Advanced Graph Theory
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Lecture - 11
Factors & Perfect Matching in General Graphs

Factors and perfect matching in general graphs.

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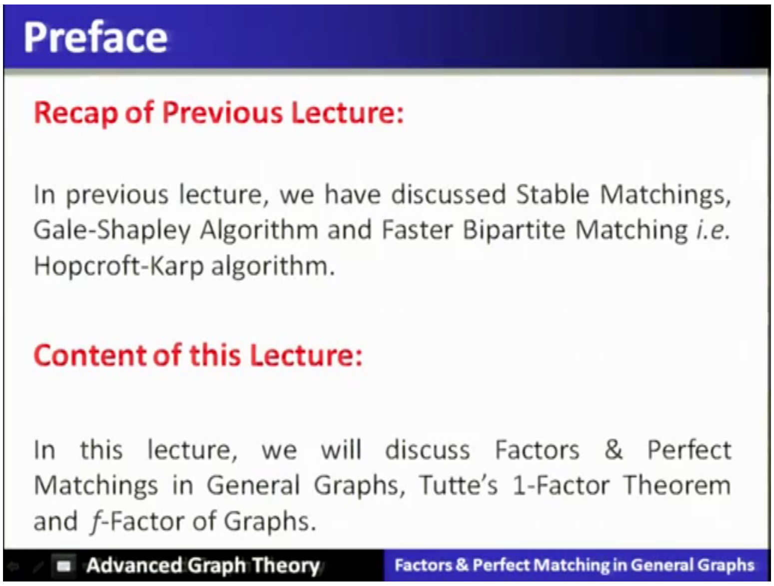

Recap of previous lecture we have discussed stable matching gale and Shapley algorithm and faster bipartite matching that is given by Hopcroft Karp algorithm, content of this lecture this lecture we will discuss factors and perfect matching in a general graph Tutte's 1-factor theorem and f factor of the graph.

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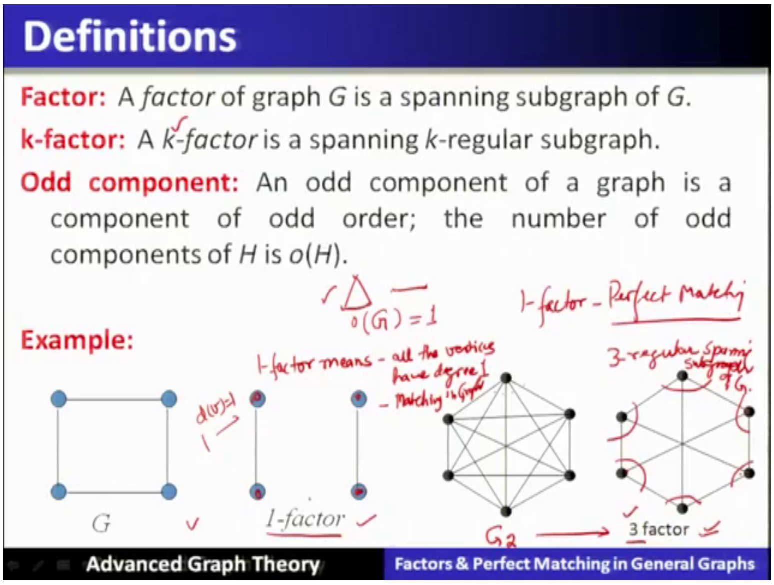

Factor a factor of a graph is a spanning sub graph of G. So, k-factor is a spanning k-regular sub graph of G, in our definition of an odd component an odd component of a graph is the component of odd order.

So, the number of odd components of H is of the order H, let us take an example, if let us say this is a graph having 2 components, then this particular component is an odd component, why? Because this it is having a odd order of this particular vertices in this component, and this particular component having 2 vertices this is even.

So, how many odd components are present here in this particular graph this is the odd component which is present that is only 1, similarity in this particular example see what is a 1-factor of a graph G. So, let us see that this particular graph which is given we want to find out a spanning sub graph. So, this is spanning sub graph will include all the vertices that is all 4 vertices, if it is a 1 factor; that means, each particular vertex should be having only degree of 1.

So, if you include the degree of 1, so this will be the sub graph of G, and hence it will be called 1 factor. So, if you recall this 1-factor is nothing, but a matching in this particular graph. Similarly, another graph let us take G 2 in this particular graph, if you want to find out the 3 factor of this particular graph. So, 3 factor means we want to find out a spanning sub graph. So, a spanning sub graph will include all the vertices present in the original graph. And so we are going to find out this particular spanning sub graph, which is basically a 3 regular that is called 3 factors. So, 3 regular means all the vertices are having 3 edges out of 5.

So, it can pick any 3 of them, so 3 vertices 3 edges are present similarly this vertex will have its degree 3, will have its degree 3 and so on. So, this becomes a 3 regular spanning sub graph. So, 3 regular spanning sub graph of G is nothing, but called as a 3 factor similarly we can extend it and we can generalize it to k factor. So, when we say 1-factor than we mean that we are talking about a perfect matching, matching in the graph let us go and see the clarification.

