Advanced Graph Theory
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Lecture - 17
Brook's Theorem and Color-Critical Graphs

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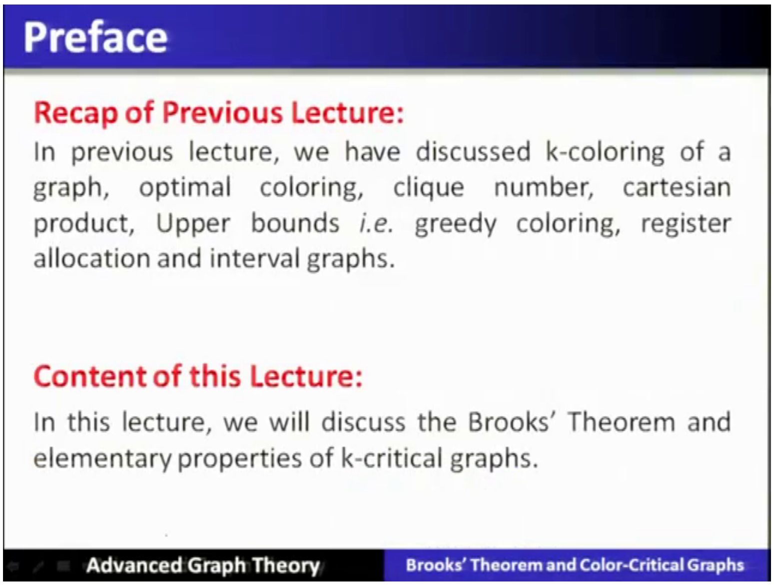

Recap of previous lecture we have discussed k-coloring of graph, optimal coloring, clique number, Cartesian product of a graph, upper bounds in terms of colorings that is upper bounds of greedy coloring algorithm.

Then we have also cover the greedy coloring when the graph is interval graph and such kind of graphs basically arises in the problems of register allocation application in compiler design.

Content of this lecture this lecture we will discuss the Brooks's theorem which will give the bound on the chromatic number of a graph and you will see the proofs of it then we will also look into the k critical graphs. So, k critical graph here means that k color critical graphs.

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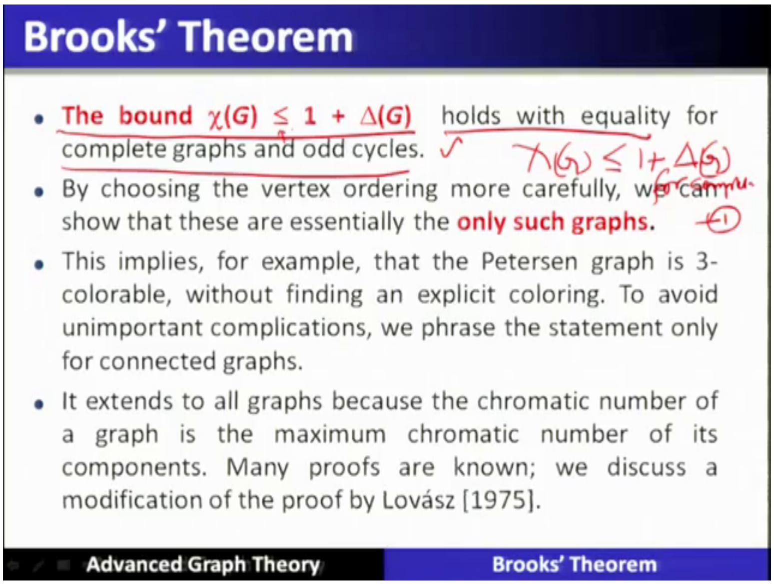

Brook's theorem let us see the background of the Brooks's theorem before we state this particular theorem.

The bound of a chromatic number in a graph is given by 1 plus maximum degree of a graph. Now this particular bound we have earlier stated. So, let us see that what are the graphs when this bound will become equal so that equality will hold for the complete graphs and the odd cycles.

