Riemann Hypothesis and its Applications
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Lecture – 9

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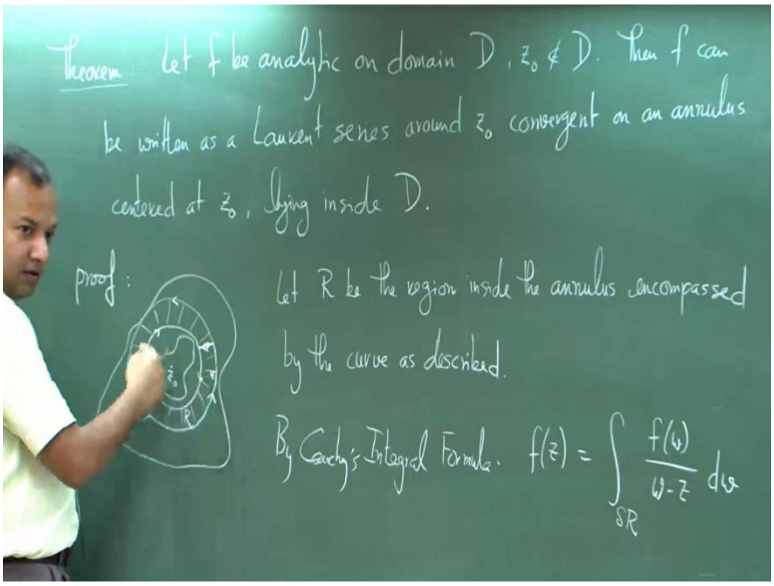

So, this is the theorem we wanted to prove last time. So, the idea is very simple this is lithe. So, this is the points are naught, and we create an analysis which should entirely lie 100 anything 0 that analysis must lie entirely inside that not, otherwise it will not make any sense to have to the f may not even be define there. So, let us pick up an annulus which is this one. So, this is the domain dif and this annulus that I am picking up, this is centered at z naught inside D.

And this I will cut by making a single slicer. So, that it becomes a single curve the boundary of this annulus, and I travels this boundary counter clock wise starting from outside going here, then coming here, then going down here. So, this is clock wise then going back on this, and then continuing counter clock wise completing this circuit. And we use the standard trick that at this point inside of running over the same cut twice, you just separate them by very tiny amount, so that it is this traversal is very well defined everywhere. I let us call this region, where is the region which is encompass by this curve R.

Now, f is completely analytic inside this region forget about completely, f is analytic inside this region. Therefore, by kashi integral formula, if there is a point let say any point z here, I can write f of z as the integration around the boundary of R of f (w) divided by w minus z later. Now, let us look at the boundary is this, this, this and this. Now, is this two parts tend towards each other, the integral around those parts will cancel each other out. So, what will be left with the limit is integral counter clock wise on this, and integral clock wise on this inside circle. So, let us, so this are board circle centered at z naught, let say this is a beta this distance and this distance is alpha.

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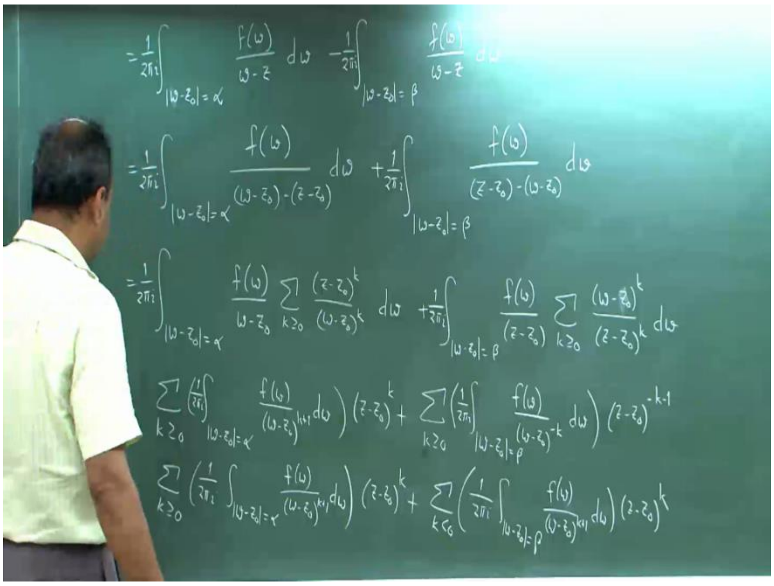

