Advanced Graph Theory
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Lecture - 18
Counting Proper Colorings

Counting proper colorings.

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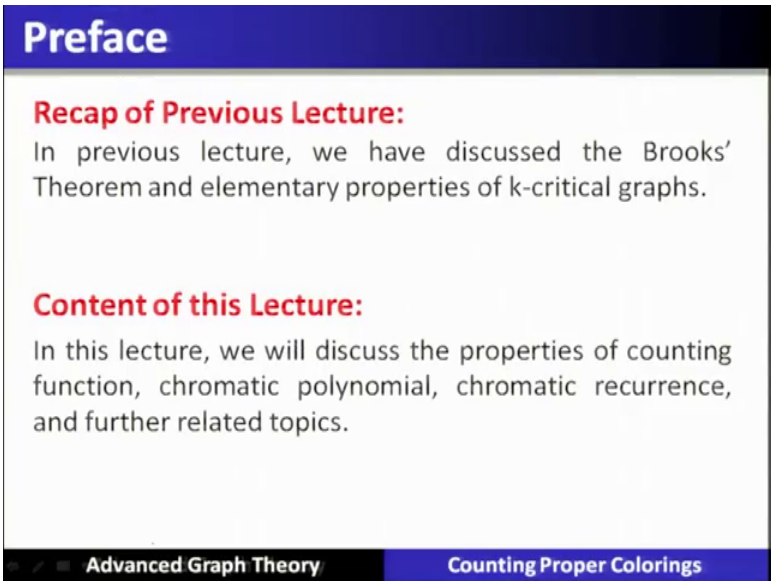

Recap of previous lecture, we have discussed the Brook's theorem and some properties of critical graphs. Content of this lecture we will discuss the properties of counting function is also known as enumeration of the colorings in a graph. We will also see the chromatic polynomial, chromatic recurrence and other further important enumerative aspects of coloring.

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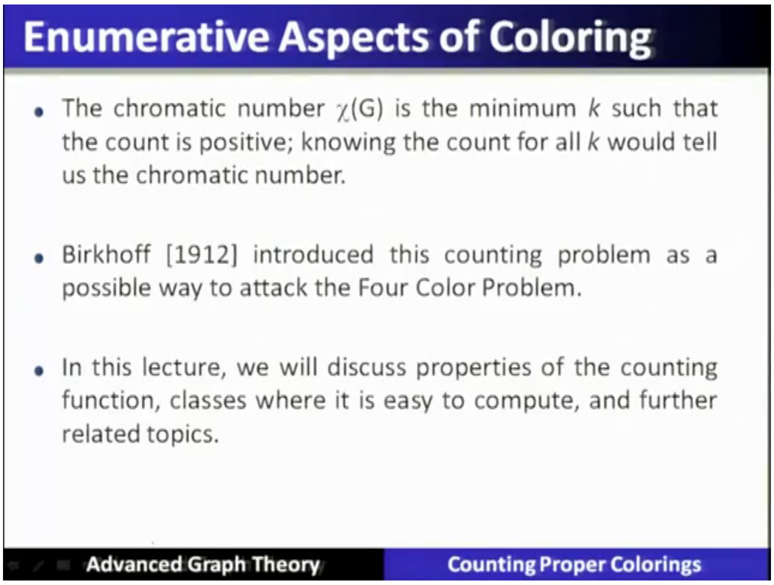

So, the chromatic number chi of G is the minimum k such that the count is positive. Now knowing the count for all k would tell us the chromatic number; the counting problem was also used to tackle the 4 color theorem. This lecture we will discuss the properties of a counting function and the classes where it is easy to compute and other related topics.

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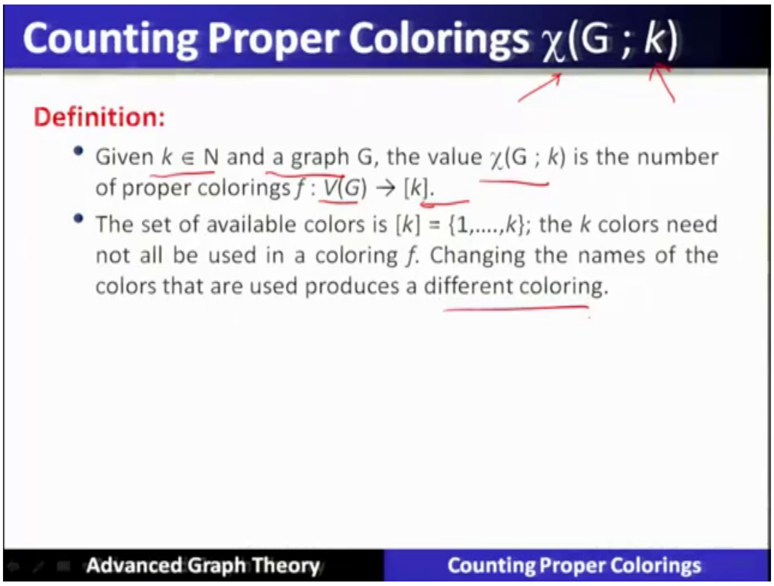

So, counting proper coloring is represented by a notation which is basically chi of G and another parameter that is called k; k is you will explain k ok. So, k is any natural number and a given graph G the value the chi of G k is the number of proper colorings. So, you know that proper coloring is represented by a function f which will label the vertices from the set of label that is the k set of labels are given, these labels are called colors because the values are not important

So, the setup available colors is k; we can represent either them in the form of labels 1 to k or we can also say that the different colors. So, the k colors need not all be used in the coloring using this particular mapping function f. So, changing the names of the color that are used will produce different colorings. So, we have to count how many such different colorings are possible? We are using this particular notation that is chi of G comma k and that is nothing, but enumeration the colorings possible with k different colors in a given graph G.