So, for complete graphs an odd cycles chi of G is less than or equal to 1 plus maximum degree of a graph for complete graphs and odd cycle so this is one point. Now by choosing the vertex ordering by carefully we can show that these are essentially the only such graphs where this equality in this particular bound will hold. If the graph is this implies for example, the Petersen graph is 3-colourable, without finding explicit coloring.

To avoid unimportant complications we state that the graphs are connected graphs for disconnected we can also extend it for finding out the chromatic number of the components and take the maximum of it that becomes. So, without loss of generality we will consider only the cases for connected graphs.

So, it extends to all the graphs because chromatic number of a graph is the minimum or a maximum chromatic number of its components so that I have already stated. We will see the proof by brook of the Brook's Theorem by given by Lovasz in 1975.

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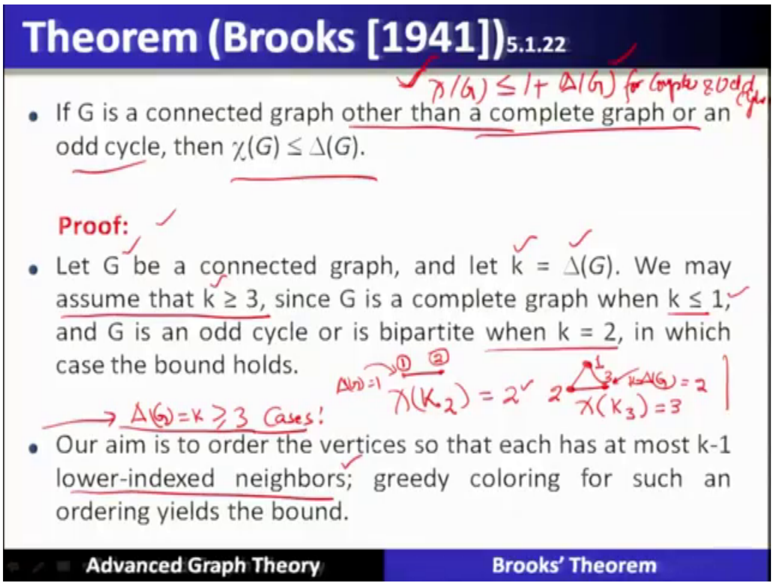

So, the Brooks theorem is stated as under which is given in 1941 and the proof will give by Lovasz in 1975. So, if graph G is connected graph other than the complete an odd cycle for complete an odd cycle this chi of G is 1 plus for complete graph and odd cycles.

So, if the graph is not complete other than the complete graph then the Brooks theorems says that the chromatic number of a graph G is less than or equal to the maximum degree of a graph. Let us see the proof given by Lovasz, let G be a connected graph and this particular symbol k small k we are using for the maximum degree of a graph G which we have considered in our discussion.

We may also assume that the maximum degree of a graph should be 3 or more in our discussion. Why because? If it is less than that then it becomes a complete graph or the odd cycle and where this particular equation will be applicable that is chi of G is less than or equal to 1 plus delta G if you want to see that if the values are 1 so this will be the complete graph k 2, here the degrees are equal to 1.

So, what is the chromatic number of k 2 so, here the big delta G is equal to 1, so this becomes 1 plus 1 that becomes 2. So, if you color this vertex with a label 1 you cannot color the other vertex with same label for a proper coloring it has to be another colour which is required hence 2 colors are required minimum to colour this particular graph. If it is odd cycle, where k is equal to 2, then k is equal to 2 that becomes odd cycle here the degree of this graph is equal to 2.

So, hence we also call it as k is equal to 2 here in this case. So, what is the chi value of k 3 this is a complete graph. So, this particular vertex can be coloured with the label 1 or index one of that colour the second vertex which is adjacent to the previous one cannot be coloured with one.

So, another colour is required third vertex which is adjacent will other previously given colour 1 and 2 cannot be coloured with either one or 2 hence another colour three is required. So, chi of G is big delta plus 1 that is 2 plus 1 that is 3 so this particular theorem or this particular bound for complete graph and odd cycles we have seen that it is applicable if the big delta is basically.

Now, we will consider that when big delta data of G which we call it as k should be greater than or equal to 3 cases only in this particular proof. Now here our aim is to order the vertices so that each has at most k minus one lower index neighbours and then apply the greedy coloring algorithm which will yield the bound for our discussion.