So, then I can rewrite this as. Now, the first part is familiar, here the integral is on w (s) which are on the outside circle and z is inside, that z is closer to z naught then this w(s) are. So, let us just rewrite this a bit with I want everything in terms of are expand around z naught. Now, w minus z naught has w moves around this circle is larger in absolute value then z minus z naught, so this is (( )) or familiar thing that we did for power series also. So, therefore I can rewrite this as having this integral w is running over the inside circle, so absolute value of w minus z naught is smaller than z minus z naught.

So, I can expand this, but taking out z minus z naught and then doing it. And now, we can here is the uniform conversion of this power series in the region of interest to swap the integral with summation, and we get summation k greater than equal to 0 integral of I got I forgot one over 2 pi i here, so there is a one over 2 pi i everywhere, have if you just change this summation hall by replacing the index k with minus 1 minus k. So, this re defines k to be minus 1 minus k. So, this becomes k this becomes k plus one, and then you get there is a bit of problem here, because this two integral are over different circles. So, I cannot write at exactly write this.

So, I would be still go with the split definition. So, we are almost there while actually we are there, because now this is a Laurent series. We also have the value of coefficient of the Laurent series that it is for the on the positive side it is this, integrals on a negative side it is this integrals. So, that completes the proof of the theorem, but actually we can do one step better, and make these two integrals over the same circle, do you see the reason?

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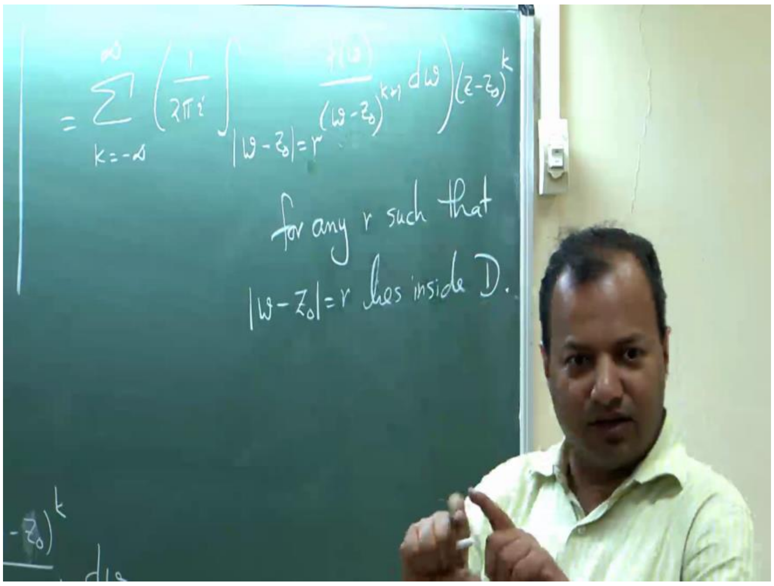

In fact, let me stay with this ((Refer Time: 14:41)). Let me just write it as the equal to sum k going to minus infinity to infinity 1 over 2 phi i integral, I can replace this integral over circles of radius alpha or beta by a circle of radius r has long way that circle lies inside the analysis not inside the region, the domain D why.

Student: ((Refer Time: 15:54))

Exactly, this function that you integrating f (w) o w minus z naught to be k plus 1 is analytic inside that annulus, because f is analytic this is analytic, why this analytic? Because it never vanishes, and it is involves of an analytic function w minus z naught absolute value is always non zero, so this is so the product of to analytic function analytic, so this is analytic. And says that analytic inside that annulus is integral over any circle inside the analysis is going to be the same, that is why kashi. And so we have a uniform definition of the constants of this expansion.

Student: ((Refer Time: 17:10))

Settle around it not yes.

Student: ((Refer Time: 17:13))

Yes it is define.

