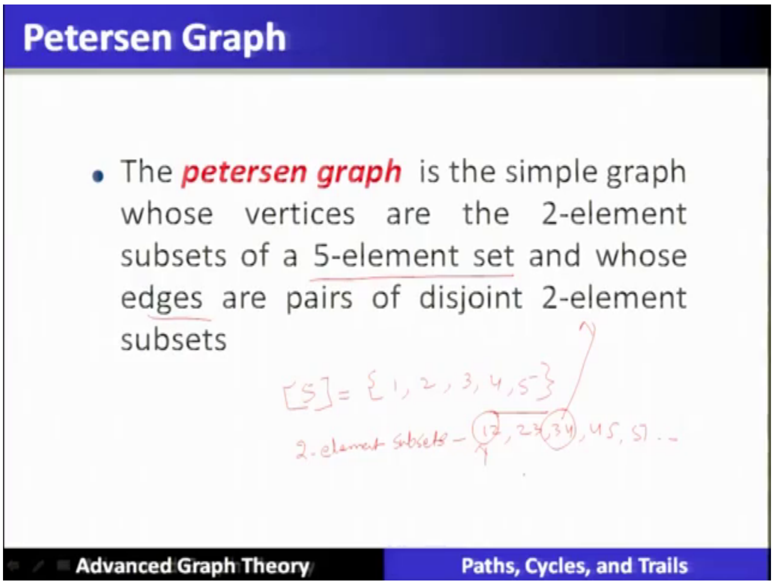

Peterson graph, petersen graph is a simple graph whose vertices are 2 elements of sets of 5 element set and whose edges are the pairs of disjoint 2 elements of sets. Why we are discussing petersen graph? Because petersen graph is a important graph and going to be used as a graph to ex to illustrate various principles of the graph theory.

So, here we have to, see here 2 things one is the 5 element set. Let us assume 1, 2, 3, 4, 5. 5 element set as 1, 2, 3, 4, 5. Now we have to find out we have to form the 2 elements of sets out of them. So, 2 elements of sets which we can form out of these 5-element set would be 1, 2, 2, 3, 3, 4, 4, 5, 5, 1 and so on.

So, here this 2-element set, 2 elements of set will become the vertices here in the petersen graph. And the edge means, the edges are basically those pairs of disjoint 2 elements of sets. For example, 1 2 and 3 4; they are disjoint 2 element subset. So, they will basically put an edge. Here in this particular graph. So, let us see the structure of the petersen graph.

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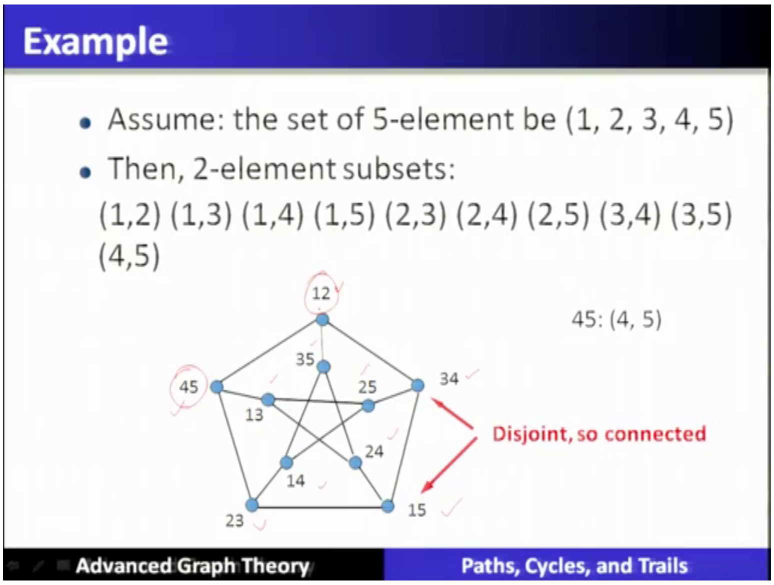

So, here in the petersen graphic, and see that all the subset of 2 elements, 2 element sets will basically form the vertices of this petersen graph. So, you will see that all these are basically the vertices of a petersen graph. And the edge will form between these 2 disjoint 2 elements of sets. So, 4 5 and 1 2 they are disjoint, subsets 2 element subsets. So, basically, they will form an edge.

So, this particular way we can construct a petersen graph.

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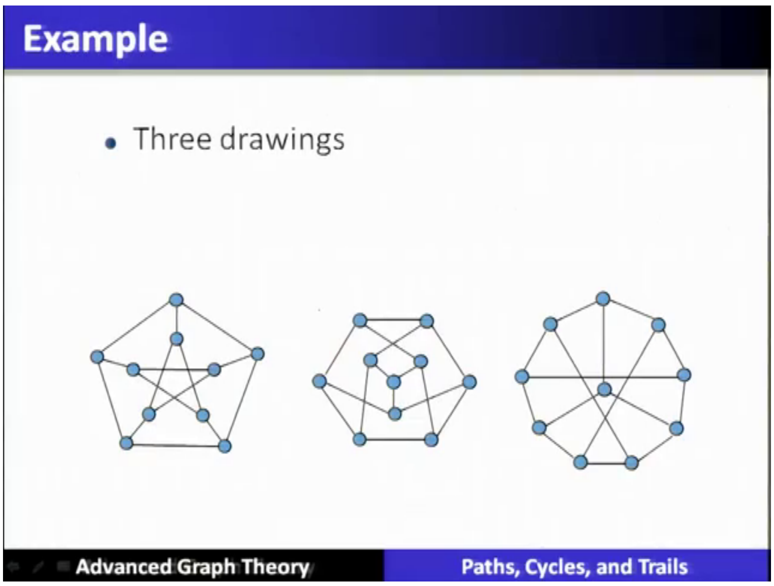

There are 3 ways petersen graph can be drawn. There are 3 drawings possible of a petersen graph, which is shown over here.

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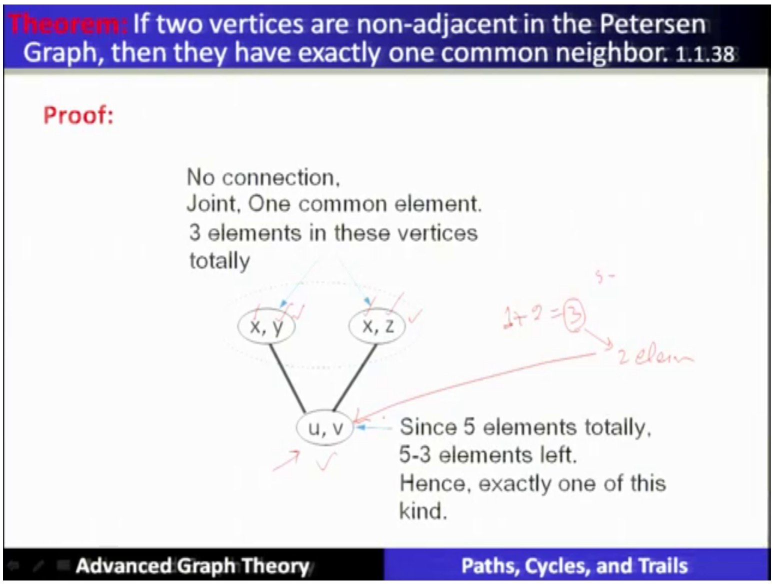

There is a term that if 2 vertices are non-adjacent in the petersen graph, then they have one common neighbor. For example, x y and x z they are basically the vertices. So, there is a common element that is called x. So, basically there is one, node which is basically the joining these 2 elements. So, there will be only at most one element. So, this you can illustrate or you can understand that. So, both of x will basically consume one element out of 5 elements. And x and y will consume furthermore 2 elements. So, together these 2 nodes will form 3 elements. So, out of 5 elements of sets, 2 element will remain, and 2 element will basically constitute only one common neighbor. So, hence the theorem is stated.

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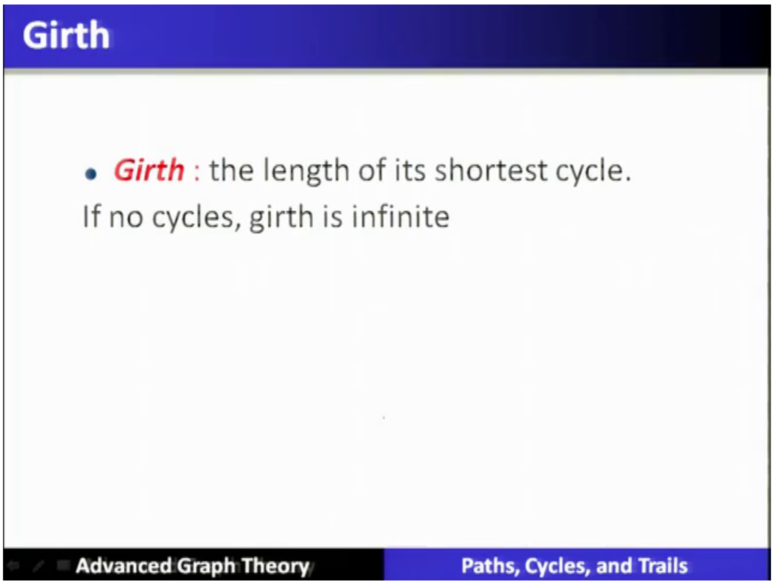

Now, girth of a graph; so girth is basically the length of the shortest cycle in a graph, so if the graph does not contain any cycle, then the girth of that graph is infinite.

