Advanced Graph Theory
Prof. Rajiv Mishra
Department of Computer Science and Engineering
Indian Institute of Technology, Patna

Lecture - 12
Matching in General Graphs Edmonds Blossom Algorithm

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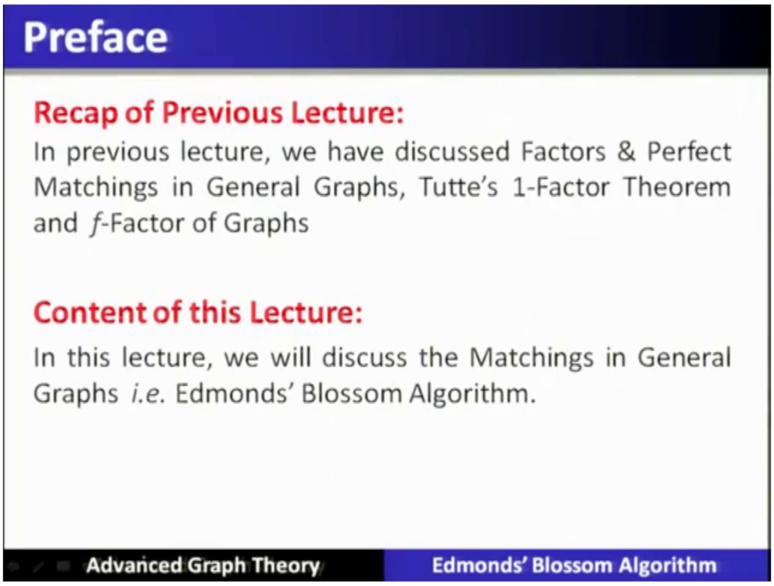

Matching in a general graphs at one blossom algorithm recap of previous lecture in previous lecture we have discussed factors, perfect matching in a general graph that is one factor theorem and f factor of graphs. Content of this lecture, we will discuss the matching's in a general graphs that is given by the algorithm, which is known as Edmonds Blossom Algorithm, Edmonds Blossom Algorithm.

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If we see the details of why a separate algorithm is now required, but we have already seen the matching in a bipartite graph using a augmenting path algorithm, for bipartite graphs why it cannot be used here for matching in a general graphs all this discussion we will see now. Now we have seen earlier that Berge theorem, stated that matching M in G has a maximum size if and only if it has G has no M augmenting path. So, the augmenting path algorithm called APA.

what it does is it finds the maximum matching using successive finding of augmenting path, this particular augmentation is carried out by the iterations of n by 2 different times and therefore, all the augmenting paths are identified and the matching cardinality or the matching size is increased every time. Now after n by 2 iterations it is proved that it cannot be extended why because it will not find any augmenting path, hence using Berge theorem the algorithm terminates with a maximum matching, in a bipartite graph.

Now Edmond in 1965 presented the first algorithm, which basically is generalized algorithm for general graphs not only confined with the bipartite graph, but it is a generalized algorithm and his paper, was called as path trees and flowers in bipartite graphs.

We can search quickly for augmenting paths, because we explore from each vertex at most once an m alternating path from u can reach vertex x in the same partite sets as u only along the saturated edge.

Hence only once we can search and explore x, take this example if this is the bipartite graph ; that means, two partite set x and y let us assume that this particular vertex is unsaturated. So, you can reach to any vertex on the other partite set and then take a matched edge, and reach on the same side and then take another unmatched edge and reach on other side let us say it is y.

So, when it reaches here, it takes in the matched edge matched to come back to the same partite set. So, there is only one possibility to reach to the other side of the partite set because matching, tells you that a particular edge is incident on only one vertex it cannot be incident on two vertex. So, if it is taking up the matching edge to reach to the same partite set then it has only one choice because, it uses the matching edge. Hence only one choice will be there and we can explore x in this particular manner.

So; that means, in the bipartite graph searching augmenting path is quite easy and fast also, but if a graph is a general graph; that means, it contains an odd cycle then reaching on the other side may have more than one options to explore. Hence the exploration is not to that particular space, in which bipartite graphs that is the graphs which are not having the odd cycles can function easily and efficiently can done.