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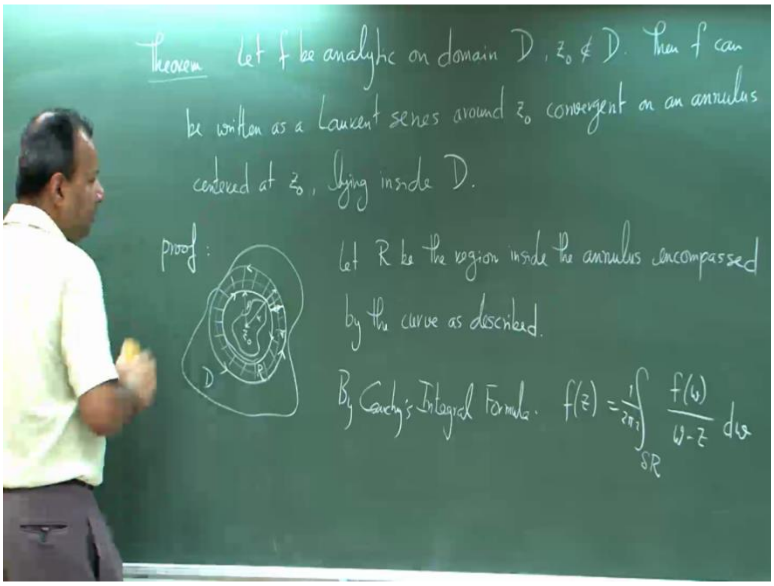

So, it is your mean, so we are if you integrate around this circle, on this circle the function is well define. And inside this, this and this and look at this annulus, inside this annulus function is analytic the complete that at f (w), but f (w) divided by w minus z naught to the k plus 1 is analytic. So, if you cut make it cut here, and look at that region the integral around the boundary of this by Cauchy the integral 0. And therefore the two integrals are same, of course the same thing does not apply here, because w minus z naught can become 0 in this integral or not this not in this integral.

But inside the region inside this disk, where we are doing the integral it can become 0 or here inside this region, the delta R it can become 0. So, you cannot this is not 0, this is non zero. So, that is the Laurent series expansion of an analytic function around singularity, at expansion make sense only around a singularity, we call it the point z naught is not a singularity. Then is expansion is simply a power series, and this expansion is define on the annulus, which lie inside the domain.

And this the inner circle we should we can shrink to as small as possible, and outer circle we can expand to as large as possible that does not matter, so that ((Refer Time: 19:04)). So, this keeps us a more general way of expressing an analytic function, especially it allows us to steady the singularities in a much nicer way. The power series always avoided singularities, whereas this actually use as a singularity to expand the function around it. So, we can study the behavior of an analytic function around a singularity using Laurent series, we cannot do that using power series.

And that is something there is going to be very, very important for us, because singularities of an analytic function are going to play a very important role. In fact you already see, this is an analytic function within one singularity had w equals z. And you already note that it has a very interesting role to play, there it actually guess me a value of f at z. So, we will see as move along more and more usages of analytic function around singularity, but first let us address this singularity them self a bit.

So, we saw 0s of analytic function, we the most important thing about them was at 0s are isolated, and every 0 has an associated order with it, that is the order of the first few term that first term number of term that as 0 in power series expansion, let see order of this way. Can we say same thing about singularities, can we say that singularity is of an analytic function or isolated make a guess.

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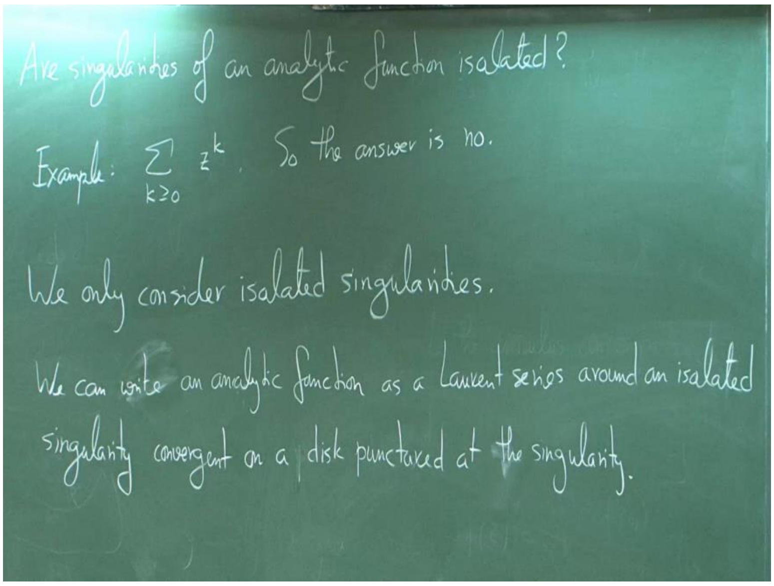

Answer is very simple.