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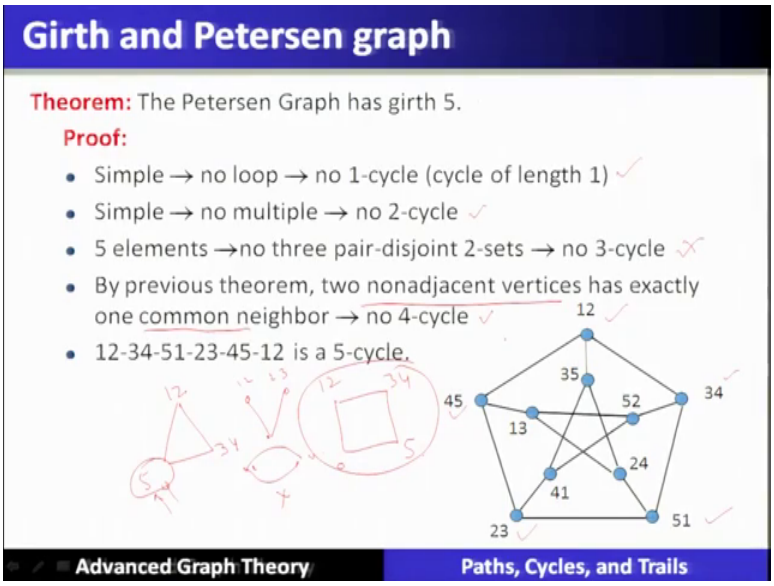

Now there is a theorem which states that the petersen graph has the girth 5 let us see the proof and it will also explain the girth of a particular graph. Now girth of a graph is basically the length of the smallest or the length of the shortest cycle. So, let us start from the shortest cycle that is basically the cycle of length 1. And we have to see whether this is present in the petersen graph or not.

Since petersen graph is a simple graph it has no loops. So, basically it will not have any cycle of length 1. Now we consider a cycle of length 2. Now cycle of length 2 will be basically the 2 edges which have the same end points. So, since this is basically a simple graph. So, such multiple, or multi edges will not be possible here in the petersen graph. Hence there are no 2 cycles.

Now, we consider the next possibility is basically of 3 cycle. Now here in the 5 elements in the previous theorem, we have seen that no 3 pair disjoint 2 sets will have a 2 different nodes joining these 3 disjoint sets. So, basically there will not be a 3-cycle possible here in this 5-element set.

Similarly, in the previous proof we have seen 2 with the previous theorem we have seen the 2 non-adjacent vertices has exactly one common neighbor. So, there will not be any 4 cycle. So, there will not be any possibility of a 3 cycle there will not be any possibility of a 4 cycle. So, there is a possibility of a 5 cycle 5 ca, there are 2 5 cycles exist in the petersen graph that can be illustrated. 1, 2, 3, 4, 5, 1, 2, 3 and 4, 5 is 1 5 cycle. So, hence the girth of this particular graph is 5.

Let us understand the 3 cycle once again. This particular petersen graph has no 3 cycle. So, 3 cycle means a triangle. 3 cycle is possible if let us say it has 1 2, then the edge can be possible wherever there is a disjoint let us say 3 4 and 5. And one more element is required to make this particular complete. So, already all the already 1, 2, 3, 4; 4 elements are already used or 5 elements are already used. So, there is no way this particular cycle can be formed, hence there is no 3-cycle possible.

Now, about 4 cycle 4 cycle is 1 2, then 3 4, now 5 and one more element is required and here also 2 more elements are required. So, in the previous theorem we have seen that for 2 different disjoint. So, 2 disjoint elements there is only one common element, 1 2 and 2 3. There is one common element possible.

Now, if there are 2 non-disjoint vertices is possible exactly for exactly will have only one common neighbor. So, we require 2 common neighbors to complete the 4 cycle which is not possible. So, this completes the proof that petersen graph has the girth of 5.

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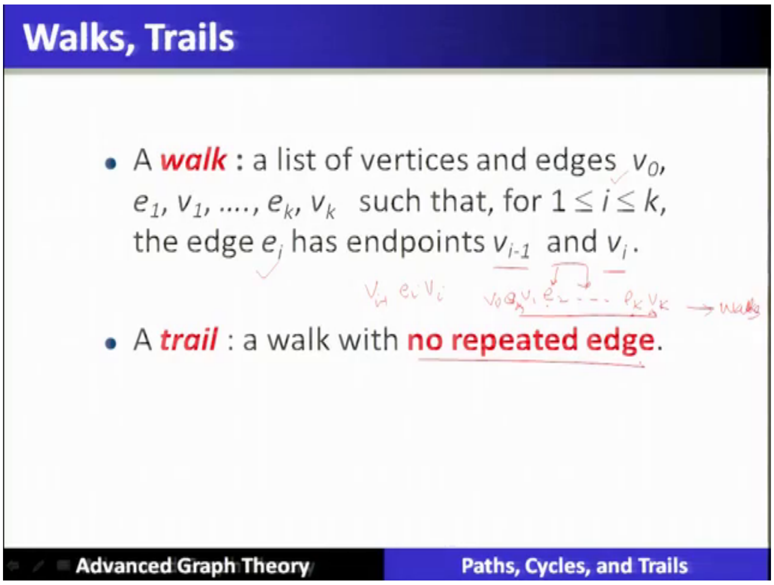

Now, let us start our most important part of this particular lecture; which will give you the definitions of walk, trails, then path, and a cycle. And we are going to see that these particular 4 different properties or terms in the graph theory how they are going to play a major role, in characterizing a special kind of graph that is called a bipartite graph or characterizing a cycle in a graph and so on, or characterizing the cut edge. So, all these things we are going to see. So, not only these definitions how these particular terms are going to be useful in building up to the further graph theory.

So, let us begin with a walk. So, walk is a list of vertices and edges which are listed as v 0 e 1 v 1 and so on, e k v k for a particular edge e i will have the end points as v i minus 1 and v i. So, in the sense for a particular edge e i it will have the vertices v i minus 1 and v i. So, we can form a list of vertices and edges v 0, v 0, e i, v 1, e 2 and so on up to e k and v k. So, this is called basically a walk. Now if you see here these particular edges which are appearing in the walk can be repeated more than once. And also, the vertices which are appearing here in this particular list; that is, in the walk that also can be repeated and there is no restriction in the walk.

Now, trail so, trail is a walk when there is no repetition of the edges allowed. Although the repetition of vertices are allowed, then it is called a trail.

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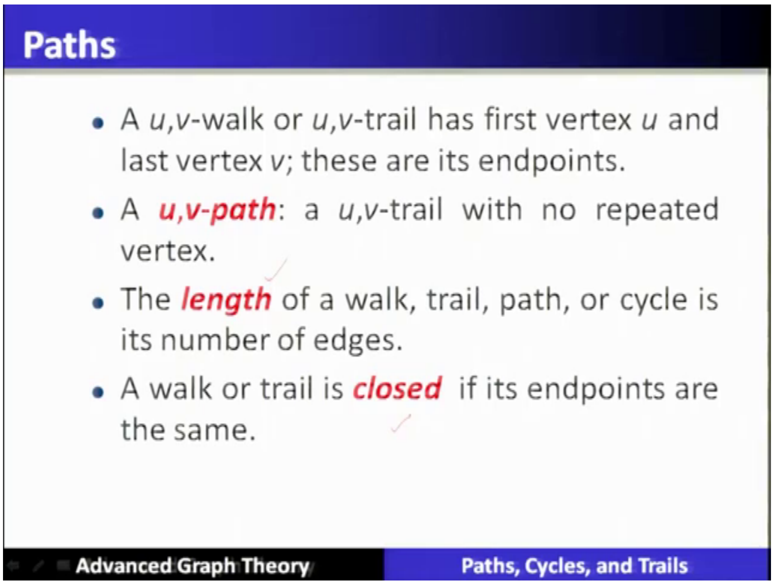

As far as the path is concerned, path is a walk which does not have the repetition of vertices or does not have the repetition of edges. So, edges and vertices if they are not repeated, then that walk is called a path. Or otherwise you can also state that a u v path is nothing but a it is a u v trail with no repeated vertices. And in the trail, we have already seen that the edges are not repeated. So, that is why I have told you that path is basically a walk without heavy any repetition of vertices and the repetition of edges. So, there is no repetition of vertices and no repetition of edges in u v walk.

Now, the length of a walk or a trail path cycle is it is number of edges involved in all these components. So, we have to count how many edges are there, and that will become the length of a walk, trail path or a cycle. Now a walk or a trail is cruised if it end points are the same.

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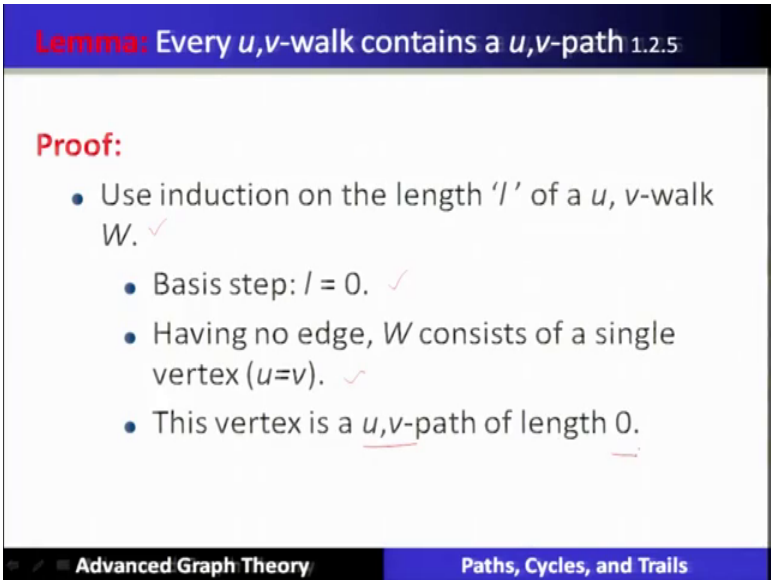

Now, let us consider a lemma every u v walk contains a u v path. So, the proof goes on the induction of the length of u v walk W. Let us see the base step of the induction when the length is equal to 0 of a walk W. Now having no edge W consists of a single vertex. So, hence the length will be equal to 0. And when the length will be equal to 0; so, u v path also will have the length 0, and we have seen that for the base case when the length of a walk is 0, the path also basically defined which will contain a path of a length 0.

