Riemann Hypothesis and its Applications
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Lecture – 7

And one of the coral view was that if an analytic function has a non isolated 0 then it vanishes. A very well or in other words if there is a extension of an analytic function in a domain beyond its definition then that extension is unique. So, let us take some examples of this, because this is very interesting property of arête functions.

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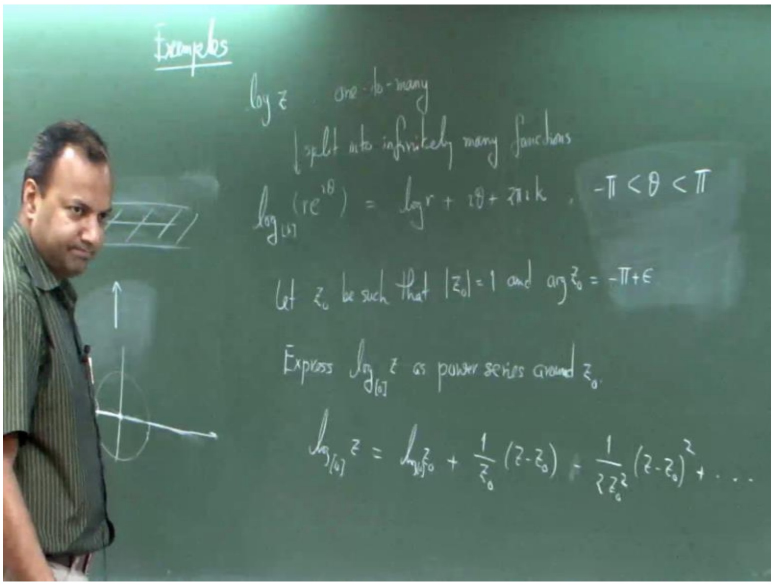

Log z function is something we will be very interested in. So and that also had a peculiarity in complex plain definition and let us try to see how that works or how that is solved. So, earlier what I said was, that look at the function log z is a want to many function and to get a sensible function out of it one too many is naught something very nice. So, we split this into finitely many functions, where log of, and this give with a specific naughtation o k. So, that is the k starting from 0 the log at them which logs r e to i theta you will log r plus that is the unique value and I theta plus 2 pi i k where k is integer. And theta allies between 0 and 2 pi o k. Actually this k can be negative also it can take an integer also; it can be any integer o k. So, pick any one of this, so specifically let’s say for k equals 0 pick the function, that function is defined over this strip, log 0 z is defined over this strip and this line is naught included the bottom line is included and this strip is mapped by this function to the entire complex plain expect 0. Wait is it correct why expect 0. So, log 0 is naught defined. So, this maps to the whole complex plain. What I need to do here is a when I take this strip I need to take out 0 also from here 0 also cannot be part of this because log 0 is naught defined ok.

Now, this strip is naught a domain because the domain has to be an open set. So, we can't even talk about this function being analytic over because we need to define analyticity at a point. We need to look at its neighborhood at this adjust does naught quite exist. So, a clean way of handling it all of these and fact of that 0 is also removed is to just look at this. That we said log set this log k is defined and precisely inside this strip naught taking the two edges in at all, ten it is a domain, and it is well defined at the domain. And it is naught to difficult to see and I think I gave it as an assignment problem to see that log z is analytic for this domain. This is all wrong; this is range of format right.

So, exchange this. So, these maps accept the point 0 here to this strip, but now let us see, what else do I need to do o k fine. So, let us re do the argument. This is the domain, this is the range. In this the question is that around this line, when we have approach this line from top or below the continuity does naught exist. So, again. Continuity is naught there analyticity is naught there. To make sure that log Z is defined in this fashion is analytic whatever copy we are considering, what we do is we take out this line entirely from the domain. So, take the positive real line out from here all the way, because this line is what is causing the problem and then it becomes analytic including the point 0 because if we can naught include the point 0 otherwise it would naught be a an open set. So, take out point 0 and all positive real numbers, and then this sort of complex plain with the cut is mapped to strip now this strip becomes an open set.

There is no upper and lower boundary and the log maps in new nice fashion to this. And this I can define for any gain. The only the difference is the domain remains the same, the range changes to a different strip. Another point to naughtice is that this choice of line to cut out which is determine by the definition I give to this each copy of log that choice is obituary. The fact that we had to choose this line is because of this definition that we say a log this k of your log is equal to log r plus I theta a plus 2 of pi I k to a theta leys between 0 and 2 pi i. If I change this range to something else say this, then what changes. Firstly in the range in this strip changes it becomes this strip in the domain.