Student: ((Refer Time: 21:55))

Notes you when to you say they are isolated.

Student: ((Refer Time: 22:05))

That is what you have to answer is, there I think not.

Student: ((Refer Time: 22:23))

So, analytic function of course is there is a domain associated where it is analytic, the rest of it. So, preservly there are it is may or may not be defined, sees possible that there analytic function can be a continued more in a larger domain. So, whatever it can be continue more and more and more, whatever maximum pass will continuation is there, after that what is left is places where it cannot be analytically continued, which means that these are singularities, so are those singularities isolated. Well, let us take an example to analytic function this is analytic inside your disk, what about singularities?

Student: ((Refer Time: 23:20))

The viewer after that this completely singular behavior, there is nothing outside there. So, this immediately shows that singularities of analytic function need not be isolated.

Now, when the singularities are not isolated those are not very interesting cases for us, because that is a complete mesh ((Refer Time: 24:02) typically not even going to be interested in those region. So, we will only been interested in those singularities which are isolated, because those are the interesting singularities often little more. So, form hence forth we will restrict or attention to isolated singularities.

We understand what an isolated singularities, is singularity so there is a smaller of neighbor it around it has no other singularity fine. Now, isolated singularities also have a very nice property that we can write a nice Laurent series expansion of the function, around that singularity in a small region. And the analysis just the puncture disk, weather we only want singularity which is right now sitting there you as puncture that. And that analysis is the Laurent series expansion of the function, which means that we can study the behavior of that through that Laurent series.

We can study the behavior of the analytic function around that singularity completely. We know the exactly how it is value behaves around that singularity. So, we can let me is write it down, because it important we can here write. And will use this observation to study the behavior of the not only analytic function, but also because we can study it is it around the singularity, we will able to say something about that singularity itself.

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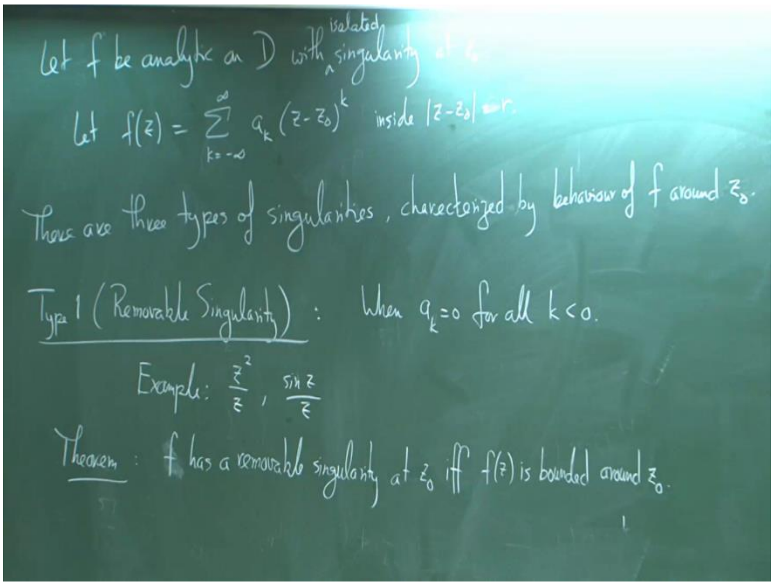

So, let f be analytic on D with isolated singularity at z naught, now using the previous observation lets write f s. Let say some disk punctured at z naught ((Refer Time: 28:07))

Now, when I say inside you take it to me in that it is punctured at z naught. So, what can we say about the singularities, well it turns out there are going to be three types of singularities, which I am going to describe now. And these singularities characterize or characterized by we are of z be aware of f around z naught, try it one is called removable singularity this is actual not a singularity.