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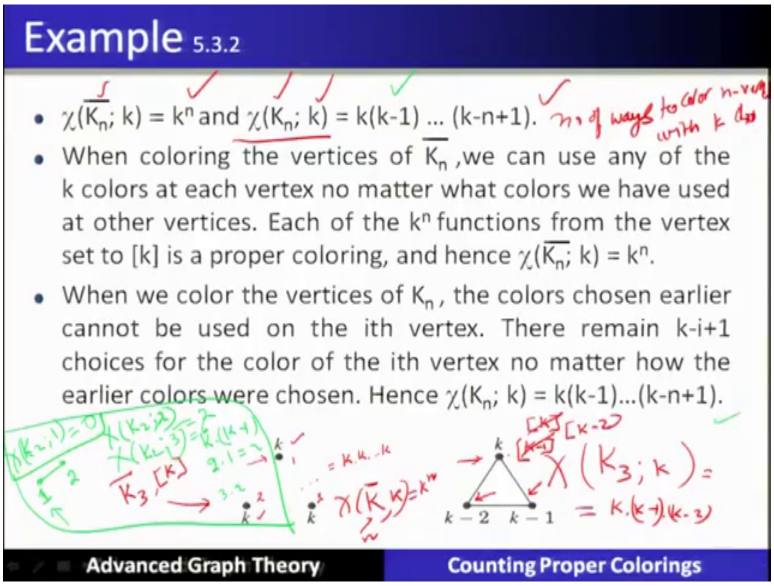

Let us consider an example in this example; we will consider a graph which is the complement of a complete graph. So, if the complete graph is k 3 complement graph will look like this way that 3 nodes are there. So, complement means in the may in the complete graph they have an edge in a complement of that particular graph they do not have any edge although only the nodes will be present

Now, if you are given color k different colors. So, let us see that a particular node we can choose any one of these k colors. So, there are k possibilities to color the vertex number 1. The second vertex is not connected; so, again we can pick any of these k colors and there are k possibilities to color the second node. Third node is concerned also it will basically a require any of the k colors; so, third node.

So, if there are n nodes. So, all the nodes will receive the same color; so, total number of colors is nothing, but the multiplication of K n times. So, the chi value of complement of a complete graph with k different colors the complete graph with n nodes will have k raised to the power n different chi value

Another example of this counting is. So, if you are given if you are given a complete graph let us say here the n is 3. So, we are given k 3 and we are also given the number of colors as k. So, we want to find out the total number of possible proper k colorings. So, the let us take the first vertex; so, since we are given k different colors none of them are already used. So, the first vertex will be colored using any of these k or k different ways

Now, having utilised the one color the remaining will be k minus 1 different colors available. So, the second vertex can be colored with the remaining k minus 1 different ways. So, having used another color; so, the remaining colors will be k minus 2. So, the third vertex can be colored with these k minus 2 different colors; so, there are k minus 2 different ways.

Hence the chi value is represented; so, there are n nodes. So, k minus n plus 1 will be that particular result. Now when; so, this will tell these are the number of ways to color a an vertex graph with k colors. Now before we go ahead let us see another important thing.

Now, if we give a less number of colors and with that less number of colors the proper coloring is not possible what will be that particular value. So, let us take k 2 complete graph of two vertices and we are given only one color. So, if we use one color the second color; if we try to use then it will not be a proper way because the other vertex is the adjacent vertex and it has to basically cannot use the color which is already given to its to its neighbour.

Another color is required. So, chi value of k 2 with one color will become 0; why because of the there is no proper coloring. Now if we increase the number of colors by one more that is 2; then let us see how many ways we can color it. So, it is one color and the other color we are going to give it to the other one and vice versa.

Similarly, if you are having; so, let us plug it this particular formula here. So, k into k minus 1; so, k is 2 into 1 that becomes 2; so, hence two colors are required. Now k 2; if it is having 3 colors, then what will happen? So, the first vertex we can color with; so, that becomes k; k minus 1. So, k is 3 into 2 that is 6 different ways we can color if the number of colors are more that is 3.

So, this is important that if the number of colors is less than the minimum required colors then; obviously, the chi value a will become 0. So, when coloring the vertices of K n complement we can use any of the k colors at each vertex no matter what color we have used at other vertex hence since each of these K n functions from the vertex at two K n is a proper coloring hence we have already told.

Now, when we color the vertices of K n; the colors chosen earlier cannot be used on the ith vertex. So, the remain there remains k minus i plus one choices for coloring the ith vertices; no matter how the earlier colors have been chosen hence this particular formula we have already derived.

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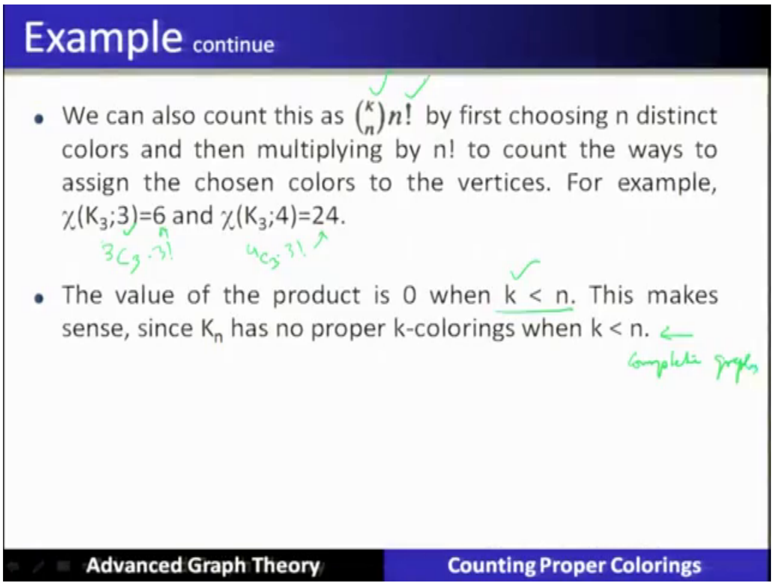

Now, we can count this also k chosen and factorial n means there are different if we change the names of the colors then we can obtain different orderings

So, hence this is the number of this to assign the colors; now we have already seen that if there are 3 different colors, then the first vertex is colored with three. So, we can apply this particular formula that K chosen n is 3. So, 3 chose 3 times 3 factorial; so, 3 factorial this becomes 1; so, 3 factorial is 6.

Similarly, 4 choose 3 and 3 factorial. So, this becomes 24; the value of the product is 0 when k is less than n that is the number of colors k is less than. And this makes sense since k has no proper colorings when k is less than n for the complete graphs.