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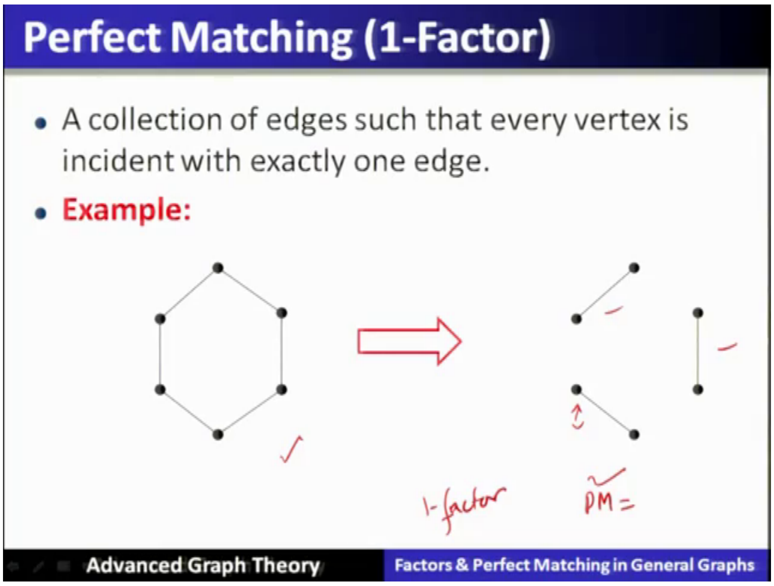

So, a perfect matching that is also we are referring it to as 1-factor a collection of edges such that every vertex is incident with exactly 1 edge. So, in this particular example, this is the perfect matching is a collection of these particular edges. So, that every vertex is incident with only 1 edge so that becomes a perfect matching and; if you take 1-factor that also comes out to be the same thing.

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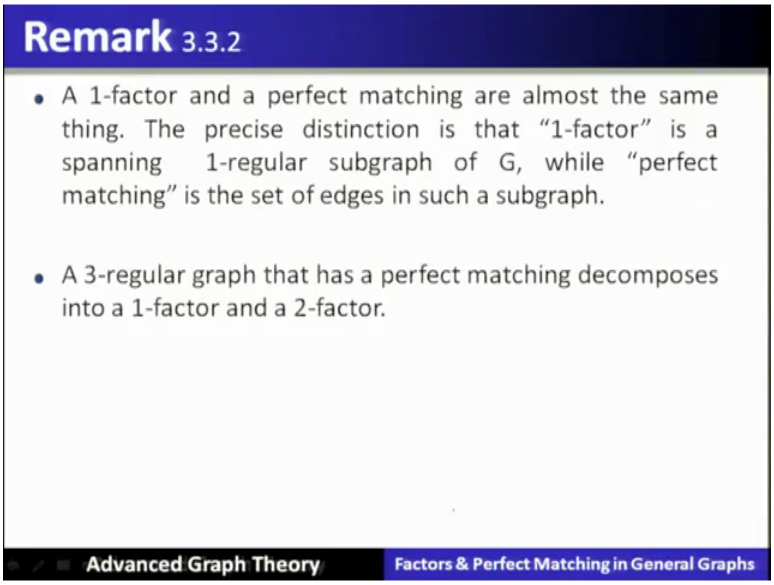

So, 1-factor and perfect matching are almost the same thing the precise distinction is that 1-factor is, spanning 1 regular sub graph of G while perfect matching is the set of edges in such a sub graph. So, there are two different definitions, but they mean to the same thing. So, 3 regular a graph that has a perfect matching decomposes it into 1-factor and 2 factors.

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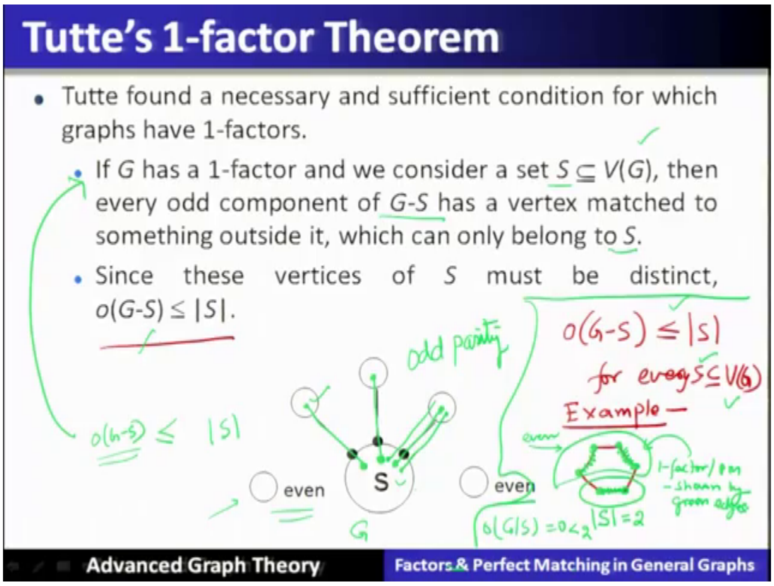

Now we will see the Tutte's 1-factor theorem, so what is that condition in which we can find out perfect matching in the graph, whether perfect matching in the graph exists that or and then we can also say that whether 1-factor exist in a graph which is nothing, but a perfect matching. So, this particular question can be verified using the Tutte's 1-factor theorem, so some of the references also called as a Tutte's perfect matching theorem. So, either we can call it as a 1-factor theorem or perfect matching theorems meaning thereby both these particular things are same.

So, Tutte found that the necessary condition for having 1-factor or a perfect matching in a general graph. So, which says that if G has 1-factor then for every subset of this vertex set the number of odd components, if you remove this S subset from the graph will have the vertices equal to that particular size of S.

So, we can basically denote it by this particular condition, so; that means, the number of odd components if in G minus S will be at most the size of S for every subset S of vertices of G take the example for example, we can see this particular graph which is having a perfect matching let us see this particular edge will saturate these 2 vertices, this particular edge of the graph will saturate these 2 vertices this edge will saturate. So, 1-factor or perfect matching is shown by the green edges.

So, this particular graph has the perfect matching, now let us see whether we can see through this particular condition that is can we prove it by the Tutte's condition. So, let us take that these 2 edges belong to S this is subset of vertices, these 2 vertices. So, this size of S is 2 here in this case the remaining portion when it is removed this becomes 1 component having 4 different vertices. So, the parity of this component is even, so how many odd components will be there in G minus S 0. So, 0 is less than 2 hence this particular condition holds and this condition holds we have all already seen that it has a perfect matching or a 1 factor.