This is a phony singularity this have occurs when in the Laurent series expansion the all the negative coefficients are 0, so actually Laurent series are power series. So, actually there is no singularity there, so then why do it even call it a singularity equal technically in the we can singularity defined, if the function appears we undefined there. So, to take a most silly example of this, if you look at the function z square by z square and z equal to 0, if you treated completely dumbly z square goes to 0 and z goes to 0, so 0 by 0 therefore un define.

There are lots of silly examples also there are for example sin z by z that is also 0 by 0, but of course we all know that we can remove it, but why do we know that we can remove it, only through there is actually even this an argument that this is not really a singularity at radical 0 is by plug in the make the Laurent series expansion of this, hence show that all these are 0 all the negative coefficient are 0. And therefore, here is not singular it is remove by.

So, this is the just formulization of this all thing clear, there is a nice character in the I said that these are characterized by behavior of f around z naught. So, how throb is removable singularities characterizes. So, f has the removable singularity at z naught, if and only if f (z) is bonded around z naught, when I say around z naught is take in too many small enough disk around z naught.

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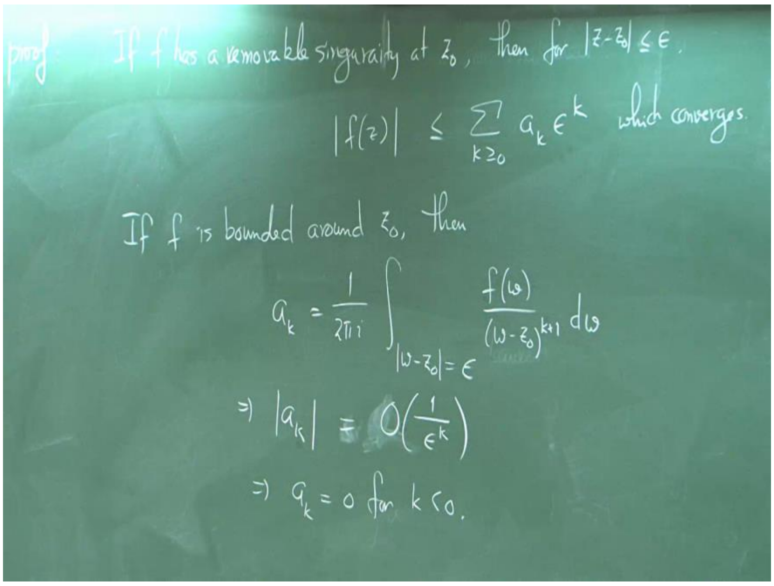

And the proof is fairly straight forward, let us tackle this or let us get the proof out of you guys, which where do a start suppose f as remove is singularity. Then how do you proof that f is bounded around z naught. Let us assume less than equal to epsilon z min us z naught the absolute value that is a very tiny disk, you need to show that absolute value of z is bounded that is very straight forward gives the definition f (z) is Laurent series expansion all that negative a k are 0.

So only the positive a k are survived a 0 will be a 0, and the other once a 1 will be multiplied within the absolute value. If you look at the absolute value of this epsilon, epsilon square, epsilon cube the higher co efficient, and since epsilon is tiny is all converges to some finite value. This is assign equal to sigma k greater than equal to 0 a k absolute E k. And this therefore converges to something value of course; here one is assume that f is analytic on D.

So, various in non trivial annulus on will this is convergent, if it is not convergent anywhere than of course there is no point talking here of anything at all, but if it is convergent inside a annulus which is this, it is epsilon is much smaller than all, so it is totally convergent and converges to some finite value. And therefore, this is bounded by some value inside, they converts if f is bounded around z naught, what?

Student: ((Refer Time: 35:59))

That is another way inside of using there is using a contra positive. If does not have removal singularity then that is sort going to work directly, because what might happen is that infinitely many a k are non zero, negative a k are non zero. And then some of those sums converges to something, well with one can argue that they can do not, but in it to measure on with infinite sums. One would like a more direct concept proof here which ((Refer Time: 36:53)) exist.