Now, we take out point 0 in this the negative real axis. Again this is also obituary; if I change this to may be minus alpha less than theta less than 2 pi plus minus alpha plus pi 2 pi whatever right. Then this will again shift and here also, the cut line will shift to an appropriate alpha angle line starting from origin. So, this choice is completely up to us.

So, since up to us, we will choose the most convenient and simplest possible choice here. And that turns out to be this one although the first instinct says to choose this one which is what I did last time, but this turns out to be a even better choice the reason is, that if you cut this out the positive real axis. Then essentially we have saying that log z is naught I am naught defining log z on positive to real axis which is, somewhat counter intuit. Because, certainly log is defined on the entire positive real axis log is naught defined on the negative real axis in the usual sense, but certainly positive real axis will define. So, you should try to include that definition at least when we generalise this definition to complex. There is another reason for it we will see very soon. So, we will choose this instead of this definition we will have this definition of the theta range and therefore, the cut that we make is from here to the negative real axis and the strip that we get here are these o k you with me so far alright.

Now, let us try to do an analytic continuation of logs it is defined over this domain right the missing part is this line. So, let us try to a extent the definition of log that over this line to do that, let us pick up a point very close to this and it will be useful to pick a point which is very close to minus 1 on the real axis, but naught quite on it with at the same time lying on the circle of reduce 1 and somewhere here. So, let say let z naught and. So, that is the point we pick the z naught. Here its absolute value 1 and this argument, which is the angle makes this minus theta minus pi plus. And now let us take a tiny circle around z naught and the circle will have radius. So, this distance is about epsilon the rate distance of z naught from the real x axis approximately epsilon, we assume epsilon is very small. And let us take a circle of radius 2 epsilon with center a z naught fine, and expand log z around z naught as the power series o k. So, what is the power series of log around z naught.

The first term is log 0 of z naught itself right plus whatever that is higher degree terms the coefficient will be the derivatives of. We know that this is analytic around z naught the coefficient will be derivative of this function at z naught. So, what is the first derivative of log 0 z. You know log x is 1 by x or why do know that log z is 1 by z over complex numbers. What that is right. So, that is why we what we can use here, is just a last time. That gives you very simply that log z must be 1 by z. Sorry derivative or log z must be 1 by z because if you derivative log z is an analytic function. Because log z is analytic right and if you look at that analytic functions log z prime minus 1 by z this function is 0 over the entire positive real axis o k. So, therefore, it must be 0 over this entire domain and that is one more reason why we want to include the positive real axis inside our domain, because when we can just lift us definition of derivative log x to derivative of log z good.

So, derivative of log this is 1 by z which is evaluated z naught is z naught, z minus, z naught plus the second derivative is the same just continue with the same to it and. So, we get minus 2 by z square divided by. There is a difference. This is the first derivative the minus 1 by z square and that is divided by 2 factorials. So, that becomes minus one over 2 z naught square plus z naught. This is also power series of expansion right. So, fact that first lets bargain for this what is this log 0 of z naught use this definition that is equal to log r which is 0 to that is pi choice of z naught.

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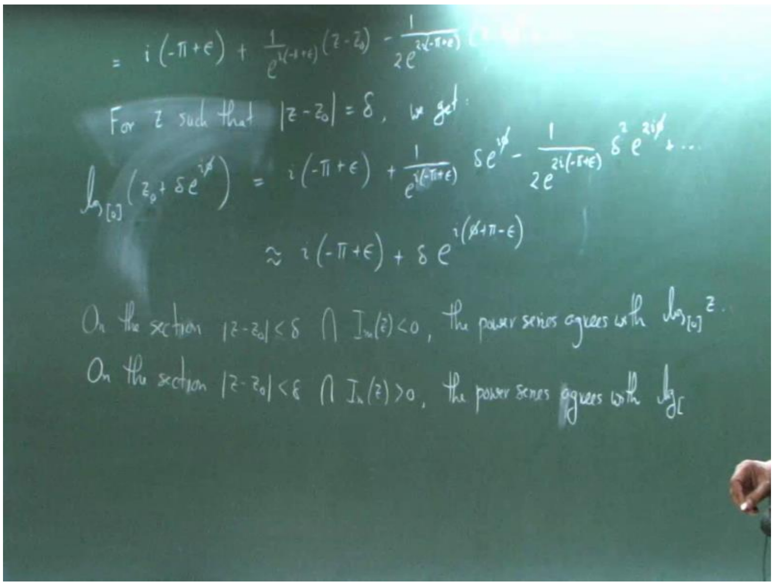