So, hence this particular necessary condition which is stated by Tutte for 1-factor in a graph, he has also proved it to be a sufficient condition that we are going to see the proof is catch. Now before we goead, we see that a important observation let us say that if this is the graph and this is the set S. So, if it is removed from the graph it will be in the form of the components.

So, some components are basically having even parity, let us say the other components are having an odd parity. Now the components having odd parity will have at least one vertices present, which has an edge and that particular as well incident on a vertex of S in a distinct vertex, similarly another component which is an odd parity will also incident on a vertex of S in a distinct vertices similarly here also it may have more than 1 also, so at least one such component exist.

So; that means, 1 vertex from, from every odd component will go and will basically have an edge which will basically incident on a. So, the size of S will have these many number of components that is why the number of odd components in after removing S will have at least that many number of or S number of elements present, why? Because for every component there will be 1 element plus some more element can be there in S. So, at least that many number of, so let us see the condition where it will violate. So, the condition where it will violate we have to see in the further examples. So, let us see that the same thing is stated over here that if G has a 1-factor than we consider a set as which is a subset of the vertices of G.

Then every odd component of G minus S has a vertex matched to something outside it and which is nothing, but that belongs to set S since these vertices of S must be distinct therefore, this particular order the number of odd components of G minus S will be bounded by the number of components present in this particular S.

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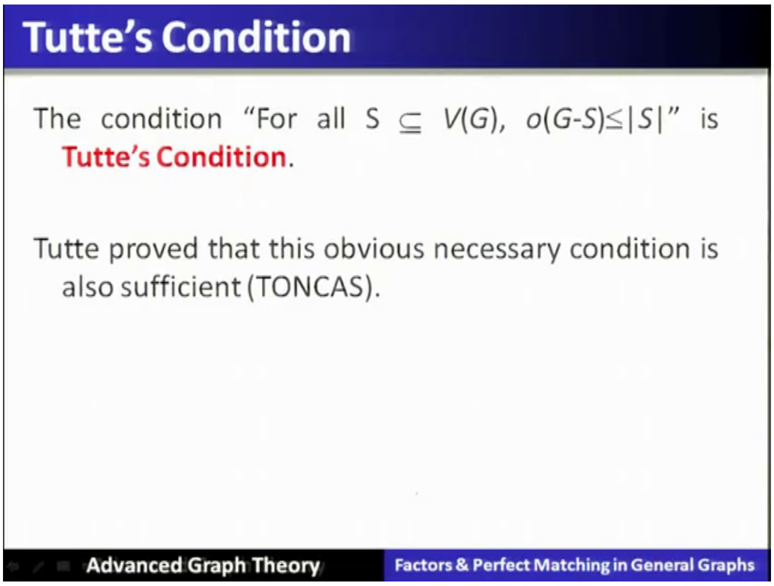

This particular condition which is called Tutte's condition and this particular necessary condition is also sufficient that is being proved here in the.

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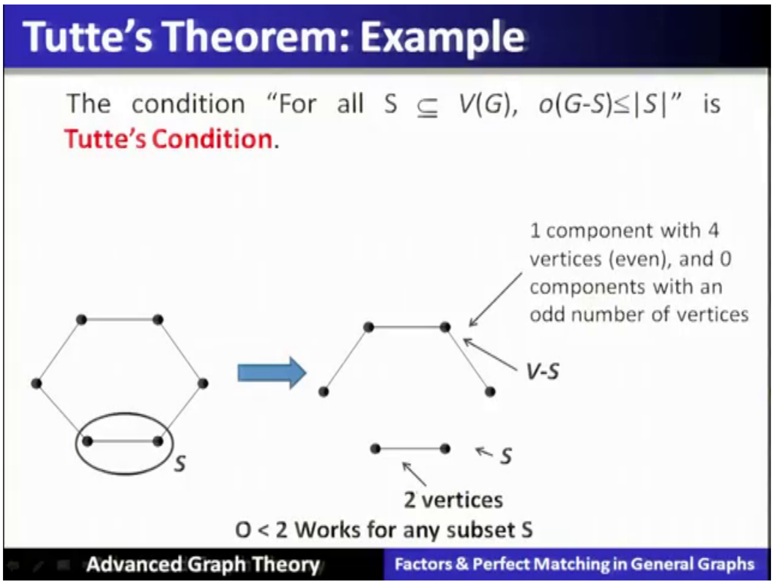

Tutte's theorem this particular example I have already stated.

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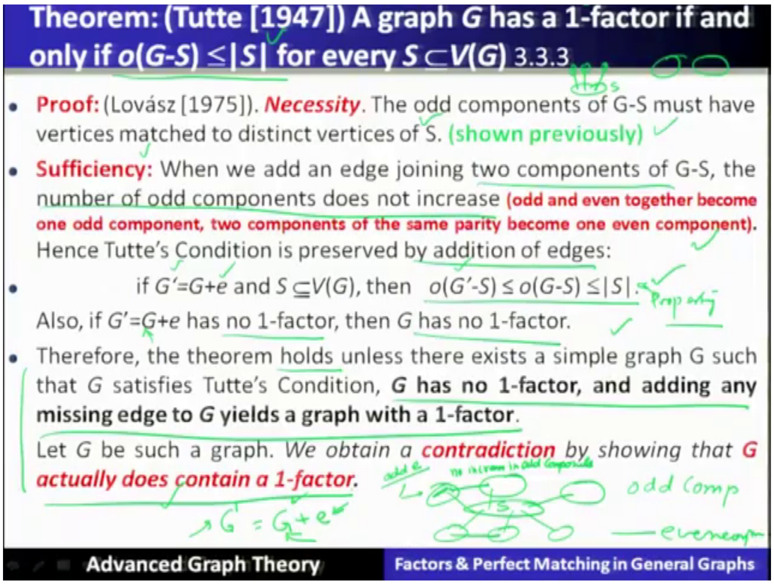

So, let us as keep it and look into the proof of Tutte's theorem. So, necessary condition we have already told you that if the graph has the 1 factor; that means, the number of odd components will have the vertices which will go into S.