So, this particular property that it has to explore once exactly one search this particular property will fail in a graphs with the odd cycle, that we will see why because M alternating path from an unsaturated vertex may reach x both along saturated and along unsaturated edges take this particular example, this is an odd cycle.

If this is unsaturated vertex u when it comes over here these are all unsaturated vertices. So, there are two different openings of unsaturated vertex where to go, if it is an odd cycle which way to go or both ways are to be explored at the same time. So, this particular property is failed here, in the general graphs the graphs especially which is having odd cycle. Hence, Edmond blossom has given a new algorithm to do the matching in a general graphs that is the graphs, which is having the odd cycle also how the augmenting path can be efficiently run out that is given here in this algorithm.

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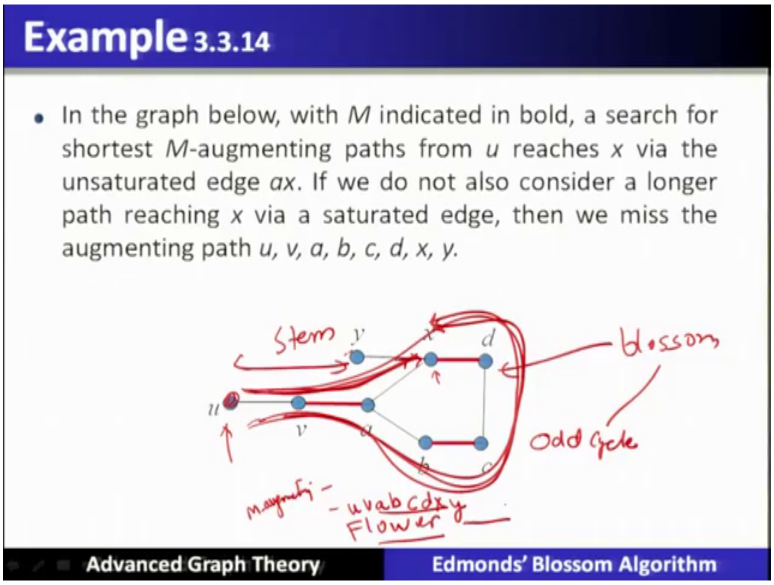

So, same example let us illustrate, it again regarding the problem in identifying the augmenting path when the graph is having an odd cycle. This is odd cycle, so in the graph below M is indicated that is the match as is indicated with the red colour. And the search for the shortest M augmenting path from u when it reaches x y are the unsaturated ax.

Then we may miss the other path which is basically a longer one, but it is reaching to x via a saturated edge. So, in this case if we choose to go directly by unsaturated vertex reaching x by unsaturated vertex then we will be missing up this augmenting path, that is u,v, a, b, c, d, x, and then y both are unsaturated vertices. So, this is an augmenting path which will be missed out if we only follow this particular path which is not a set.

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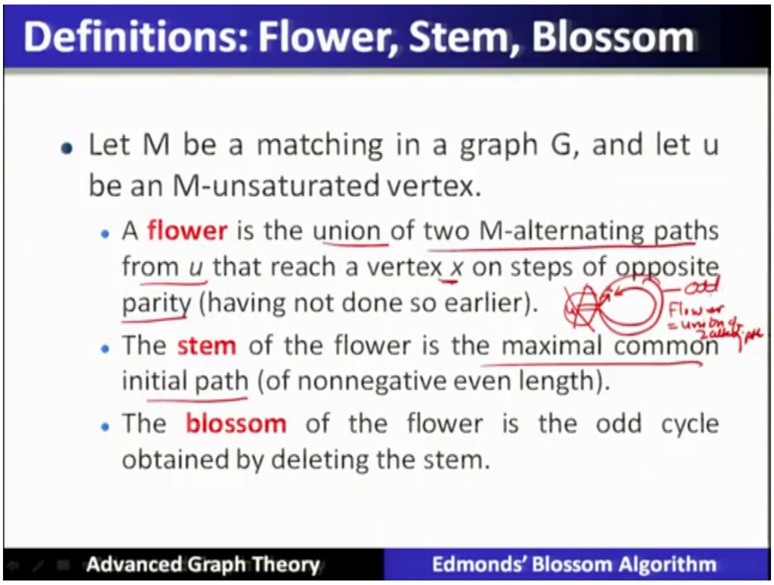

So, let us see the definitions let M be the matching in a graph G and let u be unsaturated vertex. So, a flower is the union of two alternating paths from u that reach a vertex x on by steps of opposite parity, take this particular example you start from unsaturated vertex you reach x with a different parity. So, this becomes a flower.