So, let us see how we are going to order the vertices so that this particular condition that at most k minus 1 lower index it has basically already k minus 1 lower index neighbours then only we can give a 3 D coloring or k with k number of colours.

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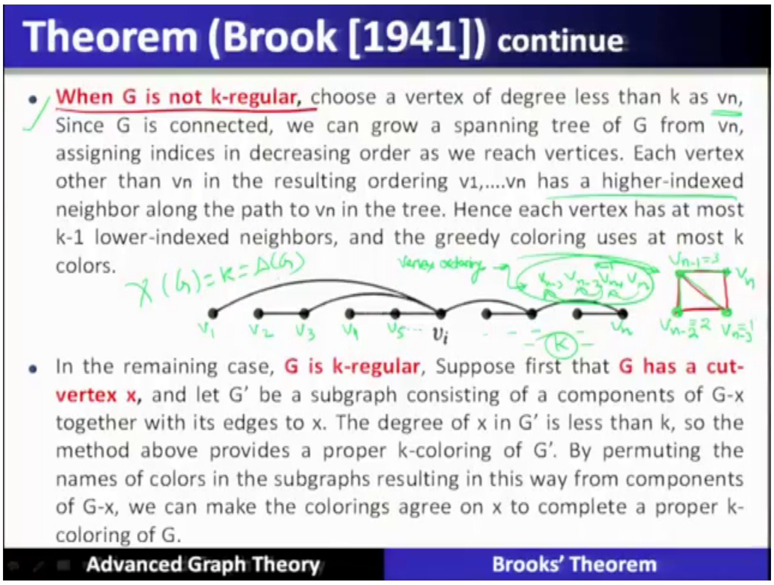

So, let us see the steps in the Brooks theorem consider that here the graph G which is given is not a regular is not k regular graph. So, what we will doing? We will choose a vortex of degree less than k so let us take an example of a graph which is not a k regular so that means, not all vertices are having a degree k in a particular graph.

Let us choose this graph and let us have this is the graph that is called a kite so it is not a regular k graph. Now here it says that choose a vertex of degree less than k here the k is 3. So, the vertex with a degree less than k is let us say it is vn which is degree less than k. Now since this particular graph is connected. So, we grow a spanning tree of G from.

Let us draw a spanning tree assign the indices in the decreasing order as we reach the vertices so it is vn. So, let us say that vn minus 1 and this is let us say vn minus 2 and this is vn minus 3 let us call it as 1, this is 2, this is 3, and this is n so let us label it accordingly. So, let us call it at as this is v 1, v 2, v 3, v 4, v 5 and so on up to vn. So, we say that this particular vn is here.

So, we have obtained the vertex ordering according to the construction of the spanning tree and then basically from vn to all other nodes connecting to that particular tree are given a lower indexes. So, this particular ordering we have obtained so in this particular graph example we order vn minus 3, vn minus 2, vn minus 1, and vn.

So, this particular way if we order it will follow the following property so higher indexed neighbour along this particular path to vn in this particular tree here each vertex has a post k minus 1 lower index neighbours. So, let us take vn vn has only one neighbour so it is on the so it is 1 neighbour on the left side so that becomes k minus 2, vn minus 1 is concern it is having 1, 2; 1 and 2 so, it is nothing, but k minus 1 lower index neighbours are basically placed on the left side.

So, if we apply a greedy coloring in this particular ordering we can use at most k colours because k minus 1 colours are already used by the neighbours that is k minus 1 lower index neighbours and so basically k-th color will be use for that purpose so at most k colors will be used here in this particular way so if the graph G is not regular. So, it will give you the k colours. So, hence chi of G is equal to k colours that is nothing, but big delta of g.