So, the size of S will be at least the number of odd components, hence this particular necessary condition is proved. Now, let us go for looking up a more detailed sufficient condition. So, if let us say that in sufficient condition if the number of odd component or this particular property exists, then we have to prove that it has the 1-factor.

Now, we will prove it by contradiction let us see from where we at what point we have to basically start our contradiction to prove the sufficiency condition of this particular theorem, now when we add an edge joining any 2 component. So, for example, these are the components let us say they are the odd components and these are the components which are called even components and this is S joining.

Now, when we add an edge joining any 2 components of G minus S, so if you remove S, so there will be a distinct components. So, if we add an edge within 1 component of an odd parity the parity of this particular component is not going to change, if we add an edge this particular parity is not going to change similarly if we add a edge across 2 odd components.

So, odd and odd will become even, so number of odd components will not increase, similarly here if 2 even components are basically joined by an edge. So, it will become a bigger component of, but even, so in any case when at when an edge is added. So, the number of odd components will not increase, so that i have already explained.

So, that Tutte's condition is preserved by adding of the edges, let us assume that an edge e is added sufficiently into the graph and we obtained a G prime graph. Than that particular condition that is the odd vertices in the new graph G minus S will be having that particular same value that is the number of odd components is basically this particular equation or this particular property is not affected.

So, that is basically sure, so if we add an edge and we see that if particular a graph G prime has no factor then the original graph also has no factor; that means, the add addition of the edges into a particular graph it will not basically make that graph a perfect matching therefore, this particular theorem holds unless there exists a simple graph G such that G satisfies Tutte's condition. Now, G has no 1-factor and adding any missing edge in G will yield a graph a 1-factor. So, let G be such a graph and we obtain a contradiction by showing that G actually does contain a 1 factor.

So, what we are going to prove in the sufficiency condition is that if a graph is given G, and if we add sufficient number of edges we form a G prime a new graph. So, if G prime do not have the 1-factor then by adding this particular edge into G. So, G also should not have the 1-factor that we have seen, but we differ from here and this will be our contradiction.

So, our contradiction will assume that in the graph G if we add an edge we will make another graph G prime, and this we will show that we will assume that this particular G prime will have the 1-factor and the contradiction to this contradiction, contradicting this particular statement will prove that the sufficiency condition and hence this particular graph satisfying this particular property will have 1-factor, that is; what we are going to prove? So, let us prove that by adding this particular edge, the graph original graph G is not having the 1 factor.

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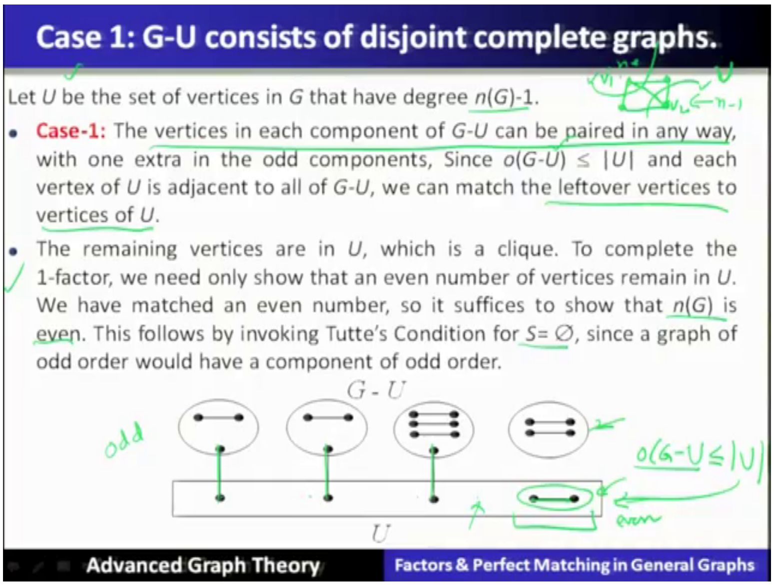

And thus it is going to do the contradiction, so case one, so let us assume that U be a set of vertices in graph G, that have the degree n minus 1; that means, in a graph we will find out let us take this particular example of a graph this particular graph has 2 vertices this particular vertex 1 and vertex 2 its degree is n minus 1; that means, total value of n is 4.

So, 3 are present here 3 are present here, so it is basically n minus 1, and this will also n minus. So, we are collecting up such vertices and let call it as U. So, this also will be part of U these vertices will be part of U other vertices are not in a part of U. So, let us see the case 1 the vertices in each component of G minus U.

So, if you remove U from the G we will have this set of vertices, so the vertices in each component of G minus U can be paired in any way with 1 extra in the odd component. Meaning to say that, when we remove U from the graph it will be in the different components, the components having the odd parity we will find out a vertex and try to match a particular vertex in U that we have earlier also stated.

The leftover vertices of U here, the leftover vertices for example, every other vertex of U will be having a matching in that odd component this is even component. So, there is no such vertex present here in U. Now the vertices which are leftover in U; that means, the number of odd components is always less than S here we call it as U.