So, in this particular flower this portion is called a stem this is called a blossom is an odd cycle and total thing is called a flower. The terminology is drawn from the horticulture why because trees were also used in the computer science and in the graph theory also, let me again read it out a flower is the union of two alternating paths from u that reach a vertex x, that I have told, let us say this is vertex x you reach via this path and you reach via this path from u.

And this is the union of two alternating path is called a flower, reaching x on the step of opposite parity by stem of a flower is the maximal common initial path. So, this both the augmenting path this portion is common and a blossom of a flower is the odd cycle obtained by deleting the stem, if the system is deleted. So, this becomes an odd cycle and this is called a blossom.

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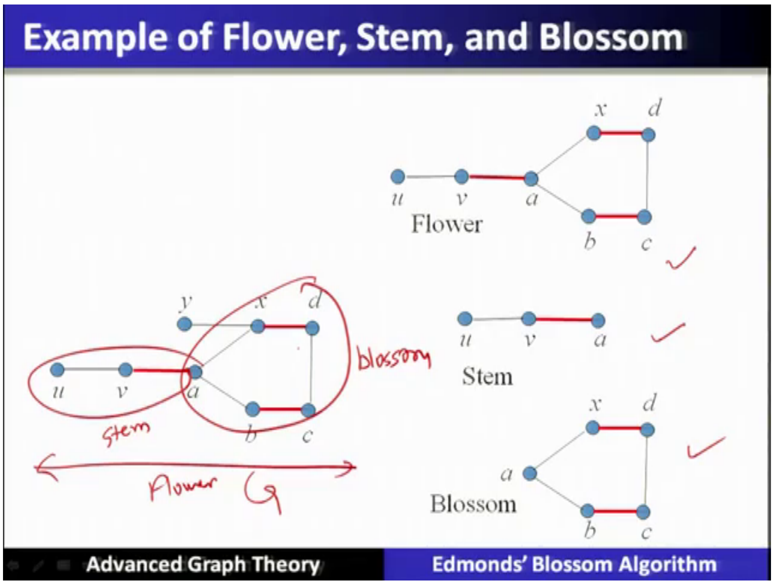

So, we have stated all these things again for your understanding that if this is the graph G this portion is called a stem, this portion is called a blossom and this entire thing is now called a flower which is highlighted.

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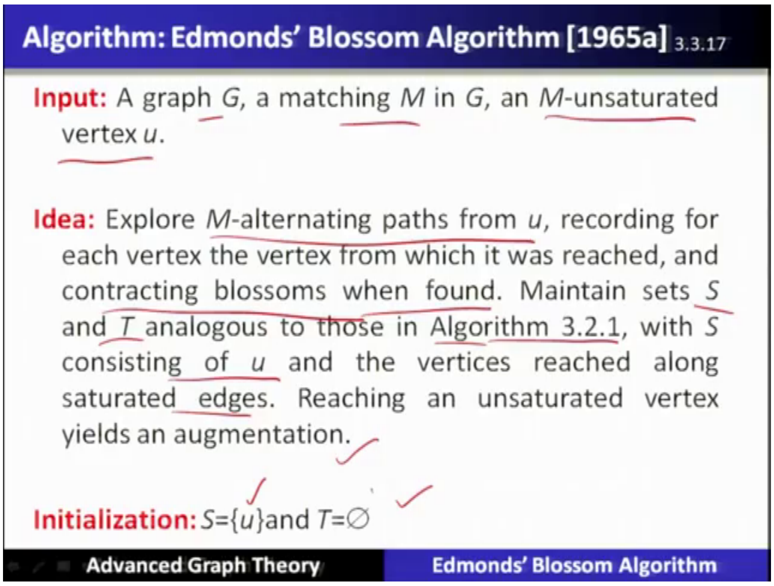

Let us see the Edmonds algorithm, we will then take an example to understand the working principle of Edmond algorithm and then again we will come back to the same algorithm for a better explanation. input to this particular algorithm is a graph G also there is a matching, and unsaturated vertex u is given as an input idea of this algorithm is to explore M alternating path from u, recording each vertex the vertex from which it was reached, and contracting the blossoms on the way.