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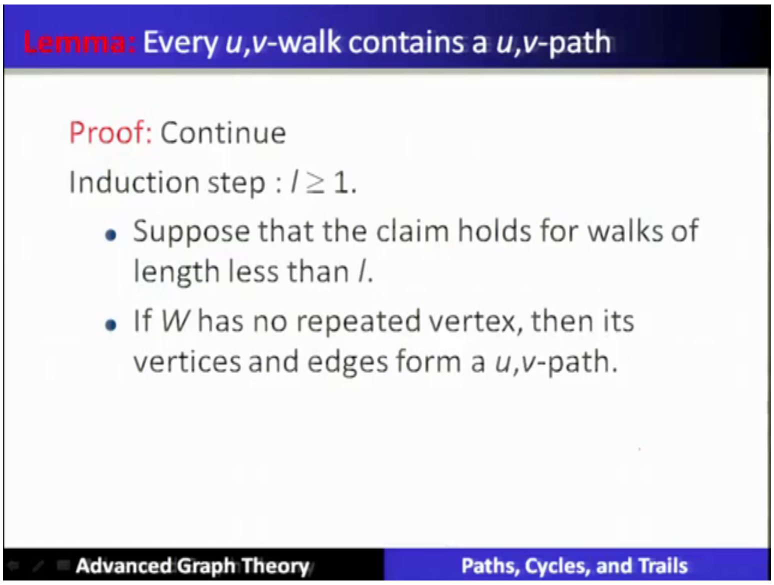

Now, we have to see the induction hypothesis, when length is at least 1. So, length is greater than or equal to 1 in that case. Suppose that the claim holds for a walk of length less than l. Now if W has no repeated vertices, then it is vertices and edges form a u v path. Now if W has a repeated vertex w and that vertex let us say a w which is shown as the bold.

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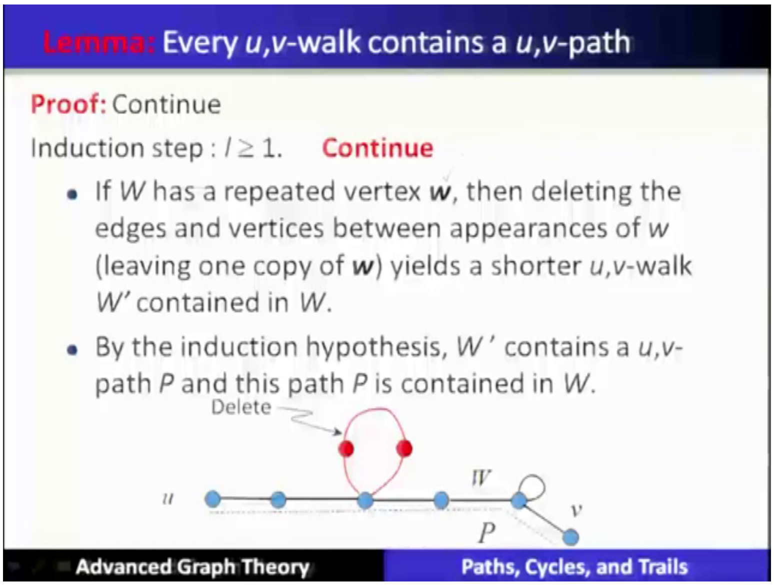

Then deleting the edges and the vertices between the 2 appearances of W, will yield a shorter u v walk W prime; which is contain in w. So, by induction hypothesis, we know that W prime which is the shorter length path, then l will contain a u v path. And this particular path is contained in a larger walk, that is w and hence it is proved connected and disconnected graphs.

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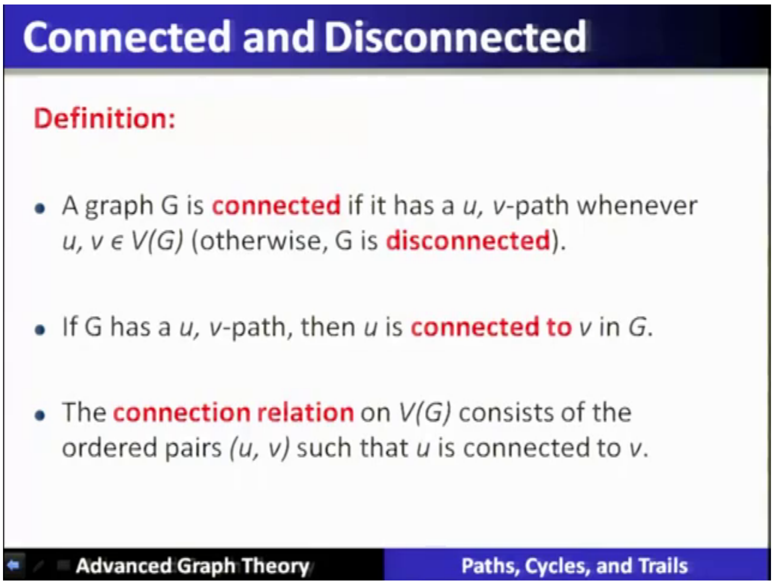

Definition, a graph G is connected if it has u v path, whenever u v is an pair of vertices of a graph. Otherwise that particular graph is disconnected. So, if G has a u v path, then u is connected to v in the graph G. This particular connectedness will induce relation on the vertex at v G, and will basically induce the relation and divide into an equivalence classes, that we are going to discuss here in this discussion.

So, again before we go ahead let us define again a graph is called a connected graph, if it hires a u v path, between any 2 pair of vertices. So, if any pair of vertices is not having a path or there is no connection between any pair of vertices or the graph becomes disconnected. Now for a particular pair of vertices u v, if there is a path exist in a graph then u is connected to v in the graph. So, this particular connectedness will induce a relation that is called a connection relation on a set of vertices, and this comprises of this connection relation on vertices consist of the order pair u v such that u is connected to v.

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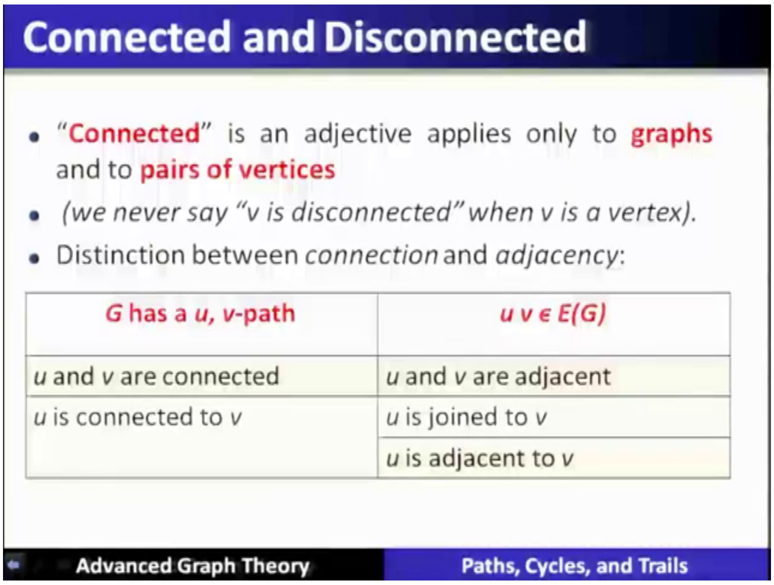

And this will induce a relation or equivalence classes that we will discuss.

This connected word is an adjective which is applied to the graphs, and also to the pair of vertices. We never say that v is disconnected. So, that means, that if it is a graph, then it is called a connected graph if it is a pair then we have to see. So, there is a distinction between the connection and adjacency.

Now, if a graph G has a u v path, then u and v they are connected that we have already discussed. And seen similarly if u and v is an edge in a graph, then we say that u and v are adjacent. Or we can also say that u is joined to v or u is adjacent to v in many way. So, that is the difference between adjacency and the connection. So, connection means a connected through a path, that is called a u v path.

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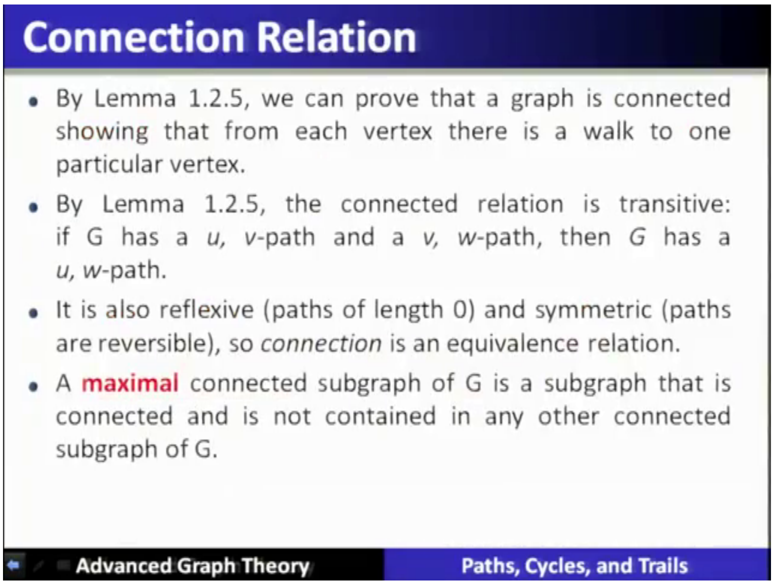

Now, we have to see the connection relation by lemma 1.2.5. We have proved that the graph is connected showing from each vertex, there is a walk to wa to one particular vertex. So, that means, if there is a walk between a pair of vertices then there must exist a path between a pair of vertices. So, whether there exist a walk or a path that will basically induce a relation that is called a connect connection relation.

So, by lemma 1.255 same lemma the connected relation is we can see that it is a transitive. That is if the graph has u v path and also at the same time it has a v w path, then that particular graph will also have a u w path, that is why transitive relation.

Furthermore, if we see the reflexive relation induced by the connection relation, then for a path of length 0 will be induced and that is called a reflexive. Similarly, as far as paths are reversible; that means, if there is a u v path v w path is also exist. Then it is also induces another relation that is symmetric relation. So, all 3 relations exist and basically when a connection relation is defined. So, hence the connection is an equivalence relation. Let me repeat again. That the connection relation will satisfy the reflexive property. So, in the sense there is a u path. That is the path of length 0. So, that means, there is no path and isolated vertex will have the path length 0. So, that will induces a reflexive property.