Since number of components are more ; that means, the number of odd vertices it will match and then the remaining ones it is talking about the remaining ones the difference and they can be paired with each other. So, the remaining vertices are in U which is a click. So, you see that that is why this clicks are there. So, if it is 2 vertices then there will be a 1-factor to complete 1-factor we need only show that an even number of vertices remains in U.

So, these set of vertices which remains is basically an even number. So, even number can be included in the 1 factor, so we have matched an even number. So, it suffices to show that n G is even this follows by invoking the Tutte's condition for S is equal to empty, since the graph of odd order would have the component of odd order.

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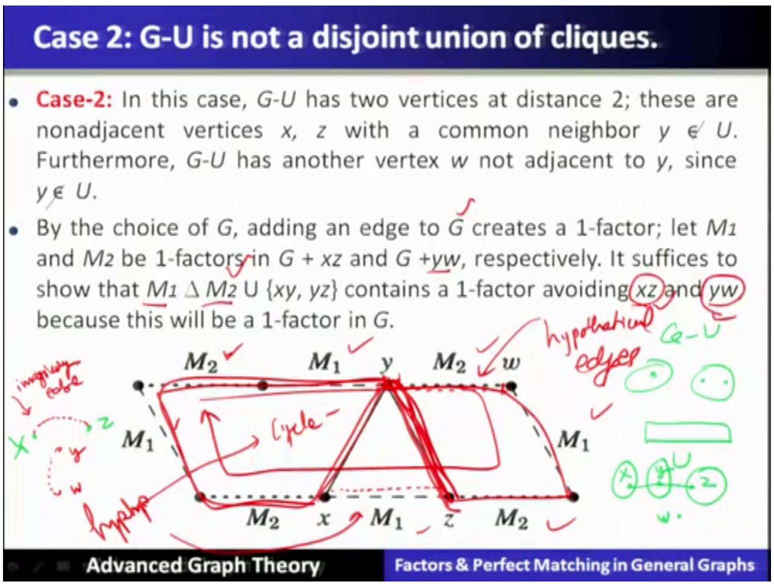

Now, let us see the case when G minus U is not the distinct union of clicks. So, the previous condition says that G minus U is a distinct complete graph, that is the clicks and we have done this particular 1 perfect matching.

Now, if these distinct vertices in U G minus U; is not disjoint union of so what will happen then in that case. So, in case 2 in this case G minus U has the 2 vertices, so G minus U, so this is U and, so this becomes G minus U these are the components has 2 vertices at a distance 2. These are non-adjacent vertices x and z and a common neighbour y and y is not in U. So, if it is not in U then it must be in some component.

So, G and x and z they are not basically having a direct edge, and they are connected through let us say they have a common neighbour y, and this y is not in U furthermore G minus U has another vertex w not adjacent to y since y is not in U.

So, we now are getting x z which is not directly connected y is there and w is also there. So, y is not in U outside U no by the choice of G adding an edge to G will create a 1 factor, let M 1 and M 2; be the 1-factor of G plus x z. So, x z they are not directly connected. So, we hypothetically we are placing an edge and we call it as x z since x and z they are not directly connected. So, we are placing a hypothetical edge, imaginary edge, similarly we are also adding y w this also a hypothetical edge we are adding up.

These 2 edges we have added, so which is shown over here let us say this edge is hypothetically added, and this edge is hypothetically added, in to G. So, it suffices to show that M, M 1 is a matching, M 1 is 1 factor, and M 2 is 1-factor union these 2 edges will contain 1 factor. And we can avoid x z and y w because this will be 1-factor in the G.

So, let us see when we take the symmetric difference of 2 matching's, or symmetric difference of 2 1-factor you know that it will become a graph with a vertex degrees either 0 or 2, 2 means it is a cycle. So, this particular cycle M 1 followed by M 2, M 1 followed by M 2 followed by M 1 followed by M 2, M 1 M 2, M 1 and M 2.

So, this becomes basically a cycle, now the question is we have added these 2 hypothetical edges, Now if you can show that this particular cycle is still we can get without these edges then this cycle will be the sufficient to show that it is having the 1-factor in the graph. So, let us assume that we can still get a cycle; that means, if we start from somehow reach to y, and y if you reach to z from z to y and then we can take around and when we reach to x.

We can come back again and take around and so on. So, either we can avoid x z or we can avoid y w in this particular way. So, when we reach x then basically we can reach to z also and so on. So, in this way we have obtained a cycle without even these hypothetical edges that we are going to see.

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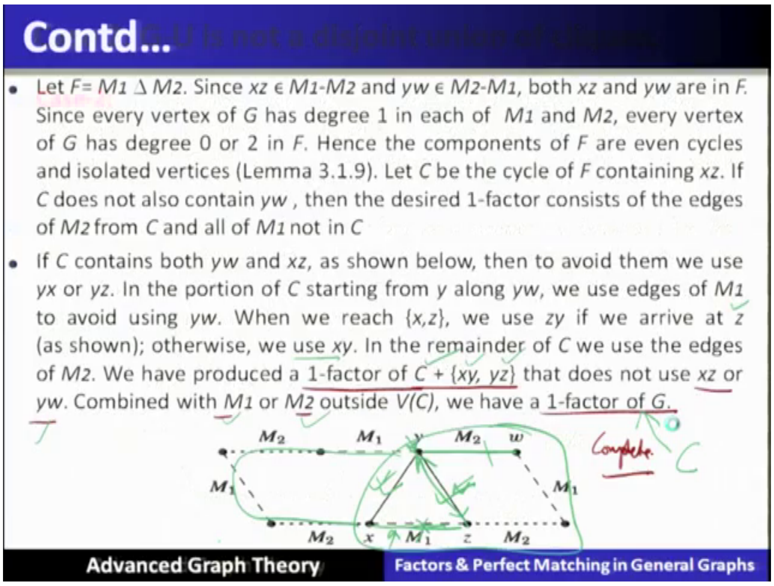

So, here we have shown that we have produced a 1-factor of this particular cycle C including x y and y x that does not use x z or y w. So, combined with M 1 and M 2 outside this particular cycle we have 1-factor we have obtained a 1-factor and we are using these edges, but we are not using these 2 edges. So, still we can complete a cycle.