When found. Maintaining sets S and T are analogous to those we have seen in the algorithm earlier, with S consisting of us and the vertices reached along the saturated edge, reaching at unsaturated vertex will identify the momentum path, let us initialized S with the unsaturated vertex and T is basically empty.

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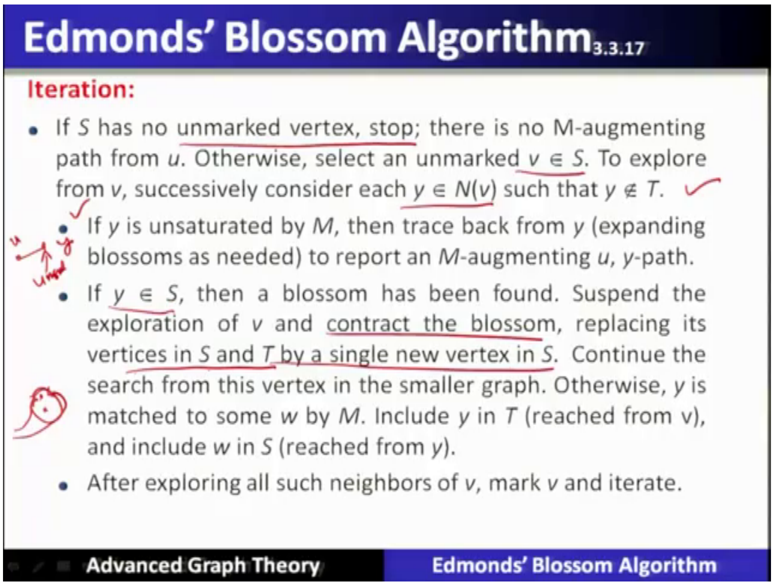

Let us see the iterations of Edmond algorithm, if S has no unmarked vertex then you stop, there is no M augmenting path from u. Otherwise, it has unmarked vertex v to explore from v successively consider each neighbour of v let us call it as y and y is not marked that is not in T. Now if y is saturated by M then trace back from y to report an m augmenting path uy.

So, if you start from two reach to another vertex y which is also unsaturated. So, immediately you will get the augmenting path. Now if S if y belongs to s then a blossom has been found because, you have already visited that node and is there in the S; that means, you have reached to a point which is already being visited and is available in S, then a blossom has been found suspend the exploration of v and contract the blossom.

So, blossom is contracted with, with one vertice one vertex replacing its vertices in S and T by a single vertex in S. Continue search from this particular vertex in the smaller graphs otherwise y is matched to some w by M. Include y in T and include w in S, after exploring all such neighbours of v mark v and iterate. Let us see all these step in an working example and then come back again we will explain you again in this example let M be the matched edges.

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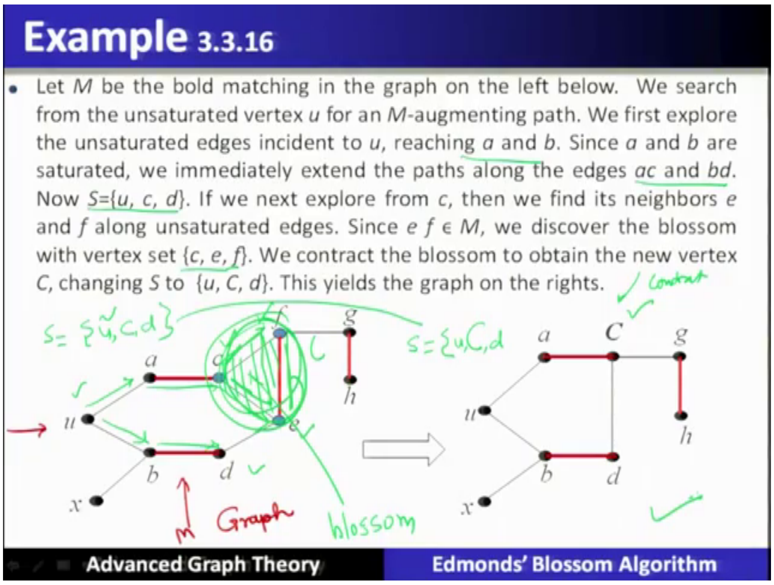