So, what I get is i and theta if minus pi by epsilon plus 1 over z naught which is e to the i minus pi plus epsilon. Now, let us say we are only going to look at those z’s which are a distance delta from z naught what do we get. So, such a z will have this form delta being the absolute value and then pi will be value over the as the circle moves at between 0 and

2 pi o k. As delta test is 0 we can ignore the higher degree terms. Because here you will see that in this delta multiplier here, delta square multiplier all other things they have absolute value 1. So, this absolute value of this term is delta square or delta square by 2 actually and delta cube by three and so on. So, they become smaller and smaller, in the limit we can throw them all away and just stay with this and off course if the delta really close to zero we can throw this also but that is don't want to do. So, this is approximately this.

Now let us follow the projectry of this point as we move around that circle ok. So, when let us say delta is 0 then we are at somewhere little above minus pi we are somewhere here when delta is 0 minus pi plus epsilon, off course delta is going to be a little bigger than epsilon actually twice epsilon or something. So, in that case where are we? So, this says 5 plus pi minus epsilon; epsilon here almost we can ignore in this phase case situation and then. So, 5 plus pi minus,, so if I am here and if I choose 5 to be 0. So, at this point what is the corresponding point here 5 0. So, you get e to the i pi which is minus one,, so you subtract delta from there. So, minus pi plus epsilon minus delta. Where does that; that is somewhere down here ok, with me so far. So, we are actually this power series is taking me out of the definitional region. That is imaginary sorry that is a real. So, this does naught take you down it takes you here, it is still inside ok good. So, now we start moving in this circle. As we move in this circle what happens to that point. That also moves in a circle of radius delta, the center of that circle is minus pi plus epsilon and as we travel around this circle, this travels around this circle of radius delta and completes the circle.

Once we complete that also completes. Now given that data is always bigger than epsilon which we can always choose. We do exceed this strip and go below it and come back again. That is nice, but question is this power series we just defined is this is this convergenent in that disk, because unless it is convergent in this disk none of this will make sense, but is it convergent. Obviously it is convergent because of epsilon is very small. So, it is value as you go here this is the power saving expansion right, if you look at even the absolutely value you go down here it starts becoming negligible. So, it is clearly the convergent series. So, what point you did naught understand. This one, that was fine ok. Why it is convergent? This is the power series we are looking at when z is in that circle of radius delta right and expansion is like this, every successive term the absolute value is delta to the k or k for the k plus first. Delta is very small and very close to zero.

So, the absolute value this power series converges right. Moreover this power series is also uniformly convergent. Because if you look out at the cut out first few terms, look at the remaining term absolute value of that, because there is big delta to power k sitting in multiply and multiply to all of them and then everything inside converges to some finite value, this whole thing is less than a finite value less than a finite value to the power some power. Since, delta is the less than one as a power increases it goes down to zero. So, this power is actually absolutely convergent. Not absolutely it is even more it is uniformly convergent, for any delta lees than one, which is the delta which is the very close to 0 little more than epsilon; epsilon is also something which is very close to 0. Delta is also let us say two times epsilon of x and z naught. Yes this is precisely at, this goes around a circle of radius to epsilon at z naught.

Since, epsilon is very close to 0; delta is also very close to 0 certainly. So, everything convergence very nicely this is somewhat funny thing that is happening now here. Because the range is very nicely defined you start at this point, look at the power series at the reasonable In fact, you can expand the disk size to quite a bit actually, look at the power series defined on this the given by the usual definition of power series. That is uniformly convergent power series defined very well and log maps. The log value according to the power series goes on a nice circle disk here. The problem is that it does naught quite match the definition that we had earlier fixed for this log 0 z to this strip. Because this strip is being violated and we are actually jumping in the strip below.

So, what kind of value is it taking? So, this is the strip below this is minus three pi. So, it is certainly taking value in the strip below. So, that is one issue, the second thing is, this circle is naught only hitting the line which is the cut out but also part of the domain below. Now, we already have a definition of log 0 z in the domain below we did naught have one in the line which we have already cut out but, we do have this definition. Now what is this definition? Now what is this value? as this is provided in the domain below. Actually I goofed up i should have taken z naught a little below this, because that is when minus pi plus epsilon is correct. Otherwise it should be plus pi plus epsilon. This angle is plus pi and this angle goes to minus pi. So, that does naught change the argument. So, this circle is x sort of cutting across this line which you have removed and going into the both sides of the domain, on one side it matches with the definition of log z for on the other side the values provided by this do naught quite match with the definition of log z.