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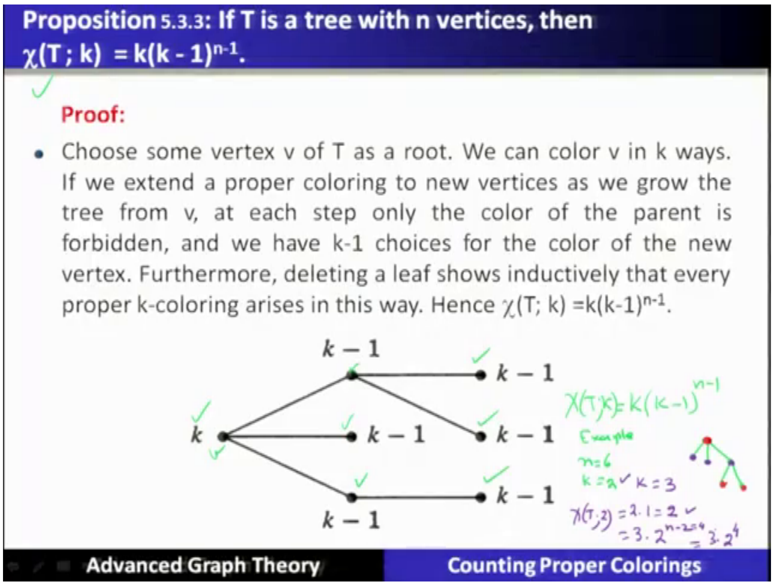

Now proposition; if T is a tree with n vertices then chi of T with k is equal to k times k minus 1 to the power n minus 1. Now let see the proof and then we will see the example proof says that choose some vertex V of a tree as a root, we can color V in k different ways. If we extend the proper coloring to the new vertices as we grow the tree from V root at each step only the colors of the parent is forbidden and we can we have k minus 1 choices for the colors of a new vertex.

So, having color this particular root when it comes to the descendant, we can color with the remaining k minus 1 colors. Similarly to this also to this also now when we go to the next level here also we cannot use this color; so, the remaining is k minus 1, so, k minus 1 k minus 1.

So, if a tree with n nodes is given; so, the first node we can color with k different base having used one of these colors at the root; in the next level can be colored with k minus 1 and if there are n different such nodes afterward. So, this will be the total number of ways we can color a tree of n nodes with k different colors.

Let us take a example ; let us take this particular tree. So, here n is equal to 6 and let us say that number of colors are given k is equal to 2. So, first we can color with the number of colors; now then we can use the other level with a different color cannot use red then red again we can use here. So, we require another color; so, next level we can use this particular color. So, two colors will basically color them.

Let us see the formula. So, the first vertex 2 times, second is 2 minus 1 becomes 1 times n minus 1 that is 5 that becomes 2. So, this is this particular tree T with 2 colors only there are two ways. Now if the number of colors is equal to 3; let us see how many ways will be there. So, the root node we can color with 3 ways and the remaining nodes that is n minus 1 different remaining nodes can be colored with other two ways.

So, how what is the value of n here? n is 6 minus 2 that becomes 4; 2 raised to power 4, 3 multiplied by 2 raised power 4. So, that will be the result here in this particular.

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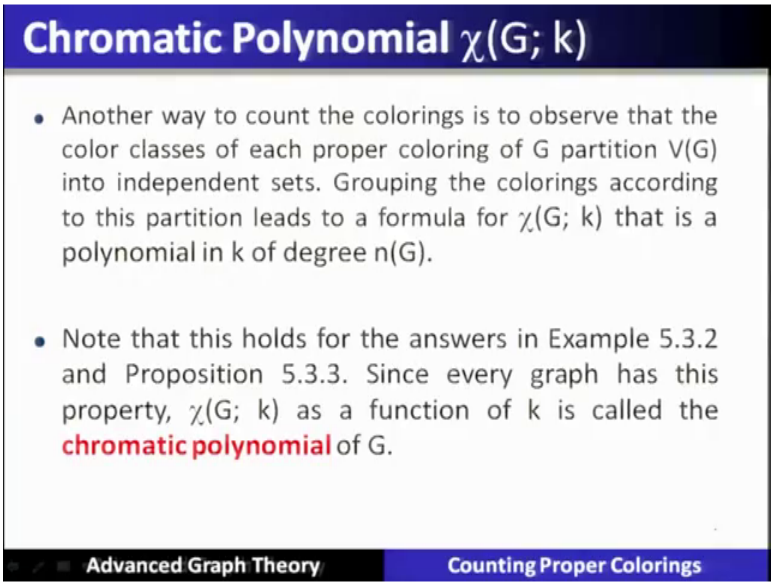

Now, we see the chromatic polynomial; chromatic polynomial is another way to count the colorings and is to observe that the color classes of each proper coloring of G partitions vertices of a graph into independent sets

Grouping the colorings according to the partition will lead to a formula for chi of G with k coloring; that this polynomial in k of degree n. Note that this holds for the answers in the examples which we are seen earlier, since every graph has this particular property as a function of k is called the chromatic polynomial of.

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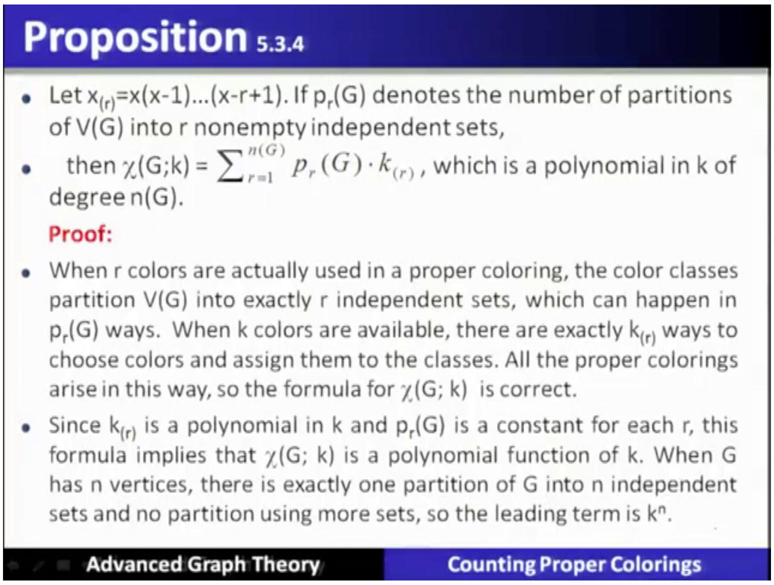

Let us see the proposition let x is equal to x x minus 1 and so, on up to x minus r plus 1 and let p r denotes the number of partitions of the vertex vertices into r nonempty independent sets. Then chi of G with k is nothing, but the summation of summation from r to n pr times k r which is the polynomial in k of degree n.

Let us see this and we will take an example to show this particular polynomial; now when r colors are actually used in proper colorings the color classes partition vertices of G into exactly r independent sets which can happen in pr G ways when k colors are available there are exactly kr ways to choose the colors and assign them to the classes all proper colorings arise in this way; so, the formula is correct.

Now, k r is the polynomial in k and p r G is the constant for each r this formula implies that that chi of G with k is a polynomial function of k; when G has n vertices there is exactly one partition of G into n independent sets and no partition using more than that set lead to the value leading value is k raised power n.

