Riemann Hypothesis and its Applications
Prof. Manindra Agrawal
Department of Computer Science and Engineering
Indian Institute of Technology, Kanpur

Lecture – 5

So, today let us and we were looking at that log z as a function as an example right. And then you saw that it is not as nicely behaved as the log over real. So, we will come back to log z again, because there is that infinitely many possible log functions can actually we stitch together into a single log and exactly how to do it we will see in a short while.

To do that we will have to look at the concept of analytic continuation with is what I will slowly the due to. Now, there is one more theorem that, I think is important, which I should prove it very easy to prove it actually. So, we saw that for analytic functions, we have infinite differentiability right, to if analytical function, f is analytic then f prime all is differential f prime, f double prime and so on remain analytic. We also saw there it is for an analytic function is integral is also analytic that was part of Morelys theorem.

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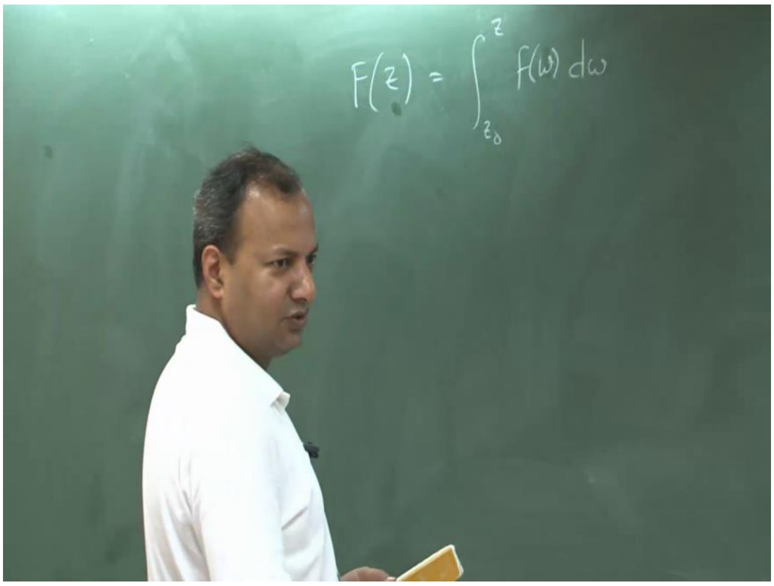

That if you define F z as then this function F is both continuous and differentiable and therefore, analytic whenever F is analytic, but there is one more way you can define integral of a function from an integral form and analytic function, which as suppose you have a bivariate function, which for on one variable is an analytic for every value of the other variable and there you integrate over the second variable.

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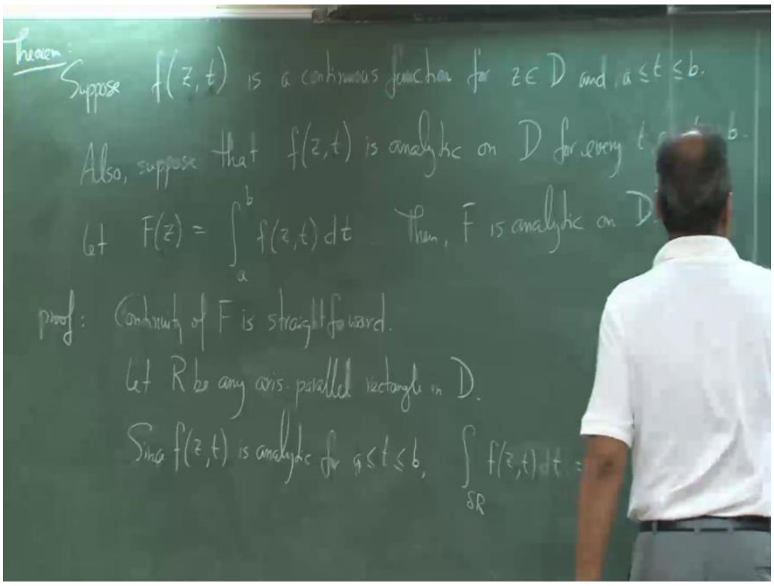

So, for example, suppose we have f z t is. So, f is a bivariate function, the first is a complex variable, second is a real variable. And it is continuous on some domain and more importantly for every value of t in the range a to b f z comma t is also analytic on the domain d.

The reason, I am choosing real variable here instead of continuous says that most of the time we will only be looking at function of this kind and when integrating over the real variable. So, this is enough and the proof here is really simple. Then if you let capital F z to be integral of f z t over d t. Then capital F is also analytic there is a integral of this small f over the real second parameter, which will analytic only. So, this is theorem.