Student: ((Refer Time: 36:54))

You can definition exactly you want to sure that a k are 0 for all k let us a 0, what is the definition of a k, a k we solve as 1 over 2 pi i integral of w minus z naught equals, and are which is can we any circle as long as within this. So, I can choose it we epsilon as long as epsilon naught 0 f (w) divide by w minus z naught to be k plus 1 d w, if you look at the absolute value of this is going to be less than equal to is forget about constants this is bounded f (w), if there is some c constant c, so z absolute value of w bounded by c this is in absolute value epsilon to the k plus 1.

So, this is and this as you integrate once you take this out and integrate you get 2 pi epsilon, to basically what will get is a k you will get to be order 1 over epsilon to the k, epsilon to the k plus 1 here 2 pi epsilon comes after integration the one epsilon cancels out forget about 2 pi this are all constant and this is also a constant, so all gets observed in this. Now I if you send epsilon to 0, what happens? For negative case they all go to 0.

Student: ((Refer Time: 39:26))

Neglecting z 0 of course, because z 0 is singularities you do not know what it is happening? But that is good enough, because you do not actually have to take this going to epsilon equal to 0. We are showing that a k cannot have an non zero value, because we can may choose an small f epsilon to make it smaller than that, so that is produce straight forward, so the are sort of silly singularity there I said that do not even exist.

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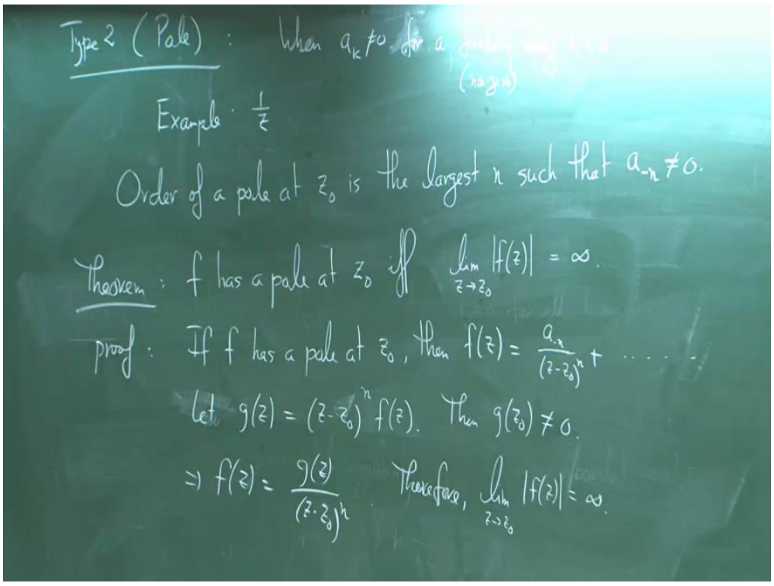

What is more interesting is here and that we the most interesting kind of singularity is are type two, we are call poles a k is not equal to 0 for, when a case non zero for the finitely many k less than 0 and that has a non zero value. So, the example of this, well 1 by z is this simplest possible example here a minus for the non zero. So, what with as with 0s we have an associated order with the pole, so is equal to the largest value of a, so that a of minus a is non zero. So, this is almost dual of the order of a 0, an how is a pole define by the behavior of f around z naught, if has pole around z naught, if and only if as you sent z goes to z naught, the absolute value f (z) goes to infinity.

Now, this is the bit over loaded statement, because when I say z goes to z naught, I can take z to z naught along any path that is very important not only one path. So, does not matter which path I take it to z naught f (z) always diverges, and the proof of this is again fairly straight forward, if f has a pole at z naught then just pulling out the definition f has a Laurent series expansion with let us a pole of order n. So, which means that there is if here is a pole z naught, then I write f (z) is a minus n by n plus had any terms. And now if you see that if you sent z naught z to z naught, all of this particular this diverges this will also diverge, but the rate of diverges here will be faster than here.

You can make this I when more precise by considering, if you let because some time these ((Refer Time: 44:48)) so one has to be very careful when making this arguments. So, just to emphasis at I define g z to be z minus z naught to the n times f (z). Now, g what is g of z naught is not 0. So, then I can write f (z) as g of z by z minus z naught the end. Now, you see that clearly has at goes to z naught this is non zero, and this goes to 0. So, clearly diverges it does not matter, what path I take.