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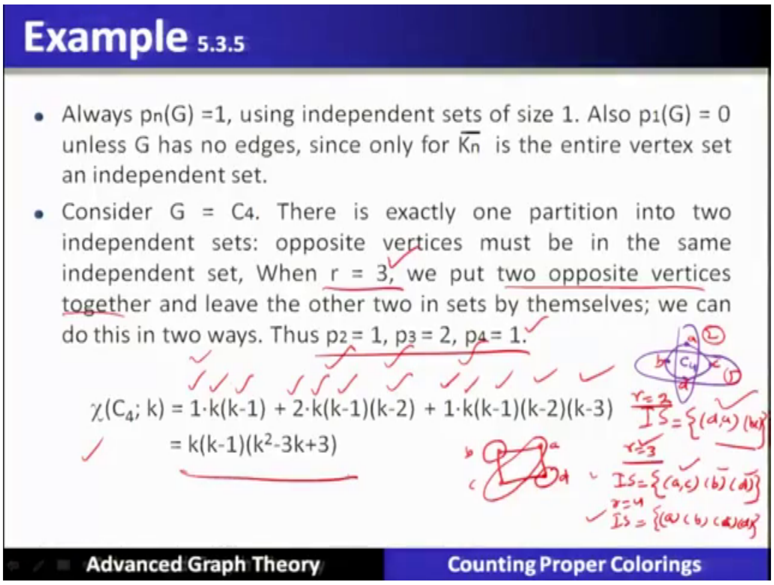

Always pn is equal to 1 using the independent set of size 1. So, p 1 G is equal to 0 unless G has no edges; since only for K n complement is the entire vertex set is an independent set.

Let us consider when graph is C 4; now there is exactly one partition into two independent sets. This is one independent set this is another independent set. So, independent set will have this is the partition one, this is another partition. So, let us see that a b c d; so, one is a d the other partition will contain b c; so, it has only one partition. So, opposite vertices will be in the same set; when r is equal to when; so, this particular when r is equal to 2. Now when r is equal to 3 r ways number of independent sets are 3; so now, we can see 3 independent sets means this is one independent, one independent set second one and this is third one; so, a b; then and d

So, when r is equal to 3 the independent sets will be a c then b and then d. So, we put two opposite vertices together here we have shown you and leave the other two sets in the set by themselves that we have done. So, we can do this in two different ways; so, two different ways means either this way or the other way. So, the total number of ways to do these kind of partitioning, when the independent sets size is 2; there is only one way of doing it when independent set of size 3; we have seen here there are two ways where independent set of size 4; 4 four means all are isolated.

Now, let us see the polynomial according to the previous relation. So, when independent set of size when independent set of size 1 is there independent set of size 2; there are one way. So, 1 into there are k ways to color one independent set. So, having chosen that color we cannot use that color the same color. So, the remaining colors will be k minus 1 and we can color to the set independent set. When the total number of independent sets are 3; so, there are two ways to do this and let us count how many colors? How many ways we can color these 3 independent sets

So, the first independent set we can color with k having utilise one color, the second independent set will be k minus 1, third is k minus 2. When the independent set is of size 4 then basically only one way is possible only one; we can have. Now let see the colorings of 4 different independent sets k k minus 1, k minus 2 and k minus 3.

So, if we solve this particular equation comes out to be which is shown over here.

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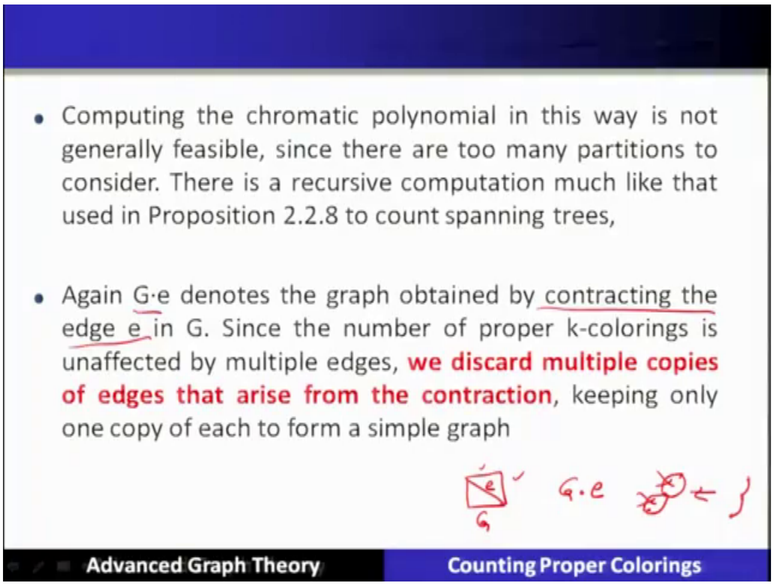

Now, computing the chromatic polynomial in this way is not generally feasible; since there are too many partitions to consider. So, there is a recursive computation much like that used in the in the counting of trees. Again we have to see the background for building it that G dot e is the representation for a graph, which is obtained by contracting an edge e. So, if this is the graph which is called a kite having this particular edge. So, G with this edge is contracted when edge is contracted that becomes that becomes a particular vertex where this edge will not be present, but these edges will be present. So, that becomes an graph

Since the number of proper k colorings is unaffected by multiple edges; you may discard the multiple copies of an edge hence this becomes the graph keeping only one copy to form this kind of graph.

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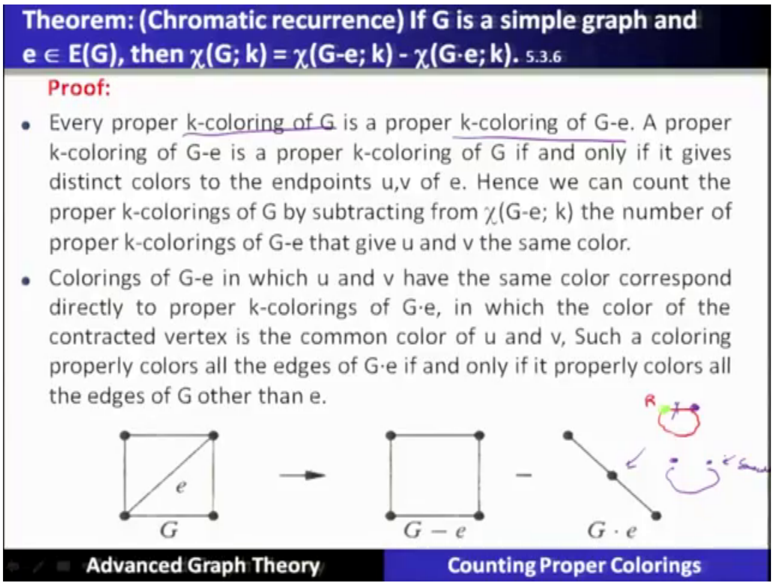

So, with this particular introduction of the contraction operation in the graph of an edge given edge e; we now define a theorem which will give you the chromatic recurrence; so, chromatic recurrence theorem.