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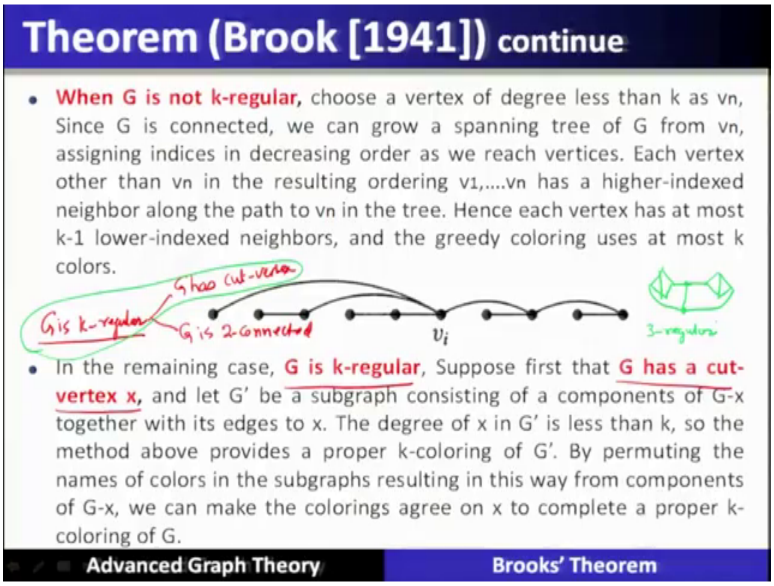

In the remaining case let G is k regular suppose first that G has a cut vertex x and let G prime be the sub graph of consisting of the components of G without x together with the edges to x the degree of x in G prime is less than k. So, the method about provides the proper k coloring of G prime by permitting the names of the colours in the sub graph resulting in this way from the components of G without x we can make the coloring agree on x 2 complete a proper k coloring.

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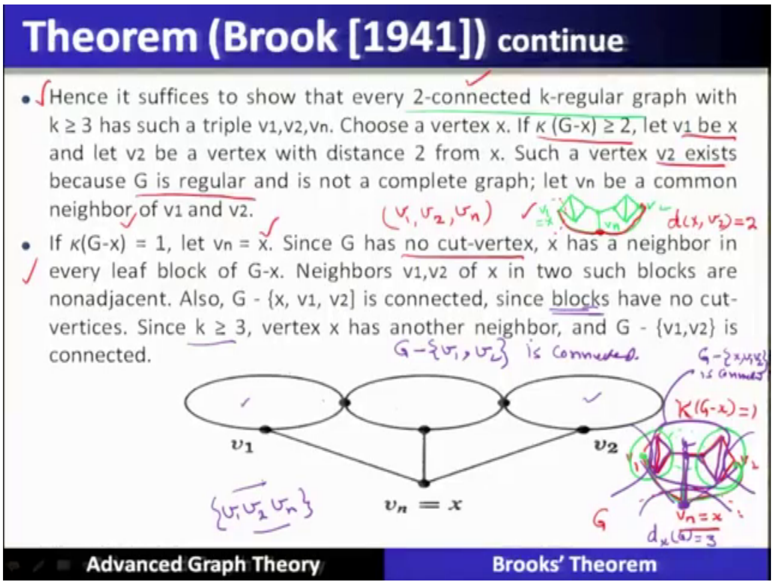

So, now we will see the cases in this particular way. Now if let us say that G is k regular in this particular case we have 2 different cases for the one is that that G has a cut vertex and G is 2 connected. So, let us take the first case when G is k regular and also has the cut vertex. Now to see this scenario let me give you an example graph which will fit in this scenario.

So, here this is the example of a of a 3 regular graph. Now in this particular hence it is suffices to show that every 2 connected k regular graph with k is equal to three has such a triple v 1, v 2 and choose a vertex x if kappa of G minus x is greater than or equal to 2, let v 1, v x and v 2 be the vertices with distance 2 from x such a vertex v 2 exists because G is regular and is not a complete graph let vn be common vertex common neighbour of v 1 and v 2

Let us take a example graph and understand this particular case. Now here this particular graph is 2 connected and k regular and we will consider 3 triples; v 1, v 2, and vn. Now chose a particular vertex x so here let us say that this is equal to x now if you remove x so without x this the remaining part of the graph is having connectivity kappa G without x of 2.