So, that means, the length of a path is the number of edges if there is no edge then; obviously, the length of a path is 0 and hence the reflexive property is satisfied.

Similarly, if u v path is there, then v w path also the reverse reversible path is also possible in an undirected graph. So, symmetric relation is also defined. So, reflexive symmetric and transitive relation based on the connection relation will induces a equivalence relation. So, a maximal connected sub graph of G is a sub graph, that is connected and if not contained any other connected sub graph, then it is called a maximal connected sub graph.

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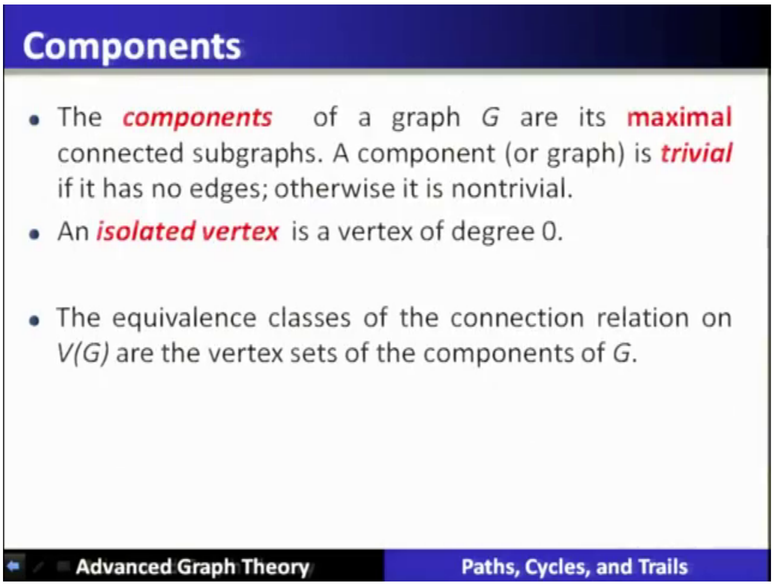

If the graph is not connected, or if the graph does not have the connection property, then it is disconnected and we say that those maximal connected sub graph are nothing but they are called the components. So, the component of a graph G is it or it is maximal connected sub graphs. A component is a trivial if it has no edges. Otherwise, it is non-trivial. So, an isolated vertex is a vertex of a degree 0. And that will basically induce or that will view a trivial component if it is present in the graph.

Again, we will repeat that equivalence classes of the connection relation on a set of vertices v G of a graph or the vertex sets of the components of G. So, the equivalence relation equivalence classes of the connection relation will induces the components of the vertex set that we have to see. So, component is nothing but the maximal connected sub graphs.

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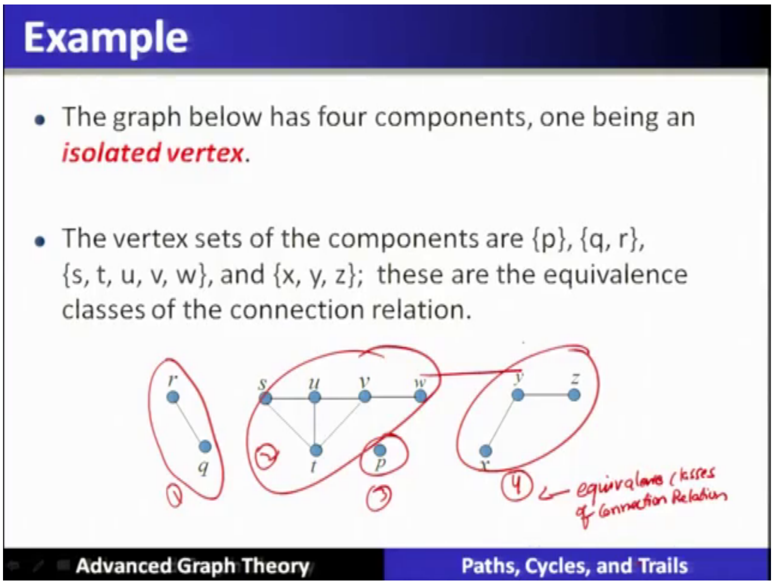

So, take this particular graph, which has these set of vertices. Now if the connection relation is basically introduced, then what it will give it will give a equivalence classes of the connection relations are nothing but the components. So, here the different components 1, 2, 3 and 4. 4 different components will become the equivalence classes. And which is equivalence classes of the connection relation.

Now, adding and removing an edge. So, these components are pair wise disjoint. You can see there is no edge which is connecting the component 1 and 2. Similarly other components. So, they are pair wise disjoint there is no edge within it. Now if you place and edge. This particular 3 and 4 2 and 3. They will join as one component and instead of 2 they will become one component. So, adding an edge will basically reduce the number of components.

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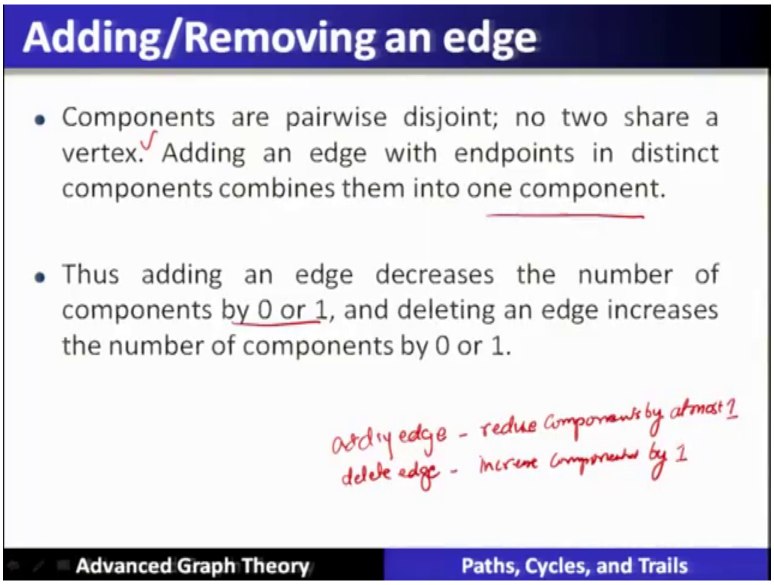

So, the components are pair wise disjoint no 2 share particular no 2 component share a vertex. Now adding an edge with the end points in the distinct components combines them into one component I have shown you in the previous figure.

Thus, adding an edge, we will decrease the number of components by at most one; that means, either if the edge is placed in the same within the same component, then it is not going to decrease the number of component, but if the edge is placed across the 2 components, then the number of components will be reduced by one. So, adding an edge will reduce the number of components by at most 1.

Now you delete the edge what will happen. So, if we delete the edge the number of components will increase by one. Or if it is within the same or if it not a cut edge sometimes it is not going to increase the number of component. So, by deleting edge the number of components will increase by there is a theorem which states that every graph with n vertices and k edges has at least n minus k components.

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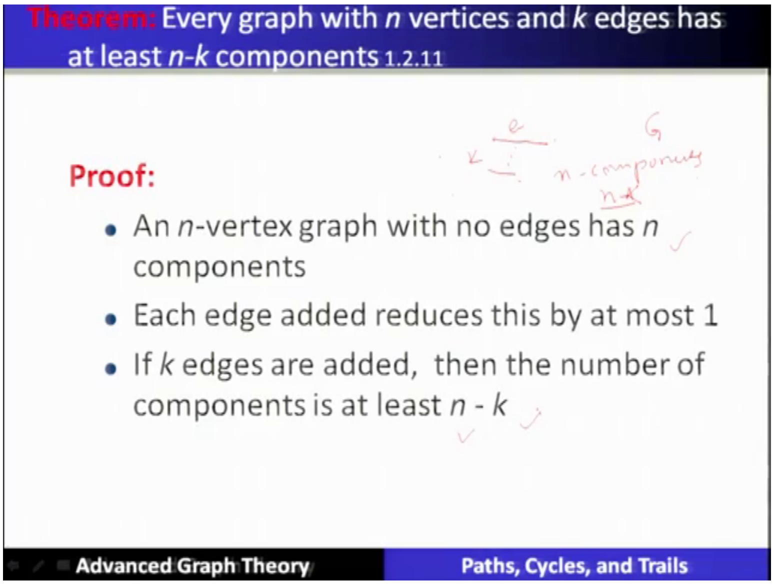

So, the proof let us see the proof of this particular theorem, and n vertex graph with no edges has n components. So, that is quite obvious, why? Because let us see that there are n vertices and if you if there is no edge which are connecting them, then how many components will be there? Then n components will be there in that particular graph G. Now when an edge is rated. The number of components will be reduced by at most one. So, when a the number of when and by if k at is r basically added, then the number of component is at least a minus k that is quite obvious.

So, again I am repeating when an edge is added. The number of components is reduced by n minus 1 is at most is reduced by at most 1. So, the number of component will be at least a minus 1. Similarly, if k edges are added. If k different edges are added, then the number of component will be reduced by at most k, and hence the number of components will be at least a minus k. And hence this proves this particular theorem.