So, if G is a simple graph and e is an edge in the edge set of that particular graph; then the chromatic value G for different colors is equal to chromatic number of G without that edge with k different colors minus chromatic number of G contracted an edge e with k different colors. So, the proof will give you the idea let us see this particular proof.

Now, every proper k coloring of a graph is a proper k coloring of G minus e that is the graph without that edge; a proper k coloring of G minus e is a proper k coloring of a graph if and only distinct colorings to the endpoints of v. Hence we can count the proper k colorings of G by subtracting from G minus e with k the number of proper k colorings of G minus e that gives u and v the same color.

Now, colorings of G minus e in which u and v have the same color correspond directly to the proper k coloring of G contraction e in which the color of the contracted vertex is the common color of u and v such a property such a coloring properly colors all the edges of

G minus e if and only if it properly colors all the edges of G let us understand this concept.

Now, if this particular graph is given then if you remove this particular edge, these vertices if they have the different colors then that many number of ways in which they are different that is to be subtracted. So, again why because let us see that if there is an edge then basically this will receive different a colors; one is let us say blue, green the other is another color let us say purple.

Now, if we remove this particular edge, if you remove this edge, if you remove this edge then there is no need of different colors. So, may be that we can use the same color in this particular graph. So, we have to find out the graph without this edge which is having the different colors that is to be calculated and removed from it. So, again I am repeating the proof; so, that will be indicated over here. So, every proper k coloring of a of graph G is a proper k coloring of G minus a G minus e.

So, if k colors are used here in G minus e and if the edge is not present then basically if the edge is removed; then basically the number of colors are going be reduced. So, the proper k coloring of G minus e is a proper k coloring of G; if and only if it gives a distinct colors to the endpoints. Hence we can count the proper k colorings of G by subtracting from G minus e with k colors; the number of proper k colorings of G minus e that gives d and e the same color. So, here we have to remove this many number of colorings which give the same color to u and v vertices for that you will do the edge contraction hence this particular formula is derived.

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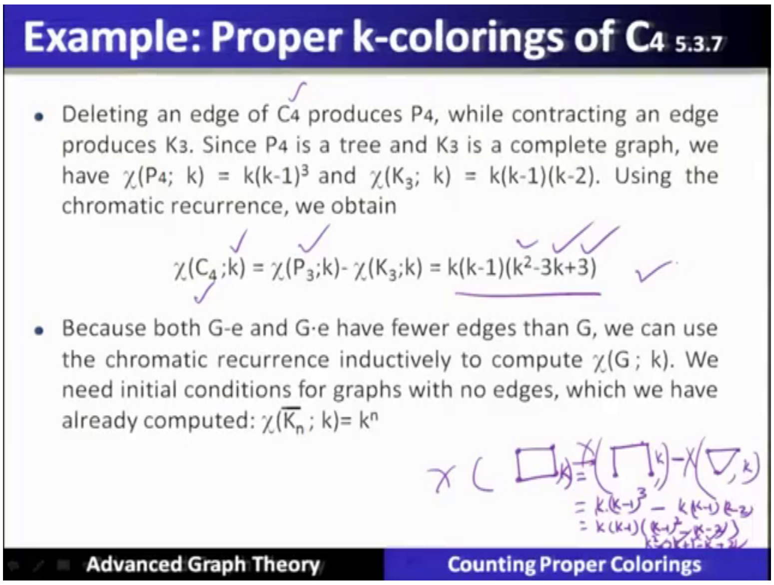

Now let us see how many colors C 4 produces with k using this particular formula. So, C 4 is this one; so, this C 4 we can if we can remove this particular edge; this will become C 4, this will become P 4, C 4 minus 1 edge is P 4 proper k coloring of P 4 wherein both are basically having the same color is to be removed; if we want a proper k coloring of G minus e.

So, that becomes that if we contract that particular edge; if we contract that edge that becomes a vertex and that is nothing, but the triangle. So, chi of G with with k colors is chi of this vertex with k colors minus why because then it will reduce the number of colors. So, how many cases where both are getting the same color is to be removed.

Now, you know that what is the value of path P 3; with k colors is nothing but first vertex; so, it is nothing, but a tree. So, tree will say that k times k times how many remaining is 4 minus 1 that is 3 minus as far as this is concerned triangle. So, the triangle will require like the complete graph; so, k then k minus 1 , then k minus 2 and if you solve this particular equation k into k minus 1 that is k minus 1 is square minus k minus 2. So, that becomes 3 and this becomes minus 3 k, minus 2 k and minus 2 k and this becomes k square. So, hence this particular equation we have derived.

Now, because both G minus e and G contraction e have fewer edges than G; we can use the chromatic recurrence to compute G k way we. So, we need initial conditions for the graph with no edges which we have already computed here in this case.

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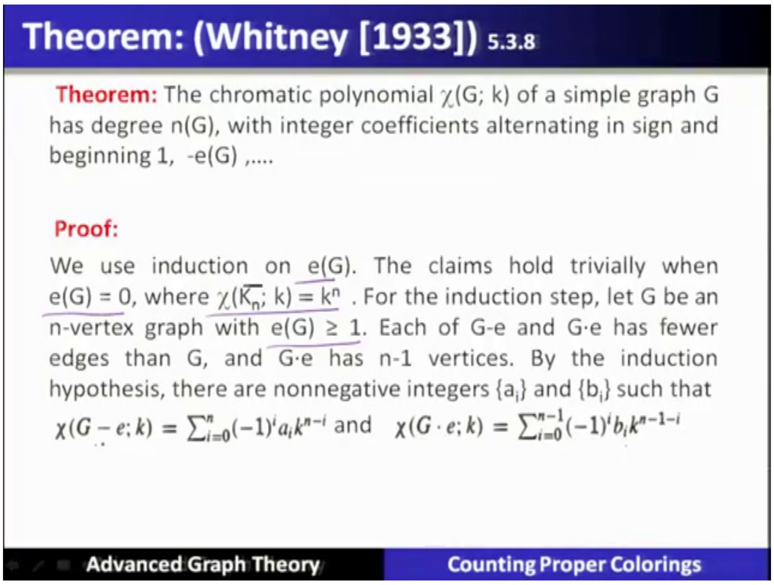

So, theorem Whitney in 1933 has given a theorem called chromatic polynomial theorem; the chromatic polynomial chi of G with k of a simple graph G has a degree n G has a degree n G with integer coefficients alternating in sign and beginning 1 minus e then plus and so, on.