So, let v 1, vx that we have already assumed and let v 2 be a vertex with a distance 2 from x so this particular distance you see that this is one half and this is another half so this distance is 2. So, distance of x and v 2 here is equal to 2 and we assume a vertex vn between x and v 2. Now such a vertex exists because this particular graph is regular and is not a complete graph.

Hence vn is basically be a common neighbour of v 1 and v 2 hence we have shown that in this particular scenario when the graph is k regular and 2 connected there exist 3 such triple. Hence this particular case will also be satisfied with k coloring of the graph so k means with the big delta G of a graph

Now, let us see that if there exist if kappa of G without x is equal to 1. So, let us say that G without x is equal to 1. Let us take another example graph. Now this particular graph if we consider without without x so this will be the connectivity of kappa value of this particular graph G without x is equal to 1. So, x is removed then basically the connectivity is 1 and let us consider vn is equal to x this is another case.

Now, since this particular graph G has no cut vertex so x has the neighbour in every leaf block of G minus x. So, meaning to say that this particular x has basically is a neighbour in every leaf block so there are 3 leaf blocks of G minus x. So, neighbours of neighbour v 1 and v 2 of x into such blocks are nonadjacent. So, v 1 so this particular block they are nonadjacent; that means, there exist another such block also G minus x v 1 and v 2 that means, if we remove; that means, if we remove v 1, if we remove v 2 and if we remove x the remaining part of this particular graph which is shown here this is this part of the graph is G without x, v 1 and v 2 is connected.

Since the blocks have no cut vertices by the definition which we can see the previous videos of blocks. Since the blocks have no cut vertex that is why removing one vertex from this particular block will not disconnect the blocks hence for k that is the maximum degree is greater than or equal to 3 that is why this particular x has another besides v 1 and v 2 has another neighbour and therefore, G minus v 1 and v 2 is connected so that means, if x is included only v 1 and v 2 are removed then x will also be have a connection with a node let us say y because the degree of degree of x in this particular graph is equal to 3.

So, this is one neighbour with v 1, another with v 2 so x is also connected. So, hence this show that G minus v 1 and v 2 is connected. So, we have shown that in this particular scenario also the 3 triples v 1, v 2, and vn they exist and thus this proves the Brooks theorem.

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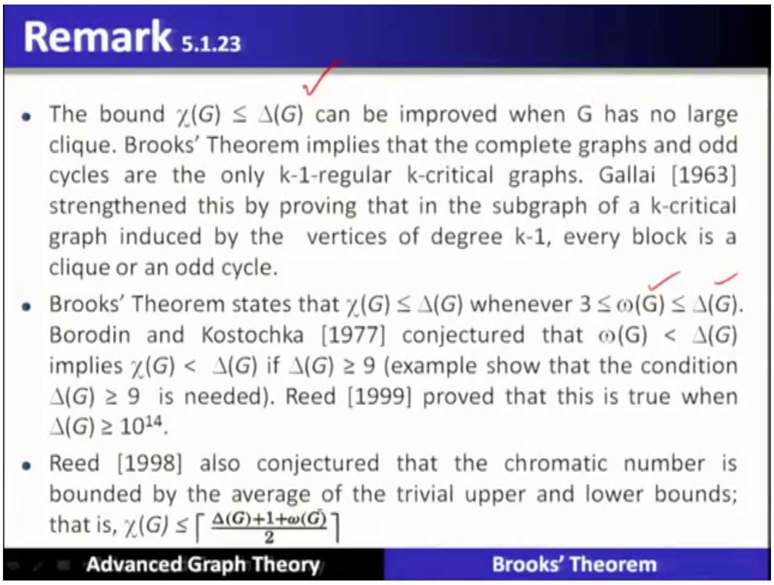

The bound given by the Brooks theorem can be improved when G has no large cliques the Brooks theorem implies that the complete graph and the odd cycles are the only k minus 1 regular k critical graphs. Gallai strengthened this by proving that in the sub-graphs of k critical graph induced by the vertices of degree k minus 1 every block is a clique or an odd cycle

So, Brooks theorem is states that chi of G is less than or equal to the maximum degree of a graph whenever omega G that is the clique number of a graph is bounded above by big delta of G and it is basically at least of 3 degree. So, there are various other readers given a conjectured about the chromatic number which is bounded by the average of the trivial upper bound and the lower bound.