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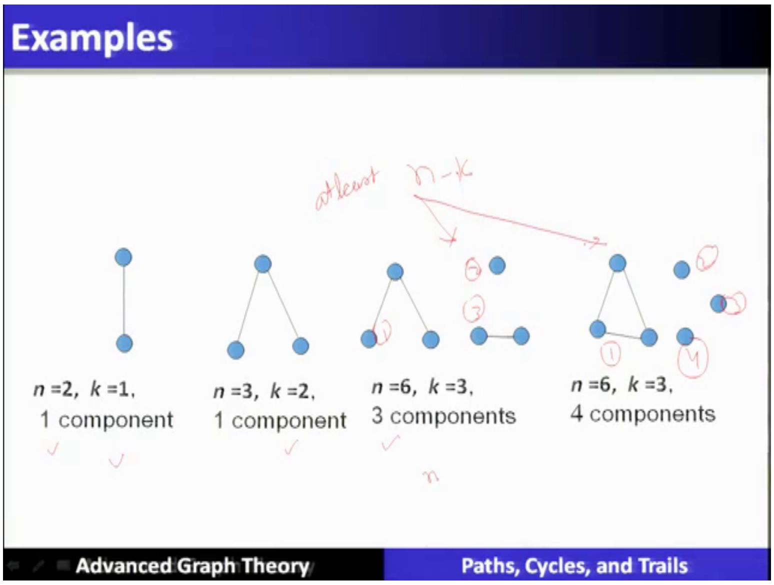

We can see the illustration here in this particular example. Here we have n is equal to 2, and number of edges is equal to 1. So, how many component? N minus k will be the total number of component, that is 2 minus 1 1 component will be there.

Here we have n is equal to 3, and k is equal to 2. So, 3 minus 2 will become one component. This is n is equal to 6, and k is equal to 3. So, how many component will be there? 3 components will be there one component 2 component and 3 component and minus k. So, here the formula which we are going to use n minus k.

Similarly, here it is 6 and it is 3. So, 1 2 3 edges are there. So, let us see how many components will be there. One component, 2 component, 3 component, and 4 different components are there. So, it says that there is at least n minus k component. So, is there. So, here that becomes equal. So, here it becomes at least means, more than 3 that is number of components are 4 here in this particular case.

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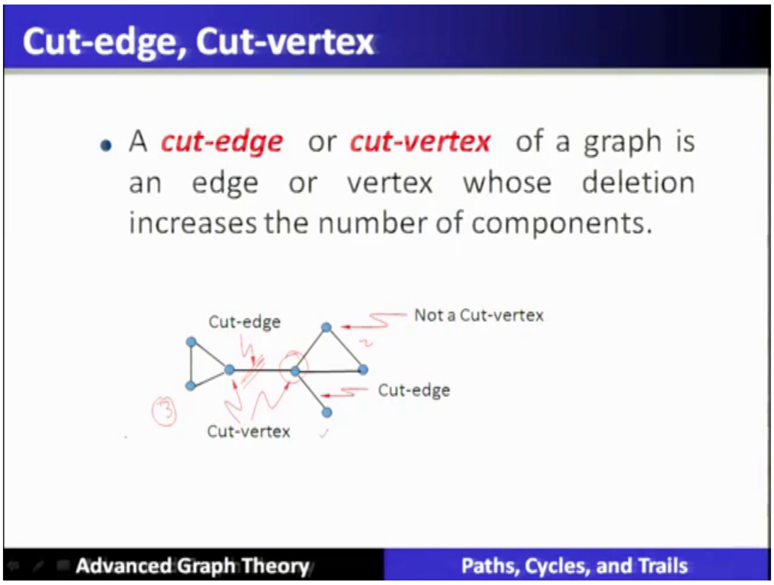

We have seen that the previous theorem basically is based on if an edge is added, in an isolated vertices it will reduces, the number of component by at most 1. So, that particular edge is going to reduce the number of components by one. And if that particular edge is removed from the graph, the number of component will increase by at most one.

So, those kind of edges are called cut edges, and if you remove the vertices the number of components is going to increase then that vertex is also called a cut vertex. We have to see 2 definitions. So, cut vertex and cut edge. So, a cut edge or a cut vertex of a graph is an edge or a vertex whose delusion increases the number of components. So, if there is a cut edge then the number of components will be increased by one, if they are if it is removed. If it is cut edge cut vertex and cut vertex if it is removed the number of components will be increased by many mo may be more than 1.

So, take this example, that if you remove this particular cut edge. So, how many components? This component will be this graph will have now, that is the graph minus this particular cut edge will have 2 components. Similarly, if you remove this particular vertex. So, how many components will basically come out, this is one component 2 component and this will be your components. So, whenever a cut vertex is removed from a graph the number of component will be increased by many that we have seen here.

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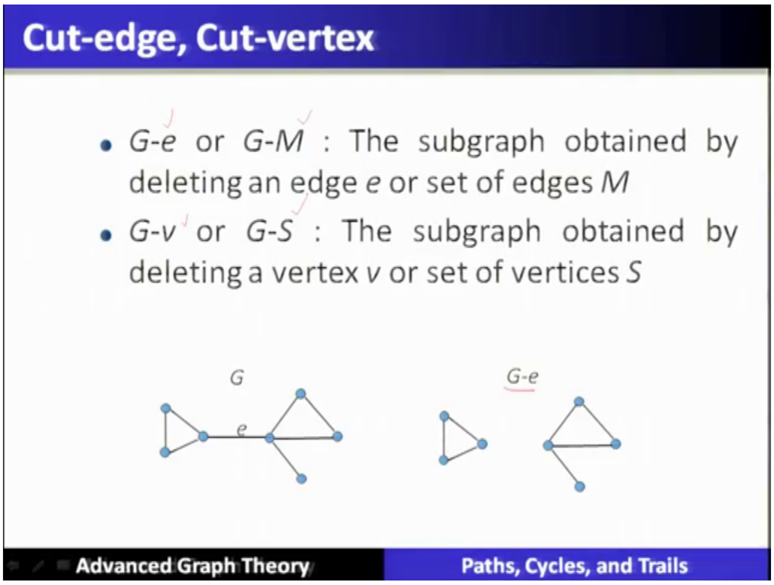

And this is represented in a in this particular form of notation that when a when a edge is removed from a graph it is represented as G minus e. So, when this edge is removed. So, this particular graph will become G minus e.

And similarly, when a vertex is removed, then it will be G minus v. Similarly, if there is a set of edges, then it is called as m and when a set of vertices, then it is called as capital s. So, these are the notations.

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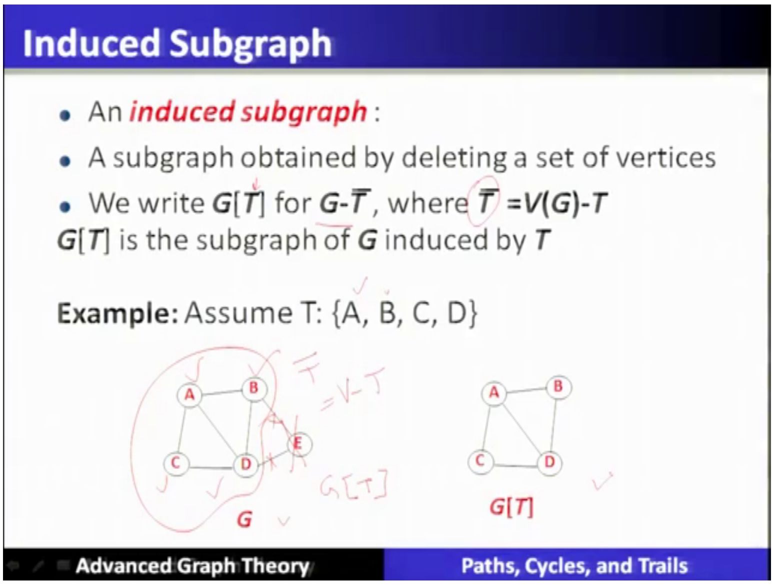

Now, we are going to introduce you the definition of the induced sub graph. An induced sub graph is a sub graph of end by deleting the set of vertices. So, when we write down G within the square within the square braces T that is the induced sub graph of T, or the sub graph of G which is induced by the set of vertices that is called capital T.

Or you can also represent as this particular induced sub graphs G T is nothing but G minus T prime. So, where T prime is v G minus T. So, that is this T prime if it is removed from the graph will remain only the T set of vertices in the graph, and the resultant graph having these set of vertices and all the connections all the edges as per the original graph T, then it is called basically the induced sub graph.

So, G T is the sub graph of G which is induced by the set of vertices. So, let us take this particular graph G and the set of vertices T as A B C A B then C and D. So, v minus T will become T prime. So, v minus T is basically e. So, e if you remove not only that vertex, but all it is edges the resulting graph which you will obtain is a induced sub graph or induced sub graph of T. So, that is what is represented over here.

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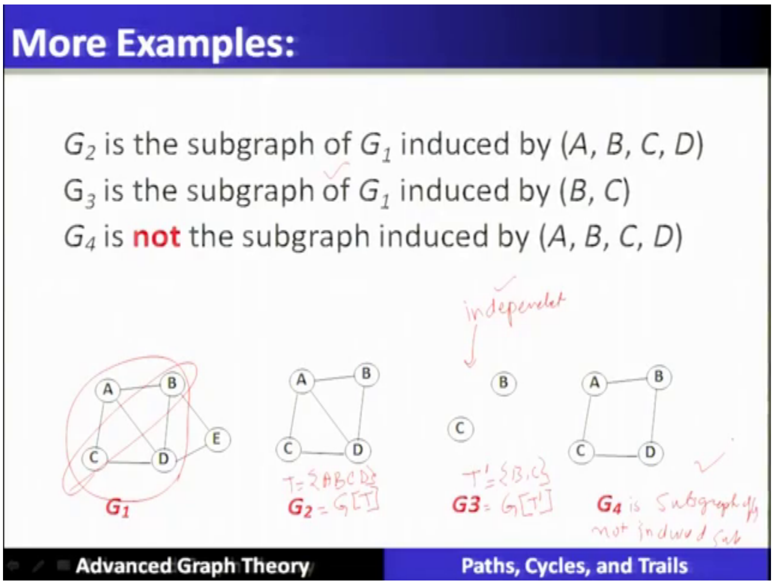

There are some more examples to it. So, there is a difference between the sub graph, and the induced sub graph. And this is illustrated here in this particular diagram. So, here you can see that this all complete thing is preserved. So, G 2 is nothing but a induced sub graph of p, and P is equal to A B C D. And G 3 is the induced by B C. So, this is nothing but the sub graph which is induced by let us say T prime, and T prime is nothing but D

C. So, this particular induced sub graph is basically an independent set. Why because in the original graph there is no edge which is joining these 2 different vertices.