Let us see the proof of chromatic polynomial. So, we use the induction on the number of edges on the edges of a graph. So, the claim holds trivially when the edges present in the graph is properties equal to 0. So, if there are no edges then this particular recurrence is basically nothing, but the complement of K n with k that is nothing k raised to the power n.

For induction step let G be an n vertex graph with e G is greater than or equal to 0; now each of G minus e and G contraction, you have your edges then G and G contraction e has n minus 1 vertices by the induction hypothesis, there are nonnegative integers a i and bi such that the chromatic polynomial of G minus e with k is equal to the summation of i which is running from 0 to n minus 1 raised power i; a i times k with the power n minus i and G contraction e with k that chi value is equal to summation of i is equal to 0 to n minus 1 with minus 1 raised power i; b i times K n minus 1 minus i.

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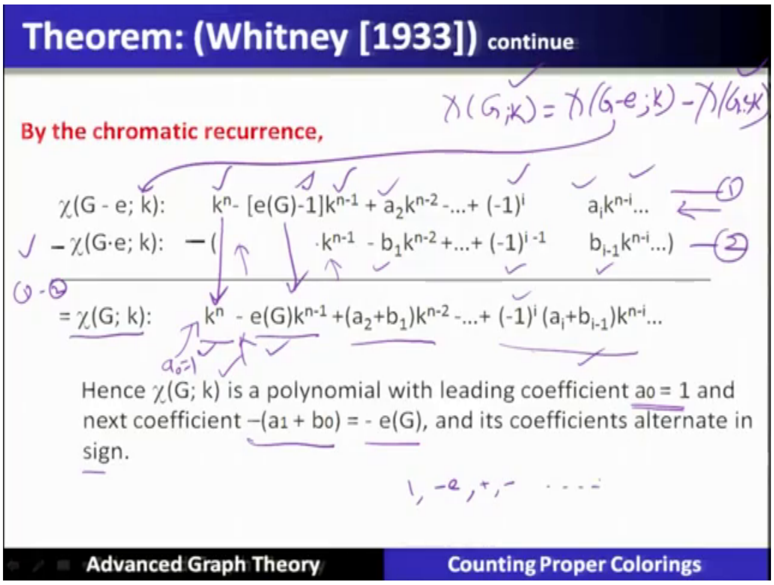

So, by chromatic recurrence both these equations we will see that chi of G with k is equal to chi of G minus e with k minus chi of G contraction e with k. Now let us see these in turn that is nothing, but k raised power n times e G minus 1, minus e G minus 1 times K n minus 1 plus a 2 K n minus 2 and so, on; i minus 1 to raised power i; a i and k raised power n minus i that we have seen in the previous slide.

Now, another there is a term for edge contraction has to be suspected; now you know that with edge contraction, the number of nodes will be n minus 1. So, here for nth node there will not be any polynomial leading term. So, the leading term will start with n minus 1 k raised power n minus 1 minus b 1 times K n minus 2 plus and so, on up to i minus minus 1 raised to the power i minus 1 times bi minus 1 times K n raised power k raised to the power n minus i.

So, if we subtract both the recurrence equation 1 and equation 2. So, 1 minus 2 you will get the recurrence for our desired formula. So, K n will come over here and this particular term both factor will be incorporated. So, here it is minus 1 and plus 1; so, that becomes e G. So, this term these terms are verified similarly we can include a 2 and so, minus and minus they will becomes plus and so, on.

So, finally, minus i and this is a i plus b i. So, this will be basically taken care of minus and minus; they will become plus. So, hence chi of G comma G with k is a polynomial with a leading coefficients a 0 is equal to 1 here. And the next coefficient is basically minus a 1 and plus b 2 that is nothing, but minus e G that is present and its coefficients alternates in the minus sign. So, 1 minus e then basically then again it will be plus 1 and then again it will minus and so, on. So, these signs will alternate in this particular manner.

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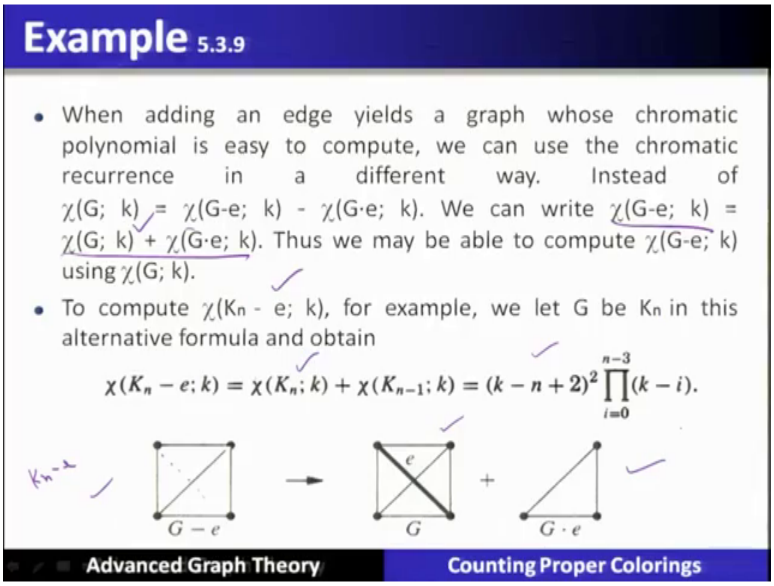

So, when adding an edge yields a graph whose chromatic polynomial is easy to compute we can use the chromatic recurrence in a different way; instead of saying that G k is equal to G minus e with k. And subtracting with G contraction e, we can write down instead of that that G minus e is equal to; that means, this term we basically put on the right side that becomes both plus.

Now, this you know that this way also we can compute; let us see that we want to compute this K n minus ; this is K n minus e that is this e is present e is absent. So, let us include it and we will form this particular recurrence; similarly here with contraction this is K n minus 1. So, we can solve this particular recurrence why because we know that this is easier to produce; this is also easier to produce and hence this is another example how we can instead of saying that an edges to be removed rather than we can say that edges added and the contraction operation.