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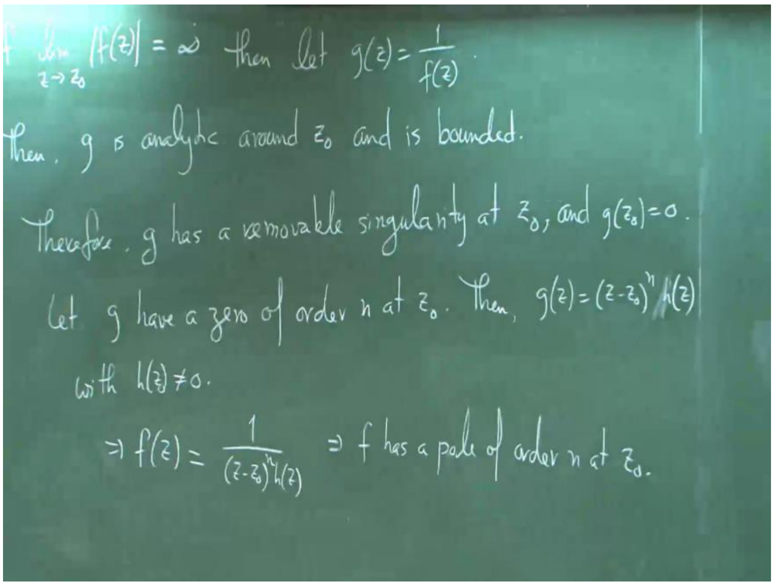

And conversely if you assume that f (z) diverges around z naught everywhere how did extent towards infinity. Then how do I show that f (z) pole at z naught this is a little trickier not straight forward as the predi simple. So, when define g go slightly differently to be 1 over f (z), then what can we say about g (z), g is analytic around z naught, because of z naught does not take value 0 around z naught and is bounded, why z bounded? It is absolute value of f (z) around z naught is very large.

So, therefore g (z) is bounded around z naught, which is analytic and bounded around z naught. Of course g z naught I am saying it is not clear, what is definition I do not I am not even defining it right now. But the fact that is bounded and analytic around z naught means that I can work the type this is the type, if at all it has singularity at z naught is a type for singularity, which means it is removable.

And if it has removable singularity z naught it means it is g z naught can be defined in a unique way. And all z define unique way well just take one path and send this is going to diverge g of z naught has to be 0, g analytic function in this disk it vanishes at z naught. So, it is has a 0 at z naught, now which 0 is isolated of course we know, and there is an order associated with this. So, let g have a 0 of order n at z naught, then we can write g

(z) as z minus z naught to be n times h (z) with h (z) not 0, thus by definition of the order of 0.

And now I can write f (z) as 1 over yes. Now, here h (z) is analytic around z naught it is not 0 around z naught, so whenever h (z) is also analytic around z naught. And this is n here and this shows that f has a pole of order n at z naught, because whenever h (z) I can write as a power series around z naught, and which does not vanish at z naught. So, this is going to be a Laurent series with a minus n being the largest or smallest non 0 coefficient.

Student: Why is g bounded?

Why is g bounded, because of this in a small disk because of this property, if you have z naught and you look it together small disk here look at this region. As this size swings absolute value of f (z) goes to infinity, so at some point let say absolute value f (z) exceeds 10 everywhere it will be because no matter vary approach from many point, no matter how you approach z naught is your going to diverge. So, far means for every point around this in a small enough disk the absolute value f (z) is say at least 10, and if approval value of z at least 10 on that disk g (z) is at most 1 by 10 in the absolute value, so is bounded.

Student: ((Refer Time: 51:46))

This…

Student: ((Refer Time: 51:59))

No I what z naught.

Student: ((Refer Time: 52:04))

Z naught not equal 0.

