Riemann Hypothesis and its Applications
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Lecture – 8

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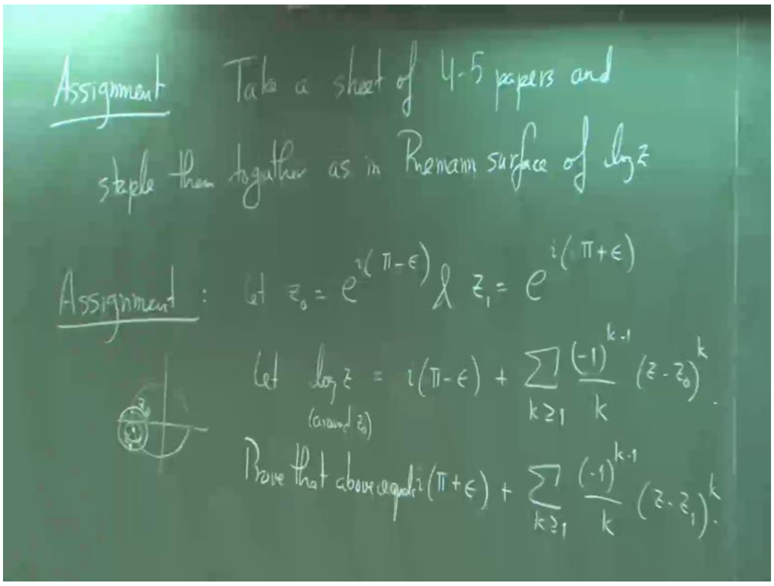

Whereas for the Riemann surface for log. But I did not get time therefore you get a home assignment. So, all you need to do is to stag them together and make a paunch cut to the center together. So, that you get perfectly a line paunch cuts and then stick them together. One more home assignment, something which I thought of hand wave not sort of, completely hand wave last time and none of you caught me there. So, let us refresh your memory. What I said last time was, there you take. And I think I started from here. Take a point z naught here, which is at this point and write the power series of log around z naught.

So, look at the disk around z naught and there I said that this disk actually intersects, goes beyond or goes above the real axis and then I also said that once the moment goes above the real axis. This power series coincides with the log 1 z definition. But I did not prove it, I just said it does and none of you pointed out also. So, as a punishment for that you get this home assignment, which is to take a point, look at the disk take a point z naught, oh sorry z 1 here. What would that be? And or let us make it simple. There are too many negative signs here. So, it is just slipping this you have to approach from here this is at pi minus epsilon and this is at as you traverse further down you get pi plus epsilon. And let log z around z naught be of course, you know what it is i pi minus epsilon, and we saw this last time all of this k greater than equal to 1 minus 1 to the k minus 1 divided by k z minus z naught to the k o k. Now this disk hits z 1 as well, and then what I want to say is that look at the disk around z 1 on that disk this power series will coincide with the power series around z 1 and the way to prove it would be, to show that this is equal to i pi plus epsilon plus k greater than equal to 1 minus 1 to the k minus 1 over k that is the power series around z1 to it. Need to show these two are equal. Of course, this equality holds for zs also. Only which are reasonably close to both z naught and z 1 for assuming all of this. So, prove that. So, this clearly establishes the fact that the power series they around z naught coincides with the power series around z 1. And therefore, that is certainly not the power series given by the log 0 z at z1, because for log 0z at z1 will have a minus epsilon plus pi here o k good.

So, now let us continue we will discuss a little bit more on this. So, this remann surface where we get for log z it is again a mental construct to allow us to nicely visualize mapping log z. Right that is you can instead looking at infinitely many logs function. You can always look at a single log function sending this Remain surface to the complex plane. Now, one could argue that this is as arbitrary as that cutting of infinitely many log functions. You could start at any point make any branch cut at any angle and then you will get different set of all infinite classes, but in one very interesting way. Remain surface associated with log z is not arbitrarily it is really extremely useful, besides allowing you to imagine the mapping log z in a nice fashion it is actually useful in a very more or as a much more concrete sense.