Which are shown in the red colour in the graph on the left side this is the graph, and the edges which are shown in the red colour they are in M. Now we search from unsaturated vertex u for an M augmenting path, we first explore the unsaturated edge incident to you there are two such edges.

So, reaching a and b using this particular edge you can reach, a and b since a and b are saturated because these edges in m they are touching up.

So, we immediately extend the path along ac and bd directly go up to this point that is not a problem. Now S will contain u, c and d ; that means, you have reached up to this point from c and d you have to go ahead and u means starting from unsaturated vertex. So, other vertices from where you are reaching c that has to be maintained somewhere else.

Now then if we next explore from c then we find its neighbour e this neighbour e and f along the unsaturated edges since ef is an M. So, we discover the blossom with the vertex at c ef c e f, why because you explore e you explore f and from f to e and e to f if you basically see which is present in S then a blossom is discovered.

So, this is a blossom, so blossom with a vertex at c e f has been discovered, because from two different parities we are basically touching either f or e. Now we contract the blossom contract means this entire thing is basically collapsed into one vertex this is called contract, the blossom to obtain a new vertex and that is called c and now you have to change this S.

So, that S will now contain u capital C and d because this will be C. So, this edge will connect to d this edge will connect to G, so this will yield the graph on the right side this particular graph we have already obtained.

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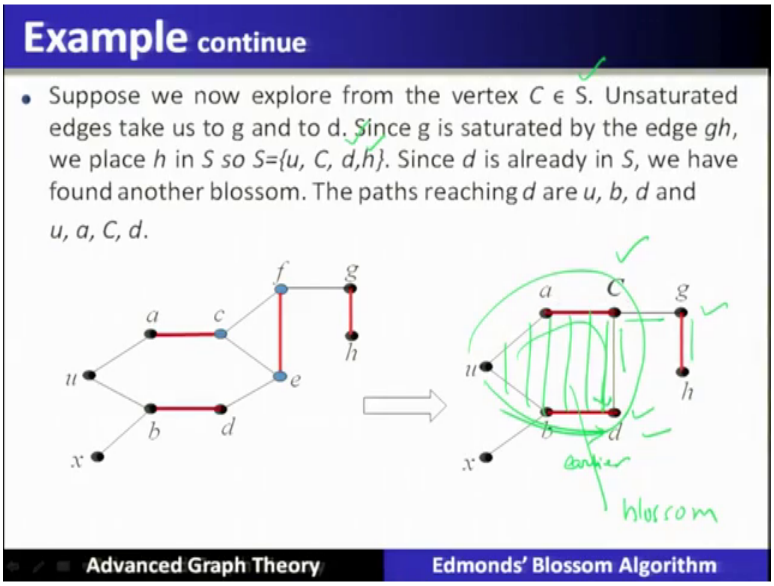

Now, suppose we now explore from the vertex c which is already present in S, unsaturated edges this and this take us to g a and to d simultaneously, we can reach here up to here using unsaturated edges since g is saturated by an edge g h we place h in S.

So, h is placed in S and also d because d we can reach. So, now, S will contain u c d h, since d is already there because we have visited d from the other side, we have found another blossom when you reach d by this way you have already visited earlier, then reaching d by different parity you will basically now identify another blossom. So, this blossom also has to be collapsed into one, so the path reaching d r u b d and u a cd. So, there that will be collapsed into a blossom.