As far as G 4 is concerned G 4 is now taking A B C D. So, A B C D induced sub graph will should contain this particular edge which is present between a and d, but it is not present. So, it is not a induced sub graph. But it is, but it is or sub graph of G. So, this is the difference we have seen that what is the difference between a induced sub graph and a sub graph.

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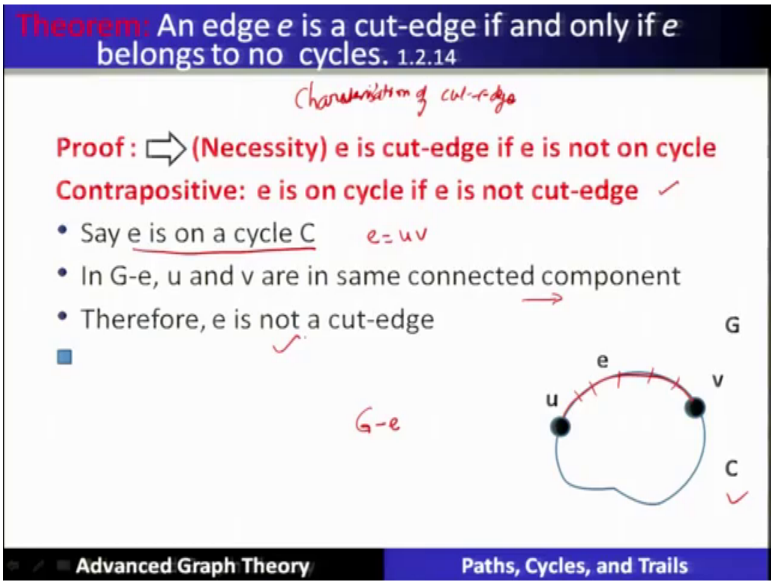

Now, there is a theorem which will use all these different concepts. So, the theorem says that an edge is a cut edge, if and only if e belongs to no cycle. So, this particular proof we have to see in both sides why because, this particular theorem will give a characterization of a cut edge. By characterization in the sense that if an edge is present on a cycle, then it is not a cut edge. So, that means, if it is a cut edge that is equivalent to saying that it is not present on any cycle, it is not on any cycle that edge is not on the cycle.

So, there goes a important theorem and we have to seek a proof of it. So, the proof we have to prove in both the directions; that is, the necessity condition first we have to see that, that e is a cut edge if e is not on the cycle. So, contrapositive statement we have to form and that particular statement contrapositive which says that, e is on the cycle if e is not the cut edge. E is not on the cut edge, if you can prove this then this particular necessity condition is proved.

So, if let us say that e is on the cycle, let us say e is on the cycle. So, let us see this particular cycle, c and e is present on this particular cycle. So, if e is there. So, the end vertices of e is basically u v. So, end vertices of that edge is u v. Now if we remove this particular e out of a graph. So, the graph which will be G minus e, will have will have a component will be one component, where u and v will be there in the same component. Since u and v are in the same component after removal of e therefore, it is not going to ? disconnect the graph. Hence, that particular e is not a cut edge, why? Because e is a cut edge then after removal of a edge the graph will be disconnected into more than one component.

So, here after removal of edge there will be only one component, it is not disconnected component it will be a connected component where u and v will be in the same component. Hence therefore, we have proved the necessity condition that e is not a cut edge. So, necessity is quite straight forward.

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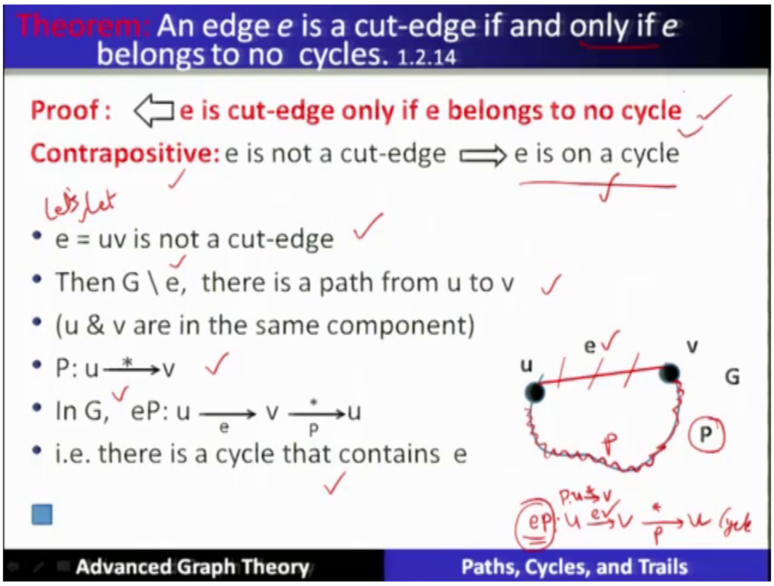

Now we have to see the other side of a proof. Other side of a proof says that, that is a sufficiency condition that edge e is a cut edge only if e belongs to no cycle. So, contrapositive says that. So, e is not a cut edge, and which will imply that e is on the cycle. So, let us assume that particular edge e which is nothing but u v pair is not a cut edge. Let us assume let us let that e is u v is not a cut edge.

So, if it not a cut edge, then if you remove it from the graph, then there must be a u v path from u to v there is a path which is shown over here. This particular path we can represent as path which is going from u to v by more than involving more than one vertices because it is a path. So, in the graph G e followed by this particular path e followed by this particular path, will basically e followed by path means from u, if we take this particular edge. We can reach v and from v we can take this particular path and you we can reach u again.

So, this particular way we can complete a cycle. If that particular edge followed by a path if we form it will it will produce a cycle. And this particular cycle will contain e within it. And so, e will be on the cycle. Hence, we have proved the contrapositive and this contrapositive will prove the other site of a proof that is the sufficiency condition. We have seen that the characterization of cut edge, that edge is not on the cycle then it is a cut edge. And the characterization we have proved also in the previous theorem.

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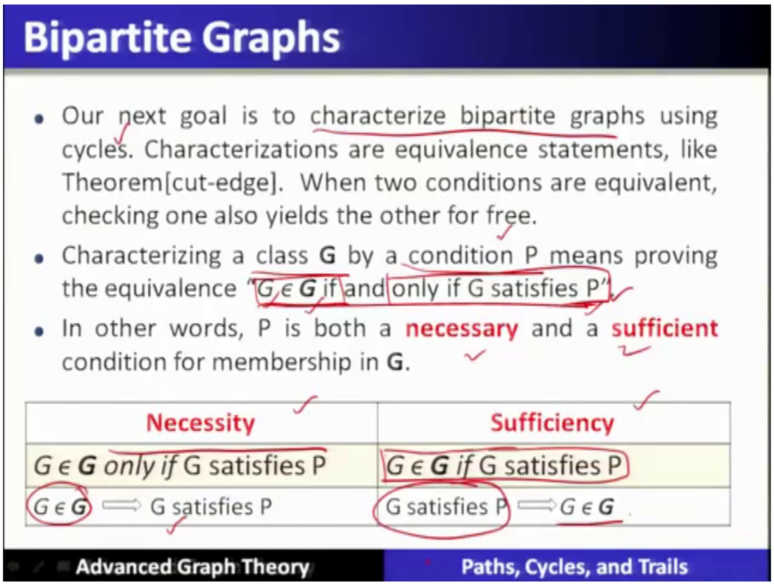

So, bipartite graph to characterize bipartite graphs using cycle is also very important. Theorem that we will see later on. So, we will also characterize similarly the bipartite graph; that means, we have to come out with the similar statements or a properties, and then we will characterize the bipartite graph according to that particular property.

To characterize a class of a graph by that particular condition, we have seen earlier in the theorem that that the cut edge was basically characterized by the condition; which basically stated that there does not exist edge on a cycle, then that edge is a cut edge. So, that condition will characterize that kind of edge of a graph.

Similarly, the bipartite graph also we can characterize with a condition. And that condition will involve the cycles. So, let us see that whenever we do like this, and prove using theorem that becomes an equivalent statements and that will be useful in various applications in the graph theory.

So, this particular condition P. So, that means, if we want to prove an equivalence condition. So, that is we have to state like this, G is a class of a graph. If an only G satisfies a condition P. So, that means, if let us say a particular graph is a class of bipartite graph, then basically it has to satisfy that it should not have an odd cycle P is an odd cycle. That we have to see and prove that.

So, in other words this condition P is both necessary and sufficient condition for the membership of for the membership of a graph in G, that we have to establish. So, there are 2 conditions simultaneously we have to establish for proving the equivalence. So, the first condition is called necessity condition. So, necessity condition means only if part of this particular equivalency statement, we have to prove this means that this G is a member of a particular kind of graph, this will this has to be proved; that means, it will mean that or it will imply that G satisfies the condition P.

For sufficiency condition, we have to see the other side of a proof, that the membership of a G is possible if G satisfies this particular condition. So, that means, we have to start with that given condition that G satisfies P and we e have to start with that given condition that G satisfies P. And we have to basically conclude that or it will imply that the membership. So, both the conditions we have seen so far in the previous theorem in the next theorem also we will see that.