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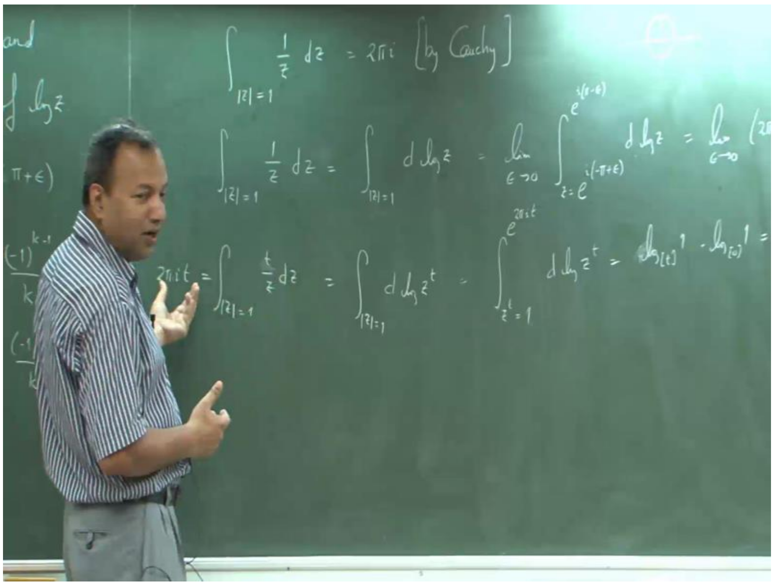

So, let us see that ok. Now I am going to work on the cautious theorem. What is this? This is integral around that is not less than equal to 1. This is integral around the unit circle of 1 by z. What does cautious theorem tell us 2 pi i right because cautious theorem in general says f z over z the integral around a disk where f is analytic is 2 pi i times a f or likely z minus z naught as f z naught . So, here it is z minus 0 gives us f 0. I can view this integral in another way, 1 by z is differentiation of log z and we have established that, it holds over all complex time right. So, 1 by z d z equals d log z. its d log z by d z times d z and d log z by d z is 1 by z o k. So, integral of d log z is what log z right and then there is a circle of course. Now in the circle there are no limits.

So, one can see that there is a bit of an issue about how do you exactly define this business of limits here that's one thing. Secondly, log z is not properly defined over this entire circle any way. Again thinking of log z over the complex plain and here let us say thinking about the differential. Now differential the log 0 z by the way differential of any variant of log function log one z, log 2 z, log three z, log 0 z. .The differential is always 1 by z because; the difference between any of these variants is only in the constant part. And that gets wiped out by differentiating. So, this differential is not defined at the branch cut. So, there is one line of in case how was the negative real line start defining. So, can't really integrate in the whole this. Both of this problem can be resolved in the following way we can say. We can integrate from z equals e to the i minus pi plus epsilon to e to the i pi minus epsilon. Now in this sort of, where this is the starting point and this is the ending point and in this region this is perfectly defined. And this also gives me a starting point and a differential point. Now by cause e while I have clear it do not would mean cause e would this is fact use the analysis of log z function to say that this integral is this log of this minus log of this, which is two pi i minus two i epsilon. And as epsilon goes to 0 this becomes 2 pi i and that matches perfectly with this coshee's formula.

So, we could do this through roundabout way of integrating and get the same value, but that is not the end of story. What about, or let us say in general t by z d z. What about this is clearly by the (( )) 2 pi i t. So, I just write 2 pi it, but if I write it in the same fashion this is equal to d log z for t right. This is simple verify it differentiate that for t you will get t by d z yeah fine. And now if you try to do the same thing we will fail why, hear the z moves around this unit circle in the counter clock wise, but we are taking log of not z, but z for t. So, if z moves around a circle in counter clock wise what happens to z to the power t; z to the t will move at t times the speed of move z. So, when z will complete one circle z to power t will complete t circles. And now we are taking log of that function which is actually the completing t circles. So, we cannot really do any limit because there is no two end points as such here which we can take a limit and get around this. So, we can't sensibly discuss this integral over the standard view. That log is infinitely when you log maps one each from a complex plain with a branch cut to a complex plain. So, it if you adopt that view you we can't really do this integration.