So, let me again state this if C contains C means the cycle, if cycle contains both y w and x z as shown below. It is containing at this particular present then to avoid them we use y x or y z any of these 2. So, in the portion of the C starting from y along y w we use the edges of M 1 to avoid y w, when we reach x z at this end, when we reach x z we use z y to arrive at z as shown otherwise we can use x y.

So, in the remainder of that particular cycle C we use the edges of M 2. So, we have produced; that means, if we look this part. So, if you are you removing M 2 than basically we are basically using this particular edge and if we are removing M 1 then we are using this edge still we can complete the cycle without them.

So, we have produced a 1-factor of C plus x y plus y z that does not use x z or y w. So, instead of x z and y w we are using x y and y z, so combined with M 1 and M 2 outside we have got we can achieve the 1-factor of a particular graph so; that means, by adding an edge if the original graph does not contain 1 factor, but we have reached to a contradiction which is shown that it is having 1 factor, thereby we are contradicting the original hypothesis this contradiction of a contradiction will prove that the sufficiency condition is satisfied of the Tutte's condition.

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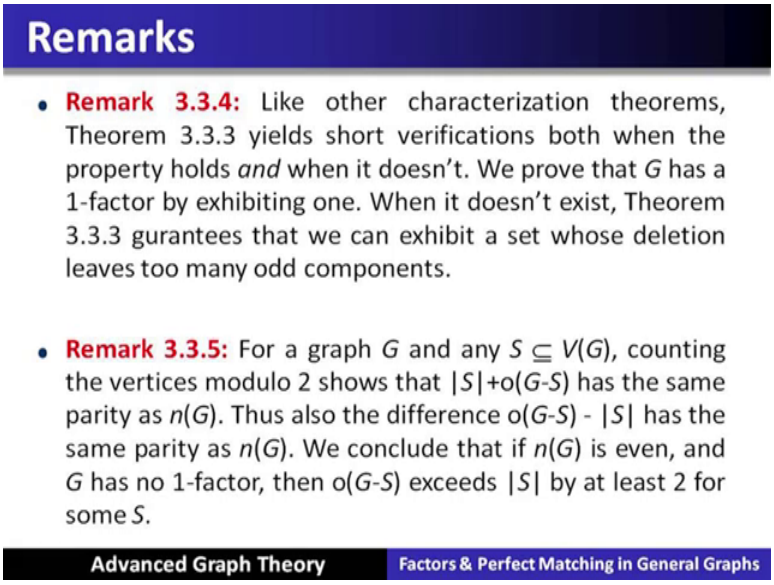

Remarks like other characterization algorithms this algorithm will yield the short verification both when the property holds and when it does not.

So, we have shown when the property holds there exist a 1 factor, we will show when the property does not hold still that particular characterization is valid. We prove that G has 1-factor by exhibiting 1, when it does not then we have to find out a subset of S that if we remove it will leave too many odd components. Hence, it will violate the Tutte's condition and hence we can state that the graph does not have 1 factor, further more for any graph G if you take a subset S of vertices and counting the vertices modulo 2 will show that the cardinality of S plus the number of odd component has a same parity as n.

If you take the difference then also it is has, has the same parity as n. So, we conclude that if n is even and G has no 1-factor then the number of odd components exceed by S odd components exceed S by at least 2 for some S. So, in this way if there is no 1-factor then we can also estimate about the size of the maximum matching by taking this particular difference.

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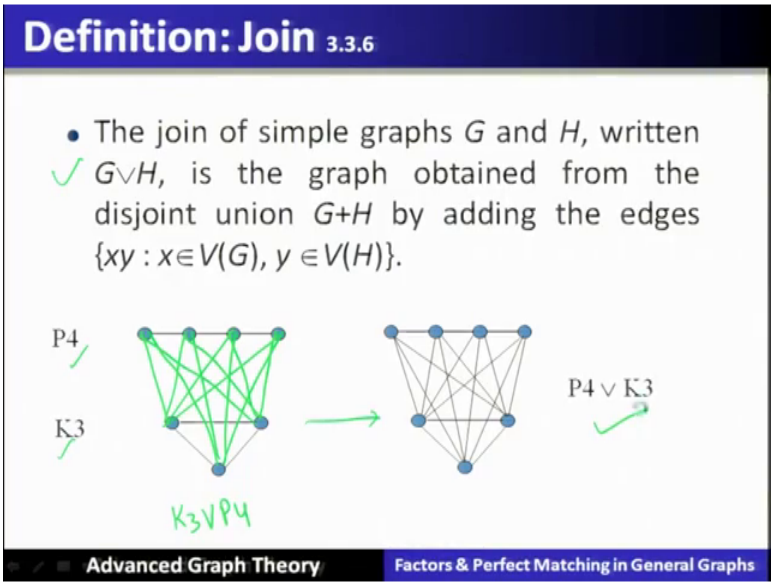

So, again we are going to give another definition of a join of 2 simple graph G and H, which is written as G join H is the graph which is obtained by the disjoint union of G plus H by adding the edges, to both x edges x y such that x belongs to a particular vertex in G and y belongs to particular vertex in H, so for all such vertices of G there will be an edge to the vertices of H.