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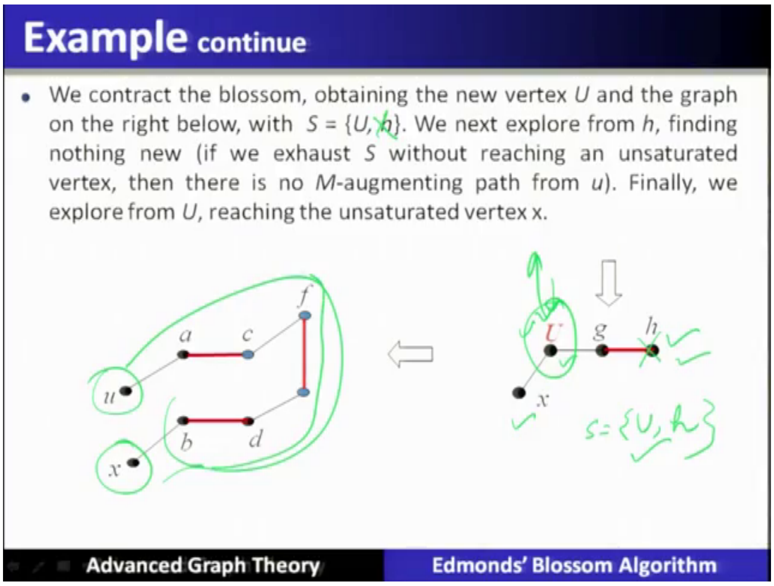

And we call it as capital U, so we contract the blossom obtain a new vertex U and the graph on the right this particular with S is equal to U and H. So, H U will reach from the other direction and u when you reach yrd.

So, we next explore from h, finding nothing new why because this is saturated. So, this is not augmenting path nothing is being identified through h finally, we explore from U reaching an unsaturated vertex. So, it will identify a path like this, so starting from unsaturated vertex you reach to unsaturated vertex now we have to trace back our path.

So, from x we have we are coming from u now we U we have to go and see within it and within U if you see you have reached from b d f a.

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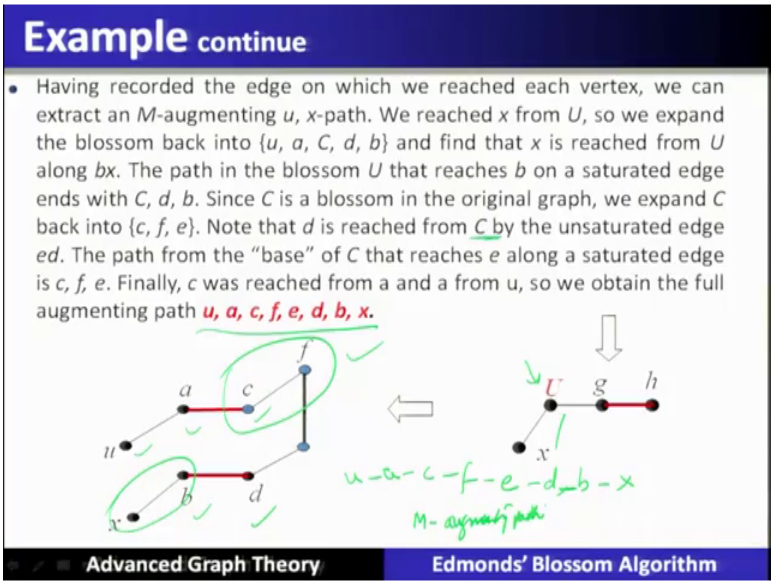

So, having recorded the edge on which we reached each edge we can extract the M augmenting path u x we reached x from u. So, we expand we reach x from u. So, we expand u, so when you expand u, so we expand the blossom back into u a then b d and c c was again and find that x is reached from u along b x, the path in the blossom u that reaches b on a saturated edge ends with c c d b.