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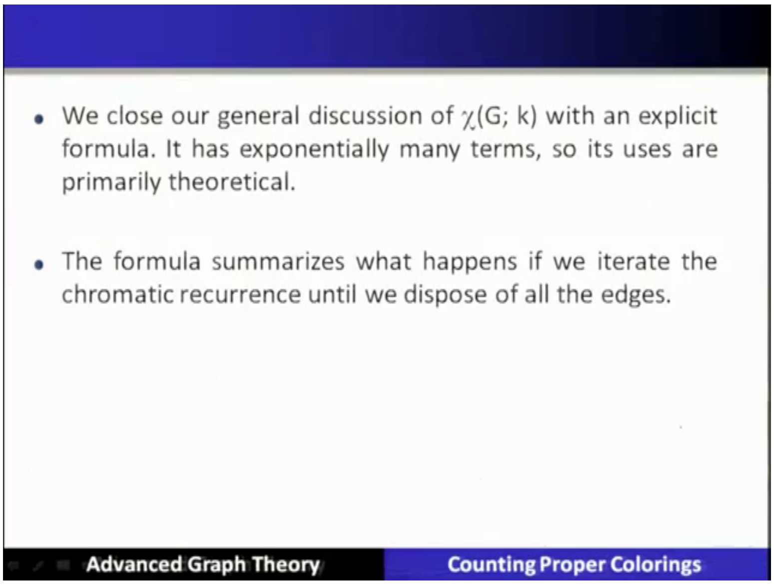

So, we close our discussion of chi of G comma k that is the number of enumeration of the colorings of a graph G with given connected with an explicit formula; it has exponentially many terms. So, its uses are primarily theoretical the formula summarizes what happens if we iterate the chromatic recurrence until we dispose of all the edges.

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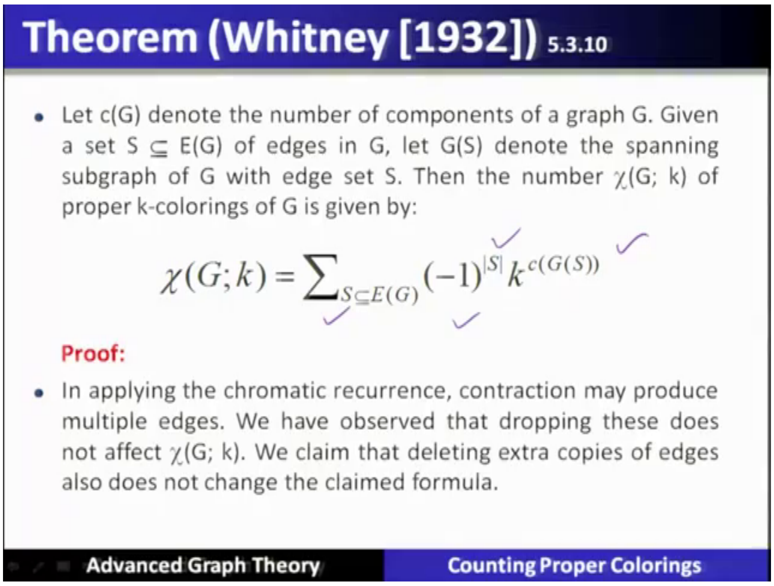

So, there is another theorem given by Whitney in 1932; let c of G denote the number of components of a graph G now given a set S which is a subset of the edges of G of edges n G; let G of S denotes the spanning sub-graph of G with the edge set S; then the number that is chi of G with k of the proper k colorings of G is given by this particular equation that is nothing, but the summation of the sub subsets of S of edges with minus 1 raised to the power that subset S times k raised to the power c that is number of components in subset of in a spanning sub-graph with S.

So, in the proof if we see that in applying the chromatic recurrence contraction may produce multiple edges; we have observed the dropping these does not affect the chi of G with k we claimed that deleting extra copies of the edges does not change.

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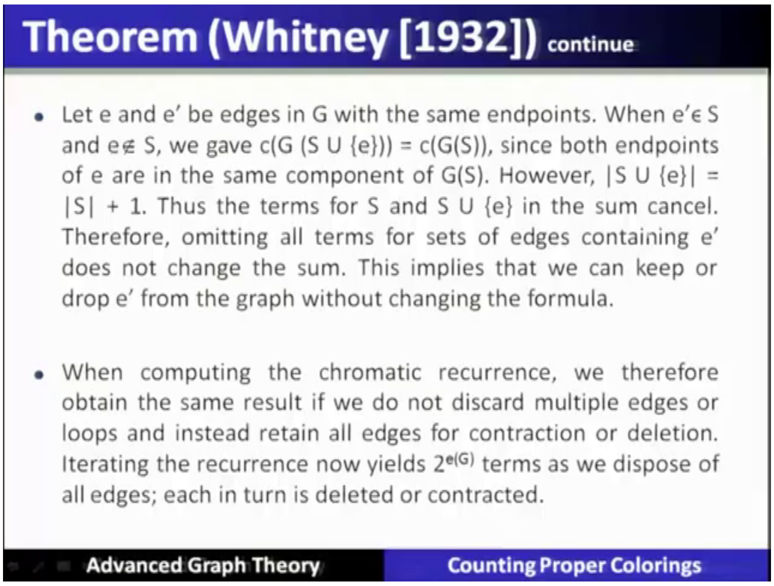

The claim formula let e and e prime be the edges in G with the same endpoints; when e prime is in S and e is not in S we give that number of components of G S union e is equal to number of components with S; since both the endpoints of e are in the same component; however, S union e is nothing, but the cardinality of S plus 1 thus the terms of S and S union e in the sum cancel.

Therefore, vomiting all the terms for the set of edges containing e prime does not change the sum this implies that we can keep or drop from the graphs without changing the formula. When computing the chromatic recurrence we therefore, obtain the same result if we do not discard multiple edges or the loops and instead retain all the edges for contraction or deletion iterating the recurrence; now yields 2 raised power e time e terms as you dispose of all the edges each in turn is deleted or contracted.

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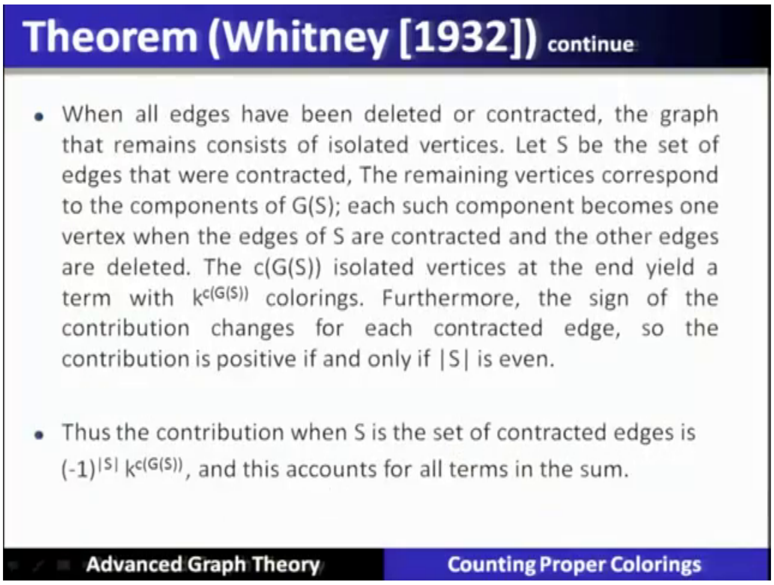

When all the edges have been deleted or contracted the graph that remains consist of an isolated vertices; let S be the set of edges that were contracted the remaining vertices correspond to the components of G S, each such component become one vertex when the graph of S are contracted on the other edges are deleted. Let c of G S isolated vertices at the yields a term with k raised power number of components of G S colorings.