Because if you are going of the positive side according to log 0 z. What we should get is the misery part should be plus pi and then a little bit more and then some little bit here also. That is naught as much, because delta is small. Instead what we are getting is minus pi and then something. So, this power series does naught even agree with this function on the domain. It only agrees with the function on part of the domain. What that is even stranger; because we just proved the last time that if two analytic functions agree on a contiguous domain, then they must agree everywhere. So, what funny thing is happening here? First thing is. That this is naught quite domain for the power series, although it looks like it is a circle which is a domain, but remember that this line is cut out in the original domain. The definition of the original domain is that this line is cut out. So, what this circle really correspondence to is two half circles, where the middle line has been taken out.

So, it is naught a domain, because it naught connected and since it is naught connected it is possible that that property is violated, that property is naught violated only when the domains are connected. So, it actually gives you a very nice example where, when a domain is naught connected. That faith your face to what ok. That is one conclusion from this. But we would like to get even something more out of it. So, exactly what is happening or after all this is artificial that we are taking out this line, but for the sake of definition we did it, but this is kind of artificial and then we have to have we have this power series is giving all the nice values on this line which you cut out as well. And, if you now think and look at it carefully that what is happening on this side it agrees with log 0 z. Let us ask a question.

What does it agree with on this side? Does it agree with some other version of log on that side, any answer is very clearly yes, on this side. So, let me write it here. This is just a formal statement of this fact. That this disk below the positive real axis agrees with the. On that region power series agrees with log 0 z .Now on the other section log what log? Minus one; minus one means that this strip and that is what it actually comes down to this and if you see that this is going to get map to this thing below. Now, this is a very interesting observation because what this is saying is, there as we travels around this circle. So, we start with this domain lets imagine that you have this infinity very long functions. Each of which is defined over the entire complex plain with negative real excess thrown out and stack them on top of each other.

There is one which correspondence to log 0. There are infinitely many planes below it, infinitely plains above it. So, there is the one we are looking at and we start at this point and start moving on this circle. And as we cross this thinking of this and keeping this in mind this mapping, as we cross this when we are looking at the mapping q n by log z then the point we end of with is naught on the same plain, instead it is on the plain below right. Again keeping in mind that when we say that traversed on the plain we are doing it in the contacts of this map given by log 0 z.

And then we expand it like a power series and then we traversed around this. And as we move around keep looking at the log log value of this as you cross this line. Number one the definition or log value changes. It becomes log of minus 1 z and we have actually moving down the plain to the points of the plain below. Similarly, if you start with the point here and start circular moving along the circle or any curve cutting across this. Then, what will happen is as we cross this line this definition will change to log 1 z and we will move on the plain above. So, a nice way of visualizing this and that really sets up all these log functions in one context is to view this map logs has naught taking a complex plain to a complex plain, but instead taking a very strange surface which I am going to define shortly to complex plain. And this strange surface is form by defined by taking inferred copies of complex plain.

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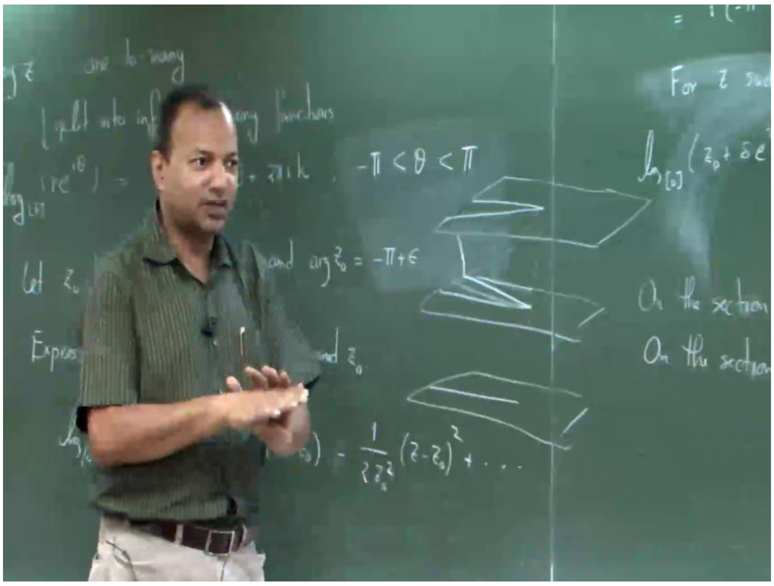