Similarly, for every vertex of H will have an edge to the vertices of G, so that gives so let us take this particular example. So, if you are given k 3 and another graph is given p 4. So, k 3 join p 4 will look like this particular vertex will have the edge to all other vertices, this also will have the vertices to all other vertices this also has the same thing and this basically will result into this particular a graph that is the join of 2 simple graph is a graph that is shown over here.

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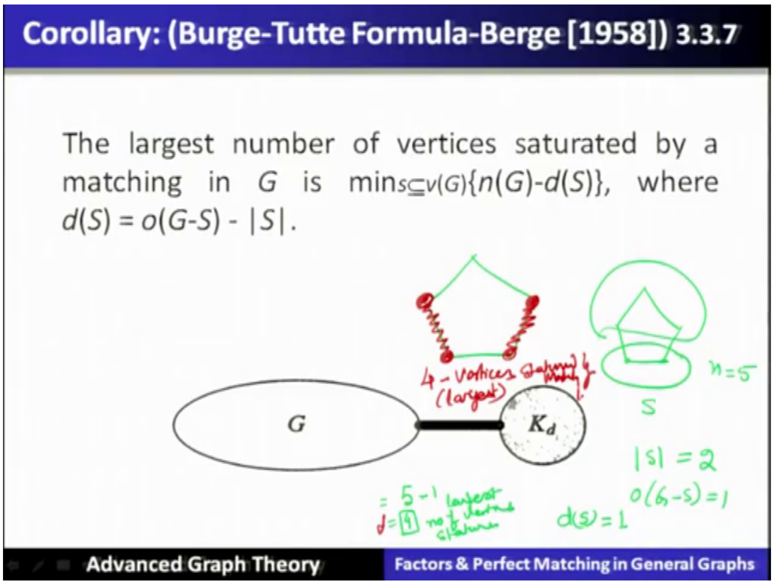

Now, there is a corollary given by Burge-Tutte formula. So, that corollary says that the largest number of vertices saturated by a matching in G is the minimum of the subsets of vertices such that n minus d S, where d S is the number of odd components in G minus S minus the cardinality of S itself, that I have earlier explained, but let us take an example and show that.

So, you know that this particular graph is not having the perfect matching. So, we have to find out the largest number of vertices which can be saturated by a matching in G. So, we are going to take S let us say this is S. So, the cardinality of S is 2, and the number of odd component this is 1. So, the d S if we compute this will be 2 minus 1 and 1 and if you plug in this particular d S value.

So, the total number of vertices is 5, 5 minus 1 this comes out to be 4; that means, 4 is basically the number of vertices, which are the large this is the largest in size, which can be saturated by any given matching let us see since this particular graph is not having 1 factor.

So, we have to find out that particular matching, which will saturate the largest number of vertices this is vertex number 1, 2, 3 and 4, and here using this particular formula also we have computed that 4 vertices this is the largest value of this vertices are being saturated by the matching.

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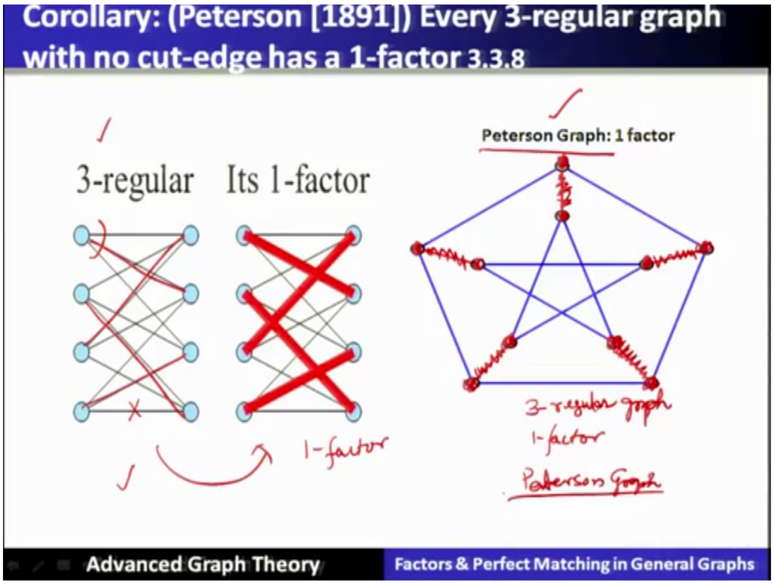

So, this particular formula can be utilized, if the graph does not have 1-factor then it can identify how many what is basically the maximum or the largest number of vertices which can be saturated by the matching. Now there is another corollary which says that every 3 regular graph with no cut edge has 1 factor. So, let us take this particular graph. So, this particular graph is a 3 regular graph, 3 regular means every vertex or every node has the degree 3. Now this particular graph is not having any cut edge so; that means, if you remove this edge the entire graph is having 1 component.

So, there is no cut edge present, so this particular graph has a matching a 1-factor and this particular 1-factor is shown over here, this is 1 such matching this particular edge will have another matching this edge will have another matching and this particular matching. So, this particular graph has 1-factor now another important graph is called a Peterson graph. So, Peterson graph is also 3 regular graphs; that means, all the vertices having the degree 3 this also has a 1 factor. So, if we include these edges as a part of the matching, then you can see all the vertices of this particular graph is saturated and this condition is called perfect matching or a 1 factor. So, this is a 3 regular graph and this is a important the name of this graph is Peterson graph, so Peterson graph has a 1 factor.