How were if you adopted remain surface then t circles are fine because every time complete a circle you move up. So, you are spiraling t times up. And as you spiral, the definition of log changes right you jump to the next log function. So, as use u do it t times you will make t jumps fine. So, this is same as integral z for t starting from and now we do not need to start from here just to make life simple we can start at one start from here and do t circle and as you two one circle where z for t goes e to the 2 pi I. And to do it twice, thrice, t times we reach here right. Now of course, this is all same there is no difference between these two values, but when talk out taking log of this and because as every time is circle around the change the value of log function itself the first log function we start with is the log 0. So, log 0 1 is 0; log 1 over 1 is 2 pi; log 2 over 1 is 4 pi i. So, as you do a t time circle. So, this would therefore be 2 pi this is log t of 1 minus log 0 of 1 which is 2 pi i t. Other has perfectly matching with the coshee's theorem. So, this allows us to view the integral in this different fashion and still make complete sense out of it. Of course, in this simple case because we can apply coshee's theorem directly here we get the value of integral without any problems but, when there more complex functions sitting here instead of z to power t let us say there is some f z sitting here then integrating. So, you do this and come here you get f times z and over b z and integrating that over a circle may not be very easy.

On the other hand if you take this vie and you look at d log f z. Then all we need to do is as z varies we need to count or see how many times does f z wind up or wind down. And that will give us the value of the integral. Without worrying too much see all we need to do is to count how many times it wound up. I do not need to worry about exactly what path does it follow while winding up, because by coshee's formula all parts will lead to the same integral, because it is an analytical function. So, just need to see that as z moves around a close control. Again this need not be a circle it could be any close control. As z moves around this close control how does f z move. And how many levels does it go up the minute we calculate the levels we got the integral. So, it provides a very convenient way of integrating the integral of this kind f prime z over b z. And when we look at zeta function this will occur very regularly. So, we need to understand this well, yes. Typically what we will have is that the function will be f z here.

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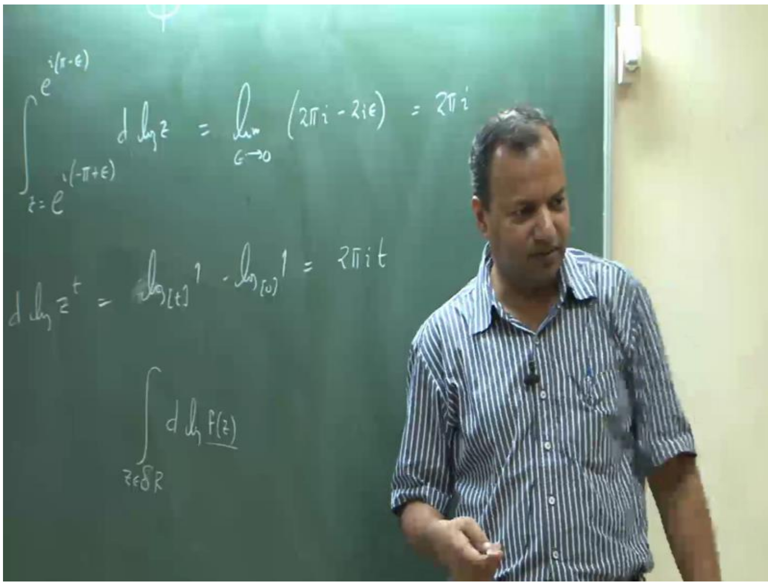

So, let us look at this. So, there will be some closed surface. Let us say some bolt again delta r d log f z. So, that kind of integral we will see. So, here let us say that z moves around the boundary of r. This is z is a complex plain, just moves around the boundary of the complex plain fine. Now z is first sent by f to f z. Now, off course z sent f z to just a complex plain, but instead of viewing f z as laying on the complex plain we think as on the remain surface, because there is a log sitting after this. And we calculate how many times does f z as z moves around this how many times does f z wind up around some point 0 that is it only yah around 0 is a k. How many times does f z wind up around 0? And that number. So, it is convenient to view f z as lying on the remain surface, because winding means going up in a circular fashion right. And that number is the value of this integral. That number ties to pi i first. So, it is still a mental construct. You can still view f z being on the complex plain you can still do this integral by say the pointed out by taking the limit. Cutting out pices where log is not defined, but it is far more convenient on thinking on this line on remain surface.