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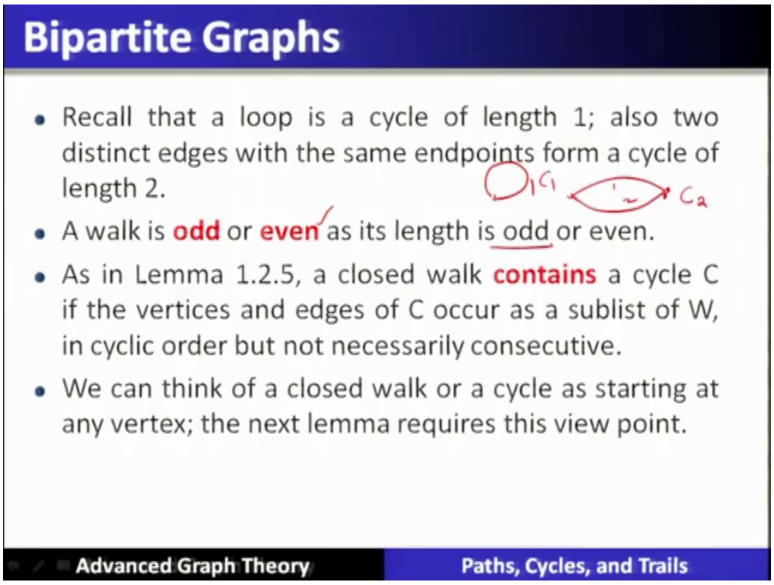

So, let us recall that a loop is a is a cycle of length 1 also 2 distinct edges with the same this is the loop is a cycle of length 1. And 2 distinct edges, 2 distinct edges one and edge 2, with the same end points will form a cycle of length, this is c 2 and this is c 1. Now the walk is odd or even as it is length is odd or even.

So, from the lemma 1.2.5 a closed walk contains a cycle if the vertices and edges of that cycle c occurs as a sub list of that walk w in the cyclic order, but not necessarily call the k 2. So, we can think of a closed walk or a cycle as the starting at any vertex, and the next lemma will require this kind of view point.

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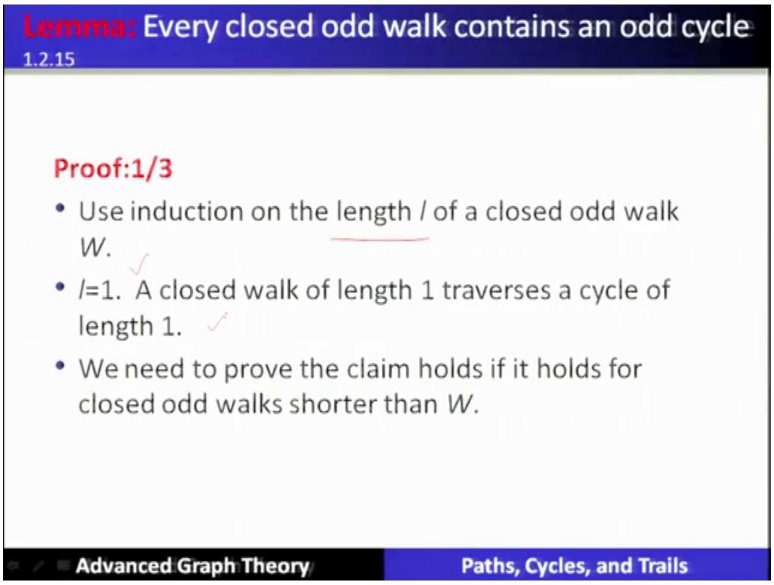

Now, we have to see a lemma. So, which is states that every closed odd walk contains an odd cycle. So, let us see the induction on the length l of a closed odd walk w. Now if length is equal to 1. So, the closed walk of length one traverses a cycle of length 1. So, this particular length one the base case is. So, we need to prove the claim course that if it hold for a closed walks.

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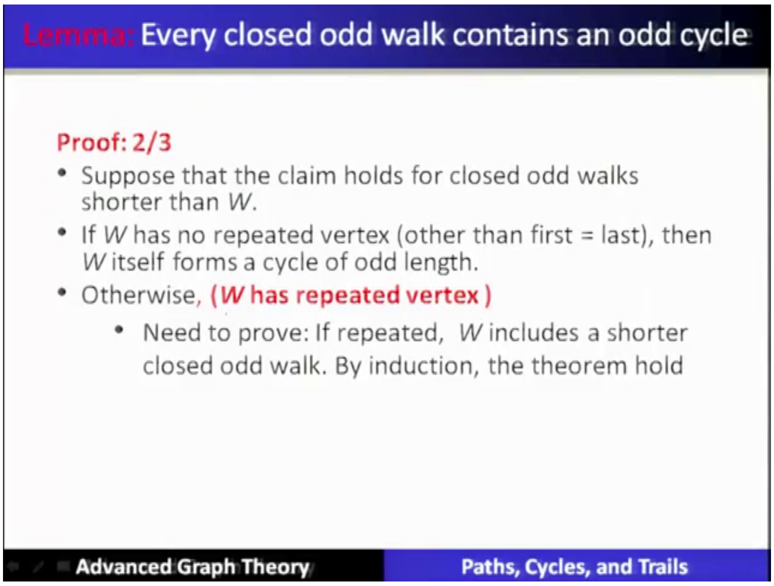

Of a shorter length w. Now suppose that the claim holds for a closed or walk shorter than w if w has no repeated vertices, other than the first and last then w itself forms a cycle of odd length, otherwise w has the repeated vertices. So, we need to prove that if it is repeated, then w will include a shorter closed odd walk by induction the theorem will hold let us see this particular case.

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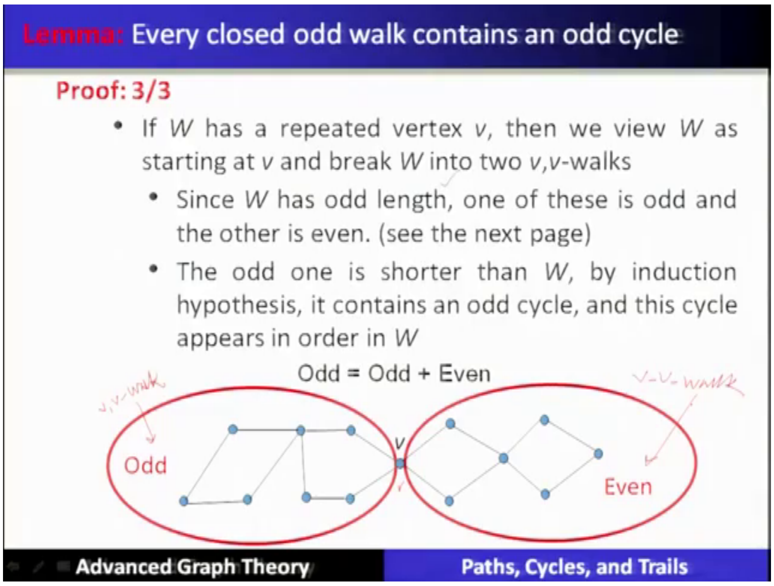

Now, if w has a repeated vertex v, then we can view this particular w that is the walk as by starting as v and will have 2 different walks. This is one v v walk, and this is the other v v walk. Since this w has an odd length. So, one of these walks is an odd let us say this is an odd walk, and the and the other one is the even. The odd one is shorter than the w by induction hypothesis. It means that this it contains an odd cycle, and this cycle appears in the order in w. So, that is why this odd plus even becomes an odd. So, every odd walk contains an odd cycle.

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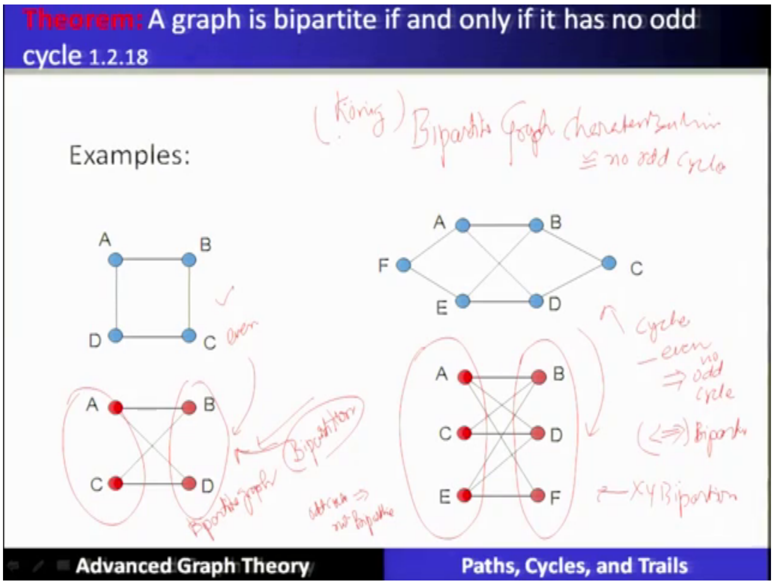

We are going to see the characterization of a bipartite graph, using cycles, or using odd cycles. So, the theorem which is given by conic is stated as a graph is bipartite if and only if it has no odd cycle. So, let us first see the illustrative examples. That this is a bipartite graph, that it has a cycle, that is a even cycle.

And it can be represented in the form of a bipartite graph. Here also another example, here there is a cycle, but this cycle is an even cycle. Hence, there is no odd cycle present in the graph. And this is equivalent to saying that this is a bipartite graph. And which is expressed in the form of a bipartition, this is one partite site this is another partite site here also.

So, this way of representing is called bipartition. So, if you want to prove that a graph is not a bipartite graph, then you have to present an odd cycle, odd cycle means the graph is not a bipartite graph. To prove that the graph is a bipartite graph, you have to come out with a bipartition, in this particular manner. So, this basically will categorize the bipartite, which is equivalent to saying that the graph has no odd cycle. And this particular theorem which states that the graph is bipartite if and only has no odd cycle is given by conic famous mathematician.