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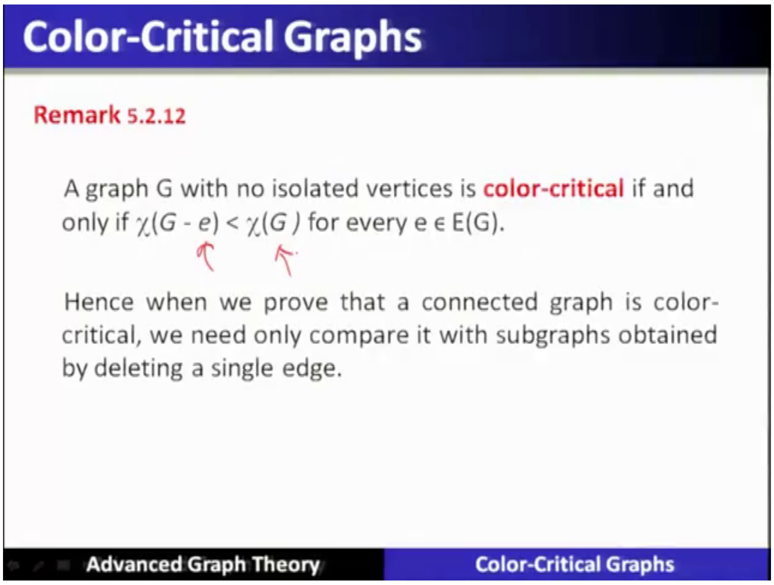

Now, you will see what is the color critical graphs the graph G with no isolated vertex is color critical if and only if chi of G minus C that means, if we remove an edge from a graph the chromatic number of that particular graph will reduce then that graph is called color critical graph; that means, it is critical in the sense it is colour critical in the sense that is every edge it is removed it will also reduce the number of colours.

Hence such graphs are called color critical graphs. Hence when we prove that a connected graph is colour critical we need only compared with the sub-graphs obtained by deleting a single edge.

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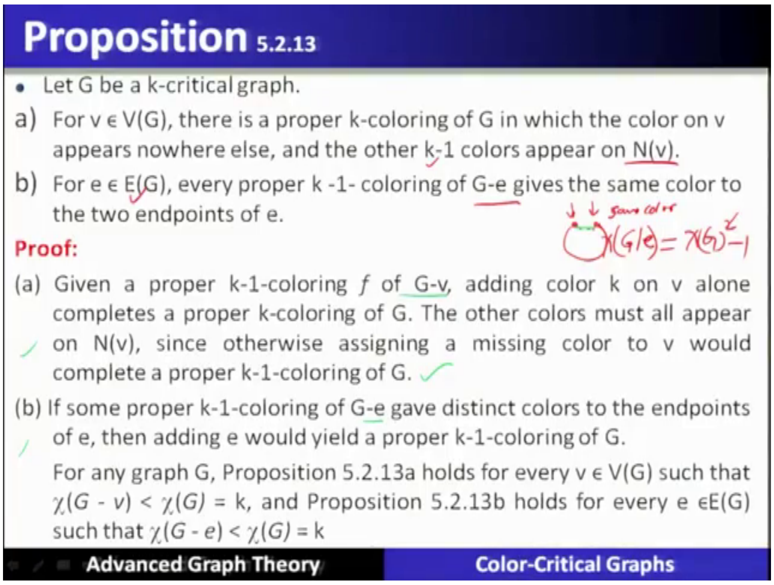

And we will see the proposition let G be a critical graph now for a vertex v in the vertices in the set of vertices of a graph G there is a proper k colouring of a graph in which the colour on v appears nowhere else and other k minus one colour appears in the neighbours of that particular vertex.