So, all you need to do is study this map z to f z to evaluate the value of this integral. An as you see as z moves in a circle. How does f z move around 0 not clear? You can also think of; if you do not want to think of remain surface. You can just think of f mapping complex number to complex number and in the complex plain itself you see how the f z moves around 0. Does it wound one time, does it wound twice, and does it wound three times. Like the z to the t wounds three times around 0. Has it moved once around 0? So, that is perfectly fine view and you can immediately conciliate the value of this integral. But to justify that this is the value of this integral we have to say that f z is actually lying on Riemann surface, because then we can say that log of f z is completely defined over this circle. So, the Riemann surface provides a way of justify the value of integrate. So, now, we have studied the analytical functions. We have also looked at the power series expansion of analytic functions and we know now that every analytic function around every point expands as a power series inside a disk.

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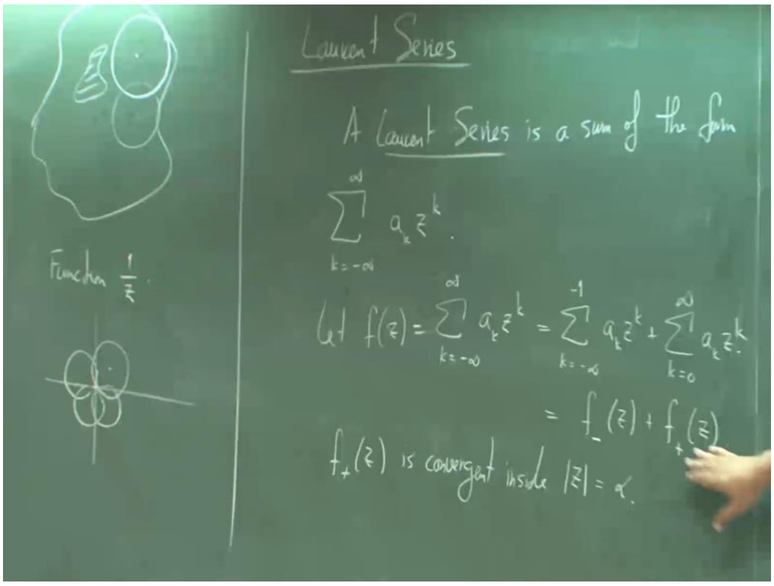

So, analytical functions can be defined in a strange domain. But, there will be sort of this is the domain of analytic function. Then in any point there will be disks over which a power series represents the analytical function. And the size of this disk can be inferred easily. You start at a point at the centre and you blow up the circle as much as you can. Until you hit one of the boundaries of the domain. And that is the size of the disk on which this is well defined. On which this uniformly converges with power series and it is equal to the function. And in this fashion, you can keep on defining this. So, one thing is clear that if it is a finite domain. No it is not clear that you will have, that is not necessary that you will have only finitely many disks here. It depends on the boundary. If the boundary is messy, you may end up having finitely many disks required to capture all of this. So, this is nice. But, this is still a little unsatisfactory and reason is that first is that this power series expansion what it says that, outside the disk everything diverges. On every point outside the disk it is certainly not absolutely convergent. So, it certainly diverges absolutely.

So, it does not quite allows us to study properly is should say, the similarity of analytical function. Certainly by definition analytic function cannot be singular in its domain, but there are certain points which are outside its domain on which the analytic formation is singular. Now, behavior of analytical functions around those points is not that properly expressed by the power series expansion. Let us take an example. Let us take this function the simplest power possible such function 1 by z. It is just has one singularity f z equals to 0. And if you, through this domain is the entire complex plain except this point, this puncture made at z equal to 0. And that is a domain on which it is analytic. Now, if I try to write this same function at the power series around this point, then z equal to 0. So, what I can do, I put a circle here which converges every year inside this power series expansions, but which touches z equal to 0. So, it is not defined than I have to cover the remaining regions. Other regions I have to define other circle, another circle.

So, I have to define multiple circles each capturing the part of the behavior of the function around the singularity. So, there are times when this is good enough, but there 3are times when one would like a simple and one single representation of the behavior around the singularity. And for example, this function, this representation itself completely defines the function around the singularity and anywhere else, but of course this is not power series. So, that is the problem, but this also says the solution. That instead of just sticking to power series if you relax our requirement a bit and allow the inverse polynomials also to occur in the power series, not the power series, in the more general series. There at list certainly for this function we can write the entire function as a single such series. And that gives rise to the notion of Laurent series. So, this is the last concept before we dive into the beta function this Laurent series then its discussion for singularity is will complete the complex analysis.