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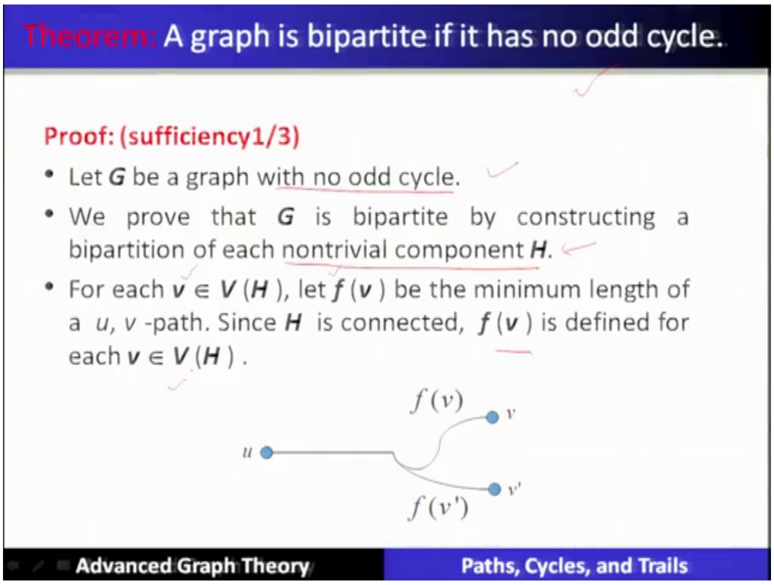

So, this is the sufficiency condition. Let us prove the sufficiency condition first. The graph is bipartite if it has no odd cycle, and we assume a graph with no odd cycle. So, here we assume the graph with no odd cycle. And we have to prove that this particular graph, which is not having any odd cycle is a bipartite graph. This is sufficiency condition.

So, here we prove that the graph G is bipartite, how we can prove the G as a bipartite? So, we have to come out with a bipartition, the construction of a bipartition. So, we prove G is bipartite by constructing a bipartition of each non-trivial component H. So, for each vertex v. So, what is known previous component h is assumed, because if a graph is disconnected, then it will be having the different components, and we are considering one such maximal sub connected graph sub graph that is call the non-trivial component. Let us say that that non-trivial component is h if the graph is connected the entire graph will become H.

Now, let us say that for a particular vertex v which is there in the graph let us say h, then we have to define a function f for a particular vertex v the function is defined to be the minimum length of u v path. Now, since h is a connected graph or a component. So, fv is defined for each vertex v which is the element of V H.

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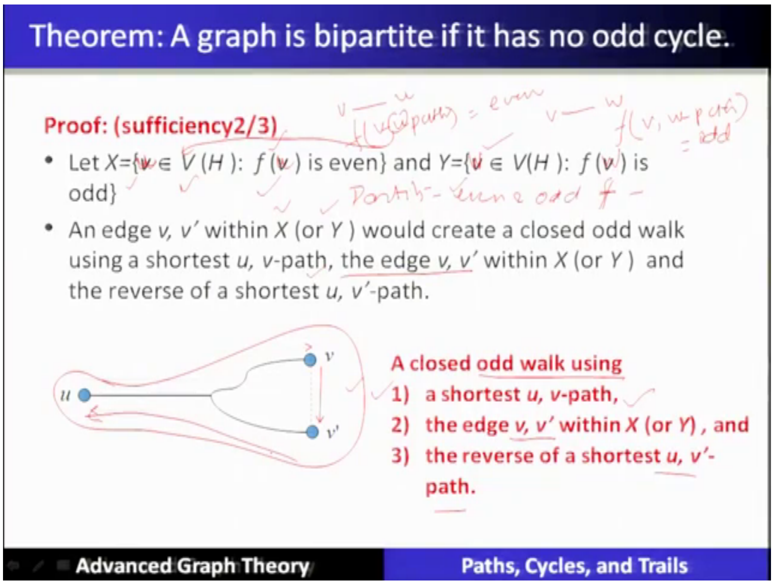

So, this way using that particular function, we are going to get 2 different sets. One set is called x consist of all the vertices, let us say u such that f u is even. So, f u means that from v to u, the v u path basically is measured in this particular function as the even. So, all such use are included in x.

Similarly, from v to let us say again w. So, we can consider all the vertices w; which are there in h such that from v to w if there is a path. So, v w path, and if we take this particular function we will give an odd length. So, this particular length it will become odd. So, based on this even and odd parity. Even and odd parity with the with applying this particular function f; which will be of the minimum length u v path it will give the 2 partite sets x and y.

Now, we have to (Refer Time: 49:43) that this particular x and y they are the partite sets; that means, no 2 vertices which are there in x, they are connected by an edge. Similarly, no 2 vertices in y is connected by an h we have to prove to complete this sufficiency condition. So, let us consider that there is an h, v and v prime whether it is within x or in y, and if this particular edge is there it will create a closed odd walk, using the shortest u v path.

And then walking through the edge v v prime, within x or y and then taking a reverse of that particular shortest path. So, this will form a closed odd walk, will have 3 components u v shortest path than an edge v v prime, whether it is an x or y and then a reverse shortage path v v prime path.

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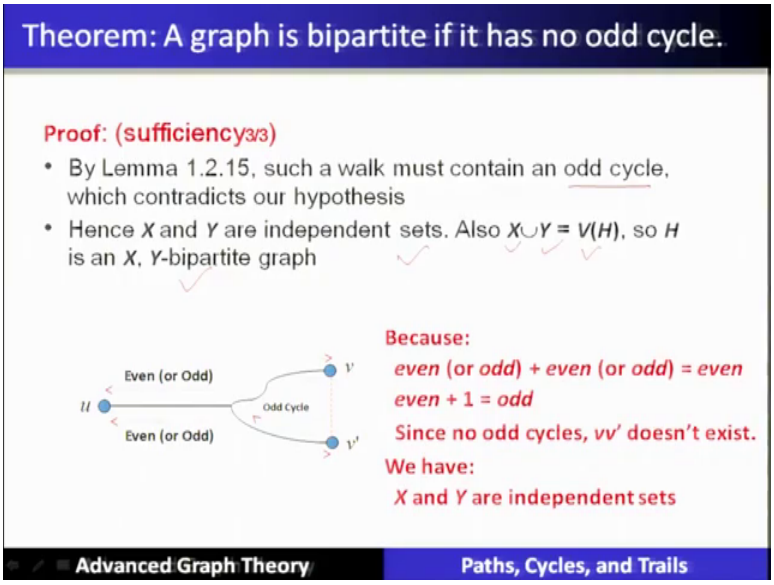

Now, if this edge v and v prime is there, then it will form a odd cycle this particular walk will contain an odd cycle by previous lemma. Hence it will contradict the hypothesis, where we assume it has no odd cycle. Hence x and y they are the independent sets, and the union of this particular partite sets is nothing but the complete vertex set of that particular graph. Hence the x y is the bipartition of that particular graph hence it is a bipartite graph.

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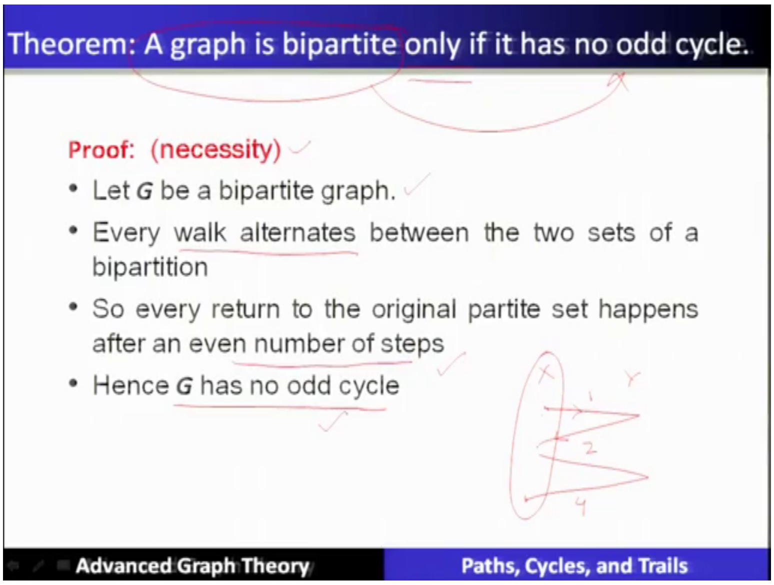

Now, we have to see the necessity condition. The necessity condition says that the graph is bipartite only it has no odd cycle. So, to prove this necessity condition, we have to assume that the graph is a bipartite graph. That is the other side of the proof.

So, let us assume that the graph is a bipartite graph, and then we have to prove we have to conclude that it has no odd cycle. So, let G b a bipartite graph. Now every walk will alternate between 2 partite sets of a bipartition. So, every return, every return to the origin partite set happens after even number of steps this is one this is 2.

So, even number of steps; so, if you go back again come back of for taking 4 steps. So, if you start from x, visit y and then come back again it will take even number of steps. Hence whenever you want to include a cycle, you have to traverse and come back to the same point, and that can be that cannot be that can be done the only in the even number of steps. Hence that the G contains no odd cycle.

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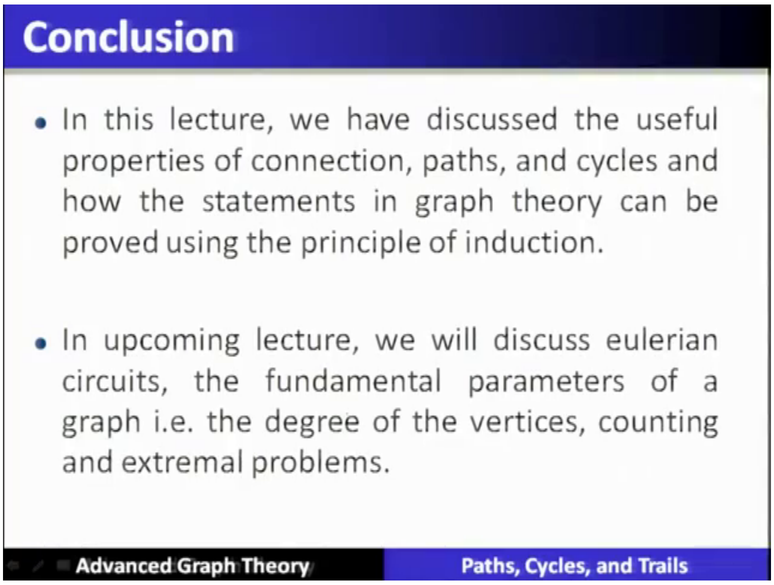

So, in the nutshell what we have seen here is the properties of the connection paths, cycles and how the statements are going to be useful in a graph theory.

So, in the next lecture we will discuss the higher circuit, and other fundamental properties of a graph; such as the degrees of a vertices counts and extremal properties.

Thank you.