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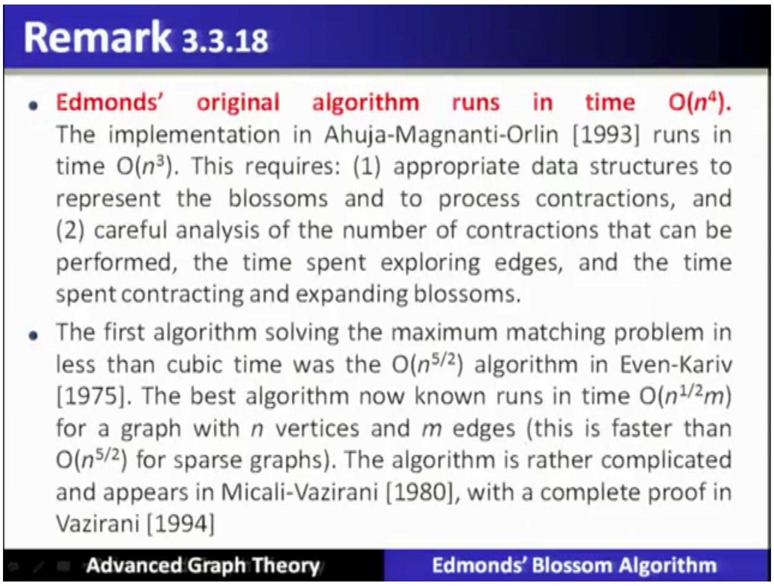

Since c is blossoming the original graph, so we expand c back to c f e e is not used here. So, we are not showing it note that d is reached from c from that blossom c by the unsaturated edge e d. So, the path from the base of c that reaches e along the saturated edge is c f e finally, c was reached from a and from u. So, we obtain the full augmenting path is like this.

So, we have started from u we have gone to a then we have gone to c then we have gone to f then we have gone to e then we have gone to d then we have gone to b then we have gone to x. So, this is M augmenting path is being identified remarks. So, Edmonds original algorithm runs in of the order n raised power 4, the implementation in Ahuja Magnanti Orlin algorithm runs in the order.

and cube that is faster way of implementing the same approach this requires appropriate data structure to represent the blossoms, and to process the contraction, and careful analysis of the number of contractions, that can be performed the time is spent exploring edges and the is time spent contracting and expanding blossoms are reduced in these algorithms.

So, the first algorithm solving the maximum matching problem in less than cubic time was of the order n raised power 5 by 2 that is 2.5 in Even Kariv in 1975, the best algorithm. Now known runs in the order and n root n m, for a graph with n vertices and m edges the algorithm is rather complicated and appears in Micali and Vazirani in 1980, and complete proof in Vazirani is given in 1994.

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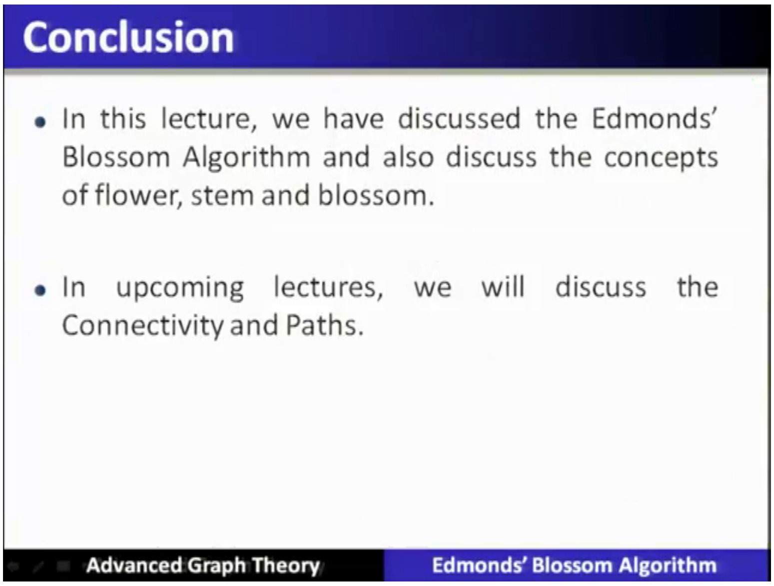

Conclusion in this lecture we have discussed Edmonds blossom algorithm and also discussed the concept of flower stem and blossom in upcoming lectures we will discuss the connectivity and paths.

Thank you.