So, Laurent series is defined as a. So, Laurent series is a infinite sum of this form, the only difference with power series is that we also allow negative part of these to occur. So, infinity value negative powers, infinity value positive powers. This is clearly a Laurent series. Now, as with power series we have to understand where is this analytic? Clearly this is this is more general in all power series. Powers, every analytic form function we can write can Laurent series where it is analytic so that is clear. So, we need to understand exact way is Laurent series convergent. So, let us start with this as Laurent series and split into two parts. This part is a power series and this part is a pure negative power series. And let us give it certain names; this is f minus z plus f plus z. For f plus z we know precisely where this is convergent there is a disk associated with this where this is convergent. What about f minus z. f z is convergent inside and whenever is write convergent I mean uniformly convergent because that is the only notion of power series that we will be interested in. Inside say some disk z equals alpha. By the way, I have again eaten up the more general form here. It should be z minus z naught to be the k. So, we are expanding series around z naught.

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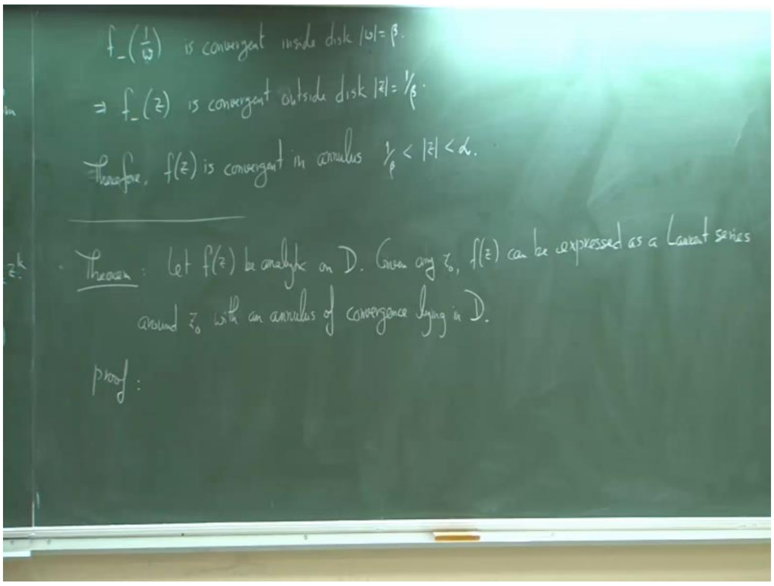

So, let us assume x s z naught 0 here. So, this we know what about this. All those are the negative powers of z, purely negative powers. So, if we flip z and replace it with 1 minus z you get a power series. So, we know precisely where this is convergent. Now instead of z let us write it as 1 over w. This is convergent inside some disk w equals beta right. So, this implies. This taking z to be one over w this is convergent outside the disk mod z equal 1 over beta; w is 1 by z. So, we replace 1 by z here , mod w replace it by 1 over mod z and then we get z equals one over beta and the inside gets flip to outside. You can see this clearly while I was writing; by replacing equal to by less than equal to. And that's it we know precisely the convergence of these two parts. So, therefore, f z is convergent in this annulus. Annulus is the name of this disk where smaller disk is taken out from inside. Yes it is possible then it is no where convergent. Then it is really not very nice definition having made. So, it will make sense only when one over beta is less than alpha only then there is some region of convergence for the Laurent surface.

Now, coming back to this example; what is annulus for convergence for this 0 to infinity? So, it is basically convergent everywhere except this single point 0 because that is punctured out and that becomes annulus. Now, outside this annulus is this convergent, can it be convergent any where no why while if you look at outside this annulus go beyond alpha; going beyond alpha this part is convergent that's not a problem. So, this is some finite value. Take any point z which models is bigger than alpha. This is absolutely convergent it is some finite value, but this diverges. So, sum diverges similarly inside below 1 minus less than 1minus beta this diverges, this converges. So, sum again diverges. So, again exactly like power series it is convergent precisely inside the annulus and nowhere else. So, again this is limited in that function, but less limited than a power series, because this is a living example here. You can capture this function just by one single Laurent series as suppose to many, many power series that you need.