So, if that particular vertex is removed what will happen the number of so the chromatic number will also be reduced why because that colour will not be present. So, number of colours will be reduced by 1. Similarly, if an edge is present so every proper k minus 1 coloring of the graph without that edge with the same colour to the 2 endpoints of v. Hence the graph of G is k critical; meaning to say that if this is the graph G minus e, if you do the k coloring of G minus e graph these 2 vertices are receiving the same colour. Hence, chi of G minus C will be chi of G minus 1 because we are in chi in G these two vertices which are having an edge they may not be having the same color, they may receive the different colours hence this is the proof.

So, let us see the proof again given proper k minus one colorings of the graph without a particular vertex v. Now adding the colour k on v alone completes a proper k coloring of the graph. The other colours must all appear in neighbours since otherwise assigning a missing colour to be would complete a proper k minus 1 colouring. If some proper k minus 1 coloring of the graph without that edge e gave distinct colours to the endpoints of v, then adding e would yield a proper k minus 1 coloring of that particular graph g.

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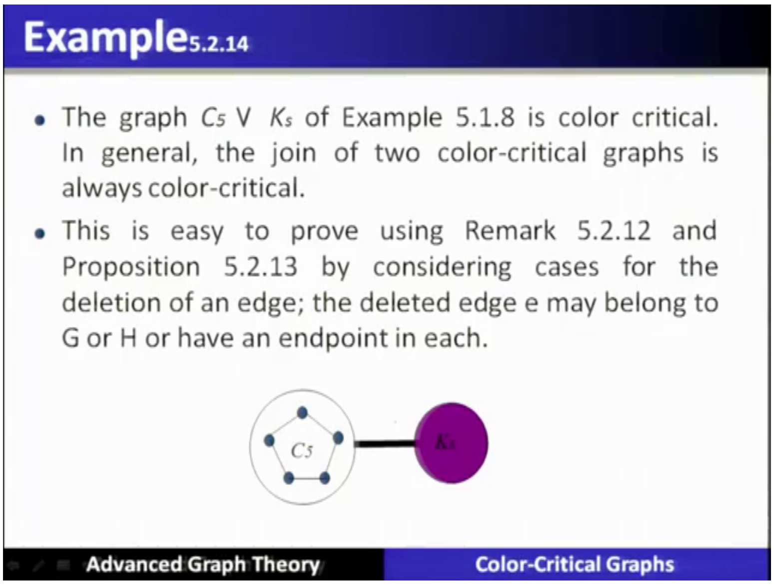

The graph C 5 join K s is an example of a colour critical graph. It is easy to prove using this particular remark and the proposition by considering the cases of and deleted edge e may belong to G or H or have an endpoint at each.

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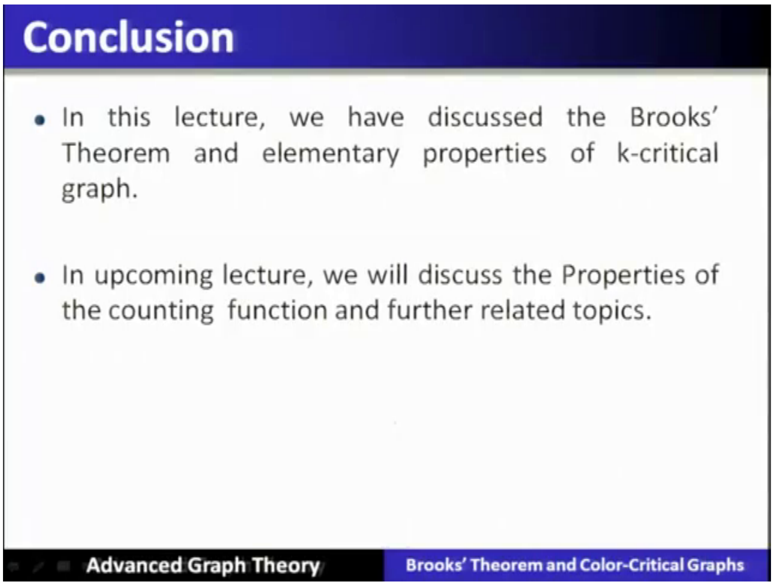

In this lecture we have discussed the Brooks theorem and we have also seen a critical graph. Upcoming lecture, we will discuss the properties of a enumeration in reference to the colorings and also some the few other topics.

Thank you.