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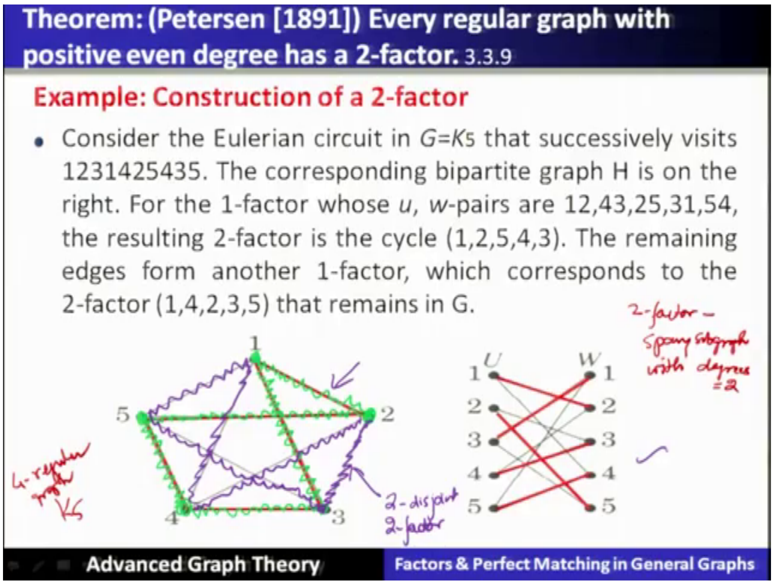

Now, there is a theorem which says that every regular graph with positive even degrees has 2 factors. So, let us consider this pentagon k 5 it is also called, so this is a regular graph with positive even degrees. So, the degree is 1, 2, 3, 4, so, all the vertices are having degree 4 it is a 4 regular graph now this 4 regular graph has the 2 factor, what is 2 factor 2 factor is a spanning sub graph with the degrees equal to 2. So, let us see in this particular already shown about the 2 factor, so let us start from vertex number 1.

We can reach vertex number 2, from 2 we can reach to 5 from 5 we can reach to 4, from 4 we can reach to 3, from 3 we can reach to 1, similarly we can also identify this is 2 factor. Another 2 factor also we can identify let us begin from we can reach to 2 we can reach to 4, we can reach to 1, we can reach to 5 we can reach to, to 3 back again. So, this is another 2 factor. So, this is this particular graph has 2 disjoint 2 factors present in the graph.

So, it has also 1-factor so; that means, 1-factor plus 2 factors also you can identify and that is shown here this particular graph. So, again i am repeating the theorem that every regular graph with positive even degrees has 2 factors, now let us see the definition of a factor.

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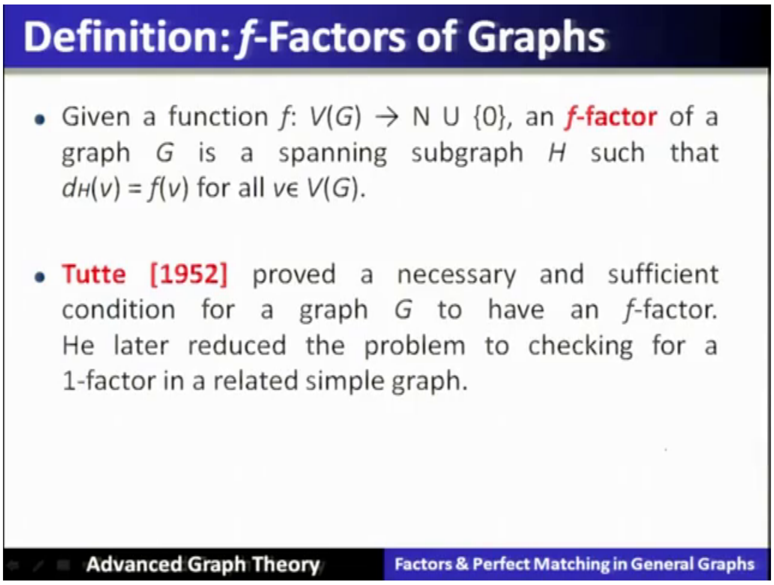

Given a function f which maps from vertices to a number n, which also includes 0 besides natural numbers an f factor of a graph is a spanning sub graph H such that the degree of the vertices is basically f, f v for all vertices and f is a number which is called a factor of a graph. So, Tutte proved in necessary and sufficient condition for a graph to have a factor he later reduce the problem to checking for 1-factor in a related simple graph conclusion.

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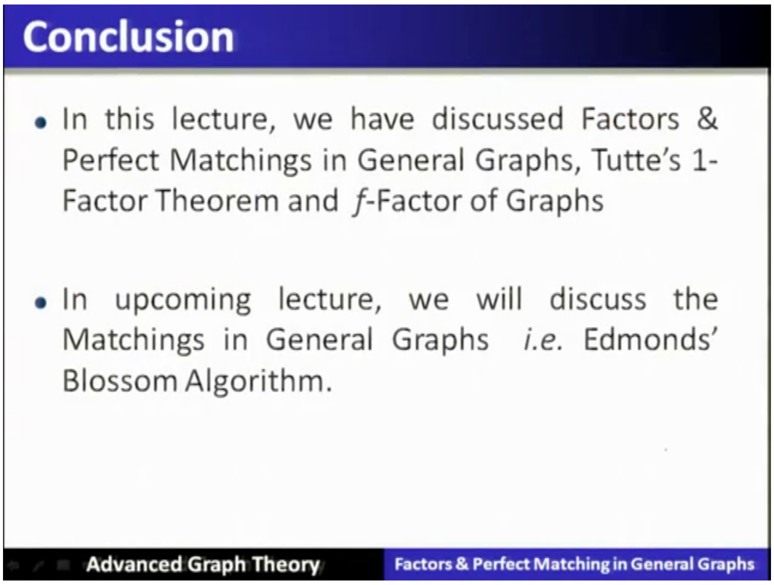

In this lecture we have discussed factors and perfect matching in general graphs, Tutte's 1-factor theorem and f factor of a graph in upcoming lecture. We will discuss the matching in a general graph which is given by Edmonds blossom algorithm.

Thank you.