In fact, to capture this function as power series in the entire complex plain you will need infinitely many power series, because any power series from any point in the complex plain can only be able take out a finite region from the complex plain. So, is a more general and most powerful form of representation of an analytic function? And it is pretty straight forward to show. That any Laurent series inside the annulus of convergence is analytic essentially the same proof that we used for power series carries. So, for in do you remember the proof how did we show that any power series inside is this of convergence is analytic. Essentially the same proof that we used for power series carried so far here. Do you remember the proof? How did we show that any power series inside its disk of convergence is analytic? So, what we did was we looked at function of power series. Limit the sum to a finitely many one that is a polynomial.

Polynomials are analytic we know right and then when you take the limit, because of uniform convergence they analyst is blizzard because that is by modules theorem when you use because every polynomial is analytic. So, the integral around any rectangular is 0, and then you take the limit of these polynomials which is the power series and due to the uniform convergence we can scrap the summation or integration. And therefore, we can. So, that is the proof for analyticity of power series and the same proof really goes through here, there is no difference. So, I will not prove that I will leave that to you to work out. Instead what I will prove is just like we showed for power series that around; if you are given any analytic function.

And around any point inside the domain we can represent the analytic function as a power series right on a disk. Similarly given any analytic function and given an annulus then inside any point. Sorry given any analytic function on a domain given any point inside the domain we can represent the function has an Laurent series inside a appropriate analysis. And proof is that pretty much mi mix the power series proof, but there is one nice twist to it that is why I want to show it you. For example, look here this is a Laurent series convergent on this domain right. This is a analytic function also on the domain inside that punctured complex plain. This is a Laurent series for the same function, and it is around a point z equals 0. That point is not inside the domain is actually a singularity. So, this Laurent series expression this one is around a point which is not in the domain; yet can be express a Laurent series around that domain. Inside the domain it always creates a disk. So, that is not really interesting anyway. Yea so anyway it is a law, because it is a power series hence it is a Laurent series. So given any z naught. Let us say, let's try to write the statement and then prove it. So, given any point on that f z can be expressed as a Laurent series the around z naught, but it's annulus of conversance will be inside the domain D. That we cannot escape from.

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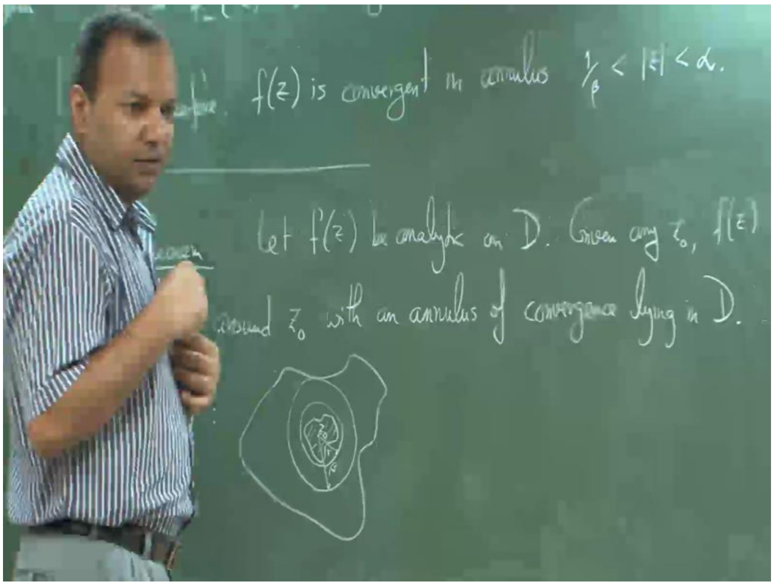

So, how do we prove this? So, suppose this is a domain and this is a point z naught which is actually outside the domain. Suppose if I take z naught then there is no annulus of convergence. Then it is pointless to define anything. No annulus of convergence which fully lies inside the domain. So, it forces me to pick a point which covered with a boundary of the domain and then I can try to define the annulus of convergence here. So, how do we handle this? Let us pick an annulus which consists of two circle which centers at z naught lying completely inside the domain d. And let us say this is alpha and this is beta.