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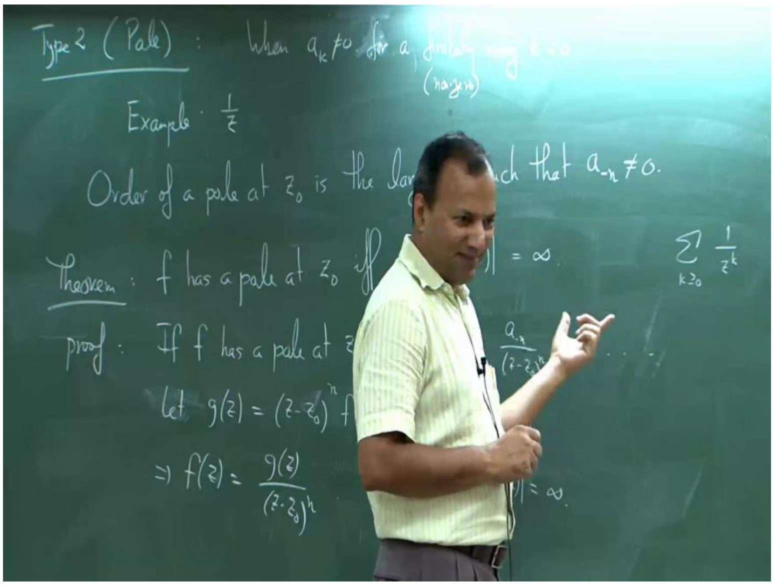

Here saying summation k greater than equal to 0 1 over z to be k, this is a infinite Laurent series yes, and a z goes to z naught thus the absolute value go to infinity.

Student: ((Refer Time: 52:29))

Thus it…

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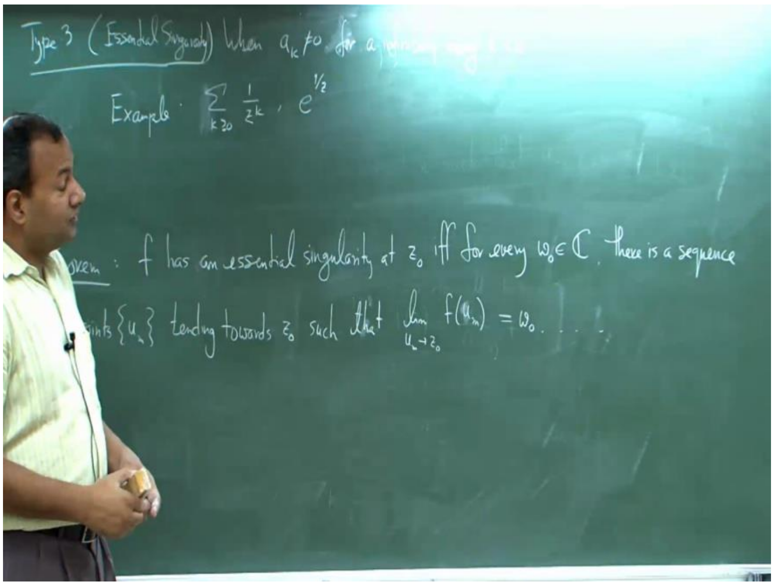

Thus type 3 singularity is there called essential singularities, when a k are non zero for infinitely many careless than 0, how this kind? The example is as just law you can are e to the 1 by z it is the 1 by z. So, what to what a want to write is at there is a sequence of points U m tending towards z naught, so each of this point is closer and closer to z naught that is what I mean here term not writing it very precisely.

So, essential singularities x a bit for completely presort behavior they diverge that on diverge, as you go from z to z naught depending on what path you follow? You can get any result that you want, will prove it in the next class, but just to convince you of this let us look at this path. If you approach it from the positive real axis, what happens? Approach towards 0 from the positive real axis goes to infinity, approach towards 0 from the positive imaginary axis.

Student: ((Refer Time: 56:02))

Minus infinity positive imaginary axis, so z is i y. So, you get e to the 1 over i y, y tending towards 0, what is the absolute value of this? e to the 1 over i y, what is the absolute value of this is always 1 does not matter about y is. So, if you project from the positive imaginary axis towards 0 the absolutely value stays at 1. And this is not the end of the story as this theorem says, you can get any value that you desire has the limit depending on what path you choose to it.

Student: ((Refer Time: 57:01))

What?

Student: ((Refer Time: 57:02))

Same, same story absolutely same story is to for every essential singularity, and essential singularities defined as infinitely many case being non zero, which is the case here. So, we will continue this in the next class.