Cutting each copy. Take inferred copies of complex plain here standing on top of each other. Cut out the negative real excess from all of them. So, make a cut, so that its sort of visually this two sides can move apart. Then take this side twist it and take this side twist it up and join the two edges. So, this side let me try to draw it, my skills in drawing are naught very good. So, let me see if I can get it right. This off course has naught come out well at all. But I hope you get the drift. That you cut out this side and then try to hold it down. Cut out this side hold it up and join the two ends, the two edges all the way from 0 to infinity, which means that this actually this point and this point should really touch each other? So, that is why it is a very strange kind of surface that you have this infinitely many plains, they all get collapse at point 0. They are really they hold together at point 0.

Then they go out like this, where there at joining strips this is joining to this and this one is fold it up and the other one is folded below and there is joint. So, let us fold it down and this fold it up and join. Now, if you traverse start any point and traverse on this strange surface. You will follow the rules given by this on this surface log z is one function, there are no infinitely any function just one function. Which is defined everywhere except the point 0 which is bias and it is completely analytic about the entire surface and it maps this entire surface to the entire complex plain. So, the range is very wide, it is just the complex plain the whole complex plain. The domain is this strange surface this surface is called a riemann surface. So, riemann was the one who realized that we can visualize such one too many functions has being one to one on a different surface which we and the name sort of came from Riemann. And there are all strange kind of Riemann surfaces depending on which function you are trying to analyze. There was complete geometry on Riemann. This riemanniean geometry is defined over riemanns. Well, one of the things to find over riemann surface uniformly convergent its only converge. Sorry, yes infinitely when it comes yes, but they converges, but it is a sort of geometric series.

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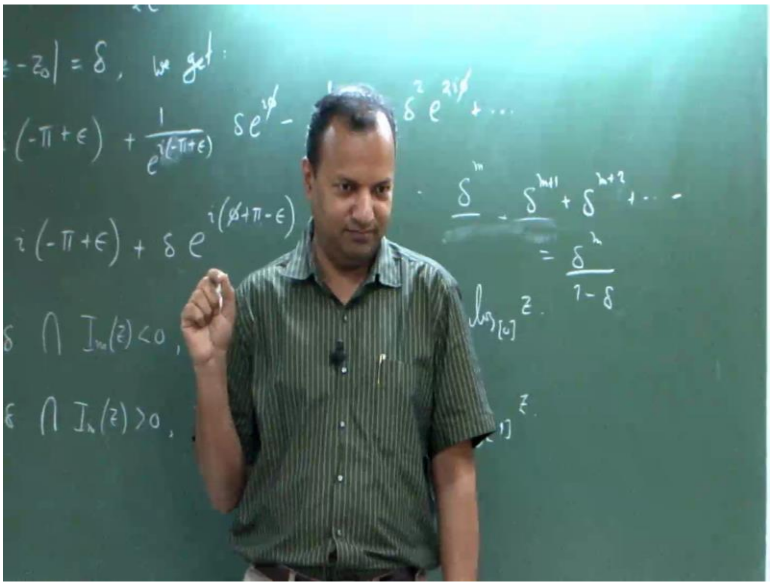

See once you take. Let us look at some delta to the m and this whatever power this is this is all absolute value one it does naught matter what this is and there is something divide by where there is m below there and there are signs plus minus. Also naught fair to it the log expansion of log is x to the m by m. Plus let us just change to make the things worst, change all the science to plus one to change all pages to one. So, the series looks like delta to the m plus 1 by m plus 1 and let us throw all of this out also, they are also reducing the series.

So, that is the plain geometric series which sums up to delta to the m over 1 minus delta that is it. No it is naught at all because you see that the strangeness of this. That these points, they all are collapsed into a single point. They are naught distinct points the 0 of this. Because this when you cut off course you can as you go down you can twist it more, but at close to this you cannot there is no place to twist. So, they all need to come together to get join there and then there is sort of branch out. So, if you traverse you make a you start on the surface and travel in this make a circle, curved circle you actually go up in the spiral. Go down in the spiral well traverse circle around 0 circle also may naught be that part.

Something for your homework. Not an assignment just to keep you busy. Consider the function square root z that is also one too many functions is one too many over real's also. So, it is certainly one too many over complex numbers. Now construct the Riemannian surface for square root side. So, that over that surface is one to one and analytic everywhere.