Furthermore the sign of the contribution changes for each contracted edge. So, the contribution is positive if and only if the cardinality of S is even thus the contribution when S is the set of contracted edges is minus 1 to the power cardinality of S times k to the power number of components of G with the spanning sub-graph of S and this accounts all the terms in sum.

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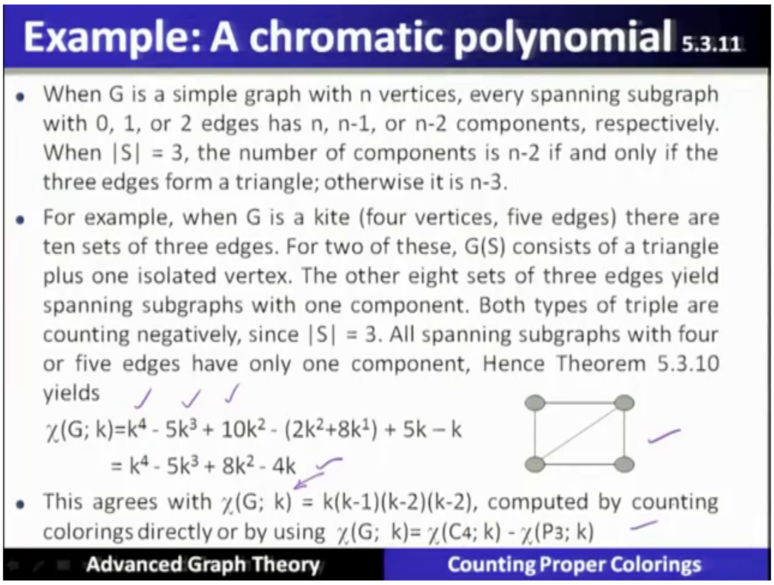

So, let us take example of a chromatic polynomial when G is a simple graph with n vertices every spanning sub-graph with 0 one or two edges has n n minus 1 and n minus 2 components respectively. When S is equal to 3 the number of components is n minus 2; if and only if the 3 edges for a triangle otherwise it is n minus 3.

For example when G is a kite with 4 vertices and 5 edges that is shown in the diagram; there are 10 sets of 3 edges for two of these G S consist of a triangle plus one isolated vertex the other 8 sets of 3 edges yield spanning sub-graph with one component both types of triples are counted negatively; since S cardinality is 3 all spanning sub-graph with 4 or 5 edges have only one component. Hence the theorem yields k raised power 4 for this particular graph k raised to the power 4 minus 5 times k raised to the power 3 plus 10 times k square and so, on.

So, if that gives the same formula that we have earlier proved. So, this agrees with the earlier stated computation method by counting different coloring directly without using this particular formula by using this particular way.

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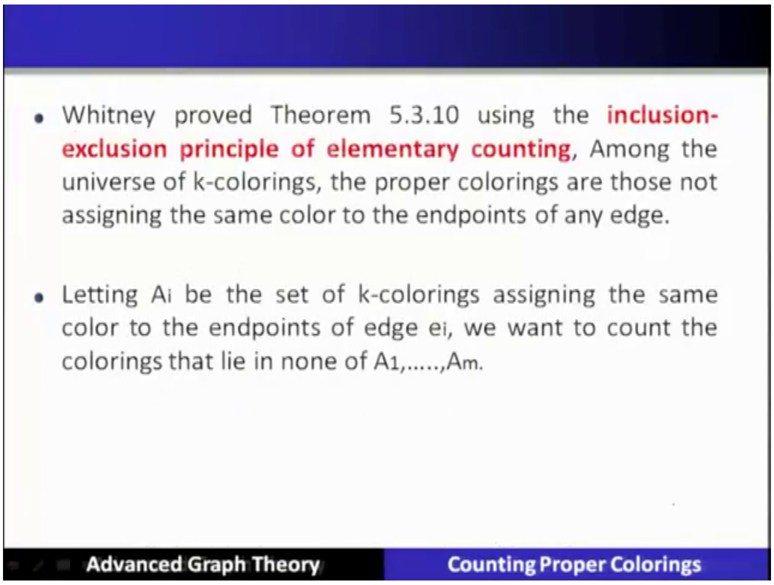

So, basically we can see that chromatic polynomial is a direct method of application without doing all enumeration steps in between Whitney proved an theorem using inclusion exclusion principle of elementary counting. So, letting A i be the set of k colorings assigning the k color to the endpoints of e i we want to count the colorings that lie in none of these A 1 to A m.

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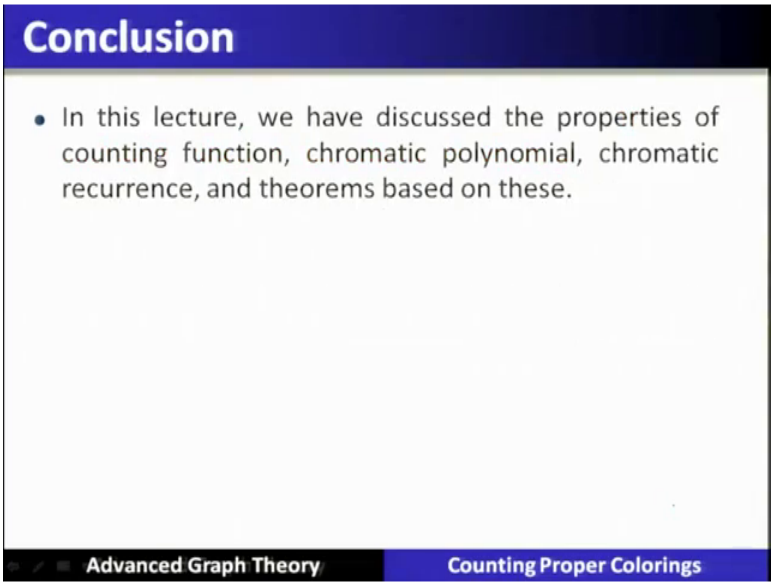

So, this lecture we have discussed the properties of counting function, chromatic polynomial, chromatic recurrence and theorems based on these issues.

Thank you.