The proof as I said is fairly straight forward, first thing to observe that, capital F is continuous and that follows by the continuity of this function. So, if we look at f z and f z plus delta z that is has delta z goes to 0 that the integrant here converges to f z t and therefore, the integral will converges to f z. So, continuity is straight forward.

And now, what we need to do to prove that capital F is analytic. Well if you go by definition, if you show that this is differentiable then we have done, but there are time when we can use alternative characterizations of analytic function for example, the one we showed last time, which is there a function will we analytic if it is continuous an any rectangle it is integral is 0 and it is for in this case that and characterization is really handling.

So, since so let R be any and we say excess parallel, because that makes an even simpler in D. Then since small f is analytic for any value of t, what we get is, this is 0. Fix any t in this range, this is analytic and therefore, this integral is 0 that is while cause and this surface for every t.

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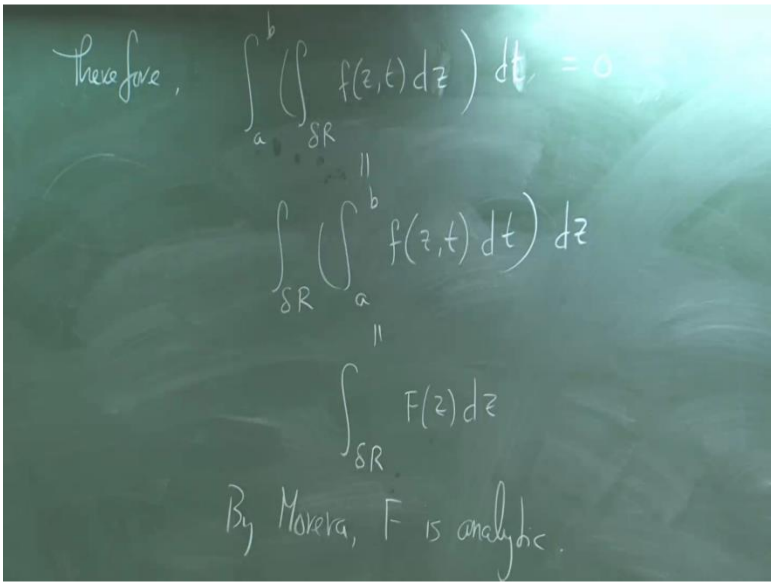

Therefore, this also 0 sorry, now, what I am missing? This should be d z and this is d t. So, this is also 0, because integrant itself is 0 here. Now, what is the left hand side of this equation equal to? See if you can slog that 2 integral we are done, but can we saw this will take series, that is a question, that is bit tricky, o reals we know if there are two integrals. O reals infinite range we can stop them, that is a classical state theorem, but here, we one integral is over reals, the other is over complex and that to over a like a rectangle, but here we can simply use fact that this is an except parallel rectangular. This integral, I can split into 4 integrals and these four will be real integrals on x when y is fixed on y when x is fixed x y. So, I can write this complex integral over a contour as four finite real integrals and then I can do the swap. And then add the four complex a four real integrals against in to a once simple complex integral. And therefore, we get that this delta R a to b, this is this.

And this is equal to the integral of capital F, which is 0 knowing invoke Morera and capital F is analytic. So, essentially we, what we should intuitively take out of this is a analytic functions are very nice function, they have very well behaved have a all differentials, all integrals of is certain kinds. And so we can play around with them with reasonable amount of freedom. At the same time there is, I will want to strike nod of caution that do not take them for granted, the analytic functions can throw up rather unexpected behavior, as we shall see also.

So, what are the examples analytic functions we have seen, well we have seen there all pal novels analytic, all sign, cosine, exponential, logarithm also at least in a strip is analytic in may not be analytic everywhere.

And yes, there is a job that it is, that it is continuity. So, it is not analytic there. So, I said last time that, I will show how to make it analytic and that is, that I will do later on. Once I discuss forward series, then I can do it, because I will need power series to discuss analytic continuation and then I will bring that back in three five forget remaining. Now, one some of these functions polynomials are pretty straight forward functions, but if you look at exponential functions for example, it is slightly more complex function and we know at least our reals are exponential function has this infinite power series representation using the Taylor expansion. Similarly log sin co sin they all have this infinite Taylor expansion as a power series.

So, at least some of this power series therefore, R analytic, the question I want address is which power series are analytic? Because power series are a generalization of certainly polynomials and they also consume or contain all the functions we have seen so far. All the analytic functions we have seen so far atleast right, because all of these analytic functions can be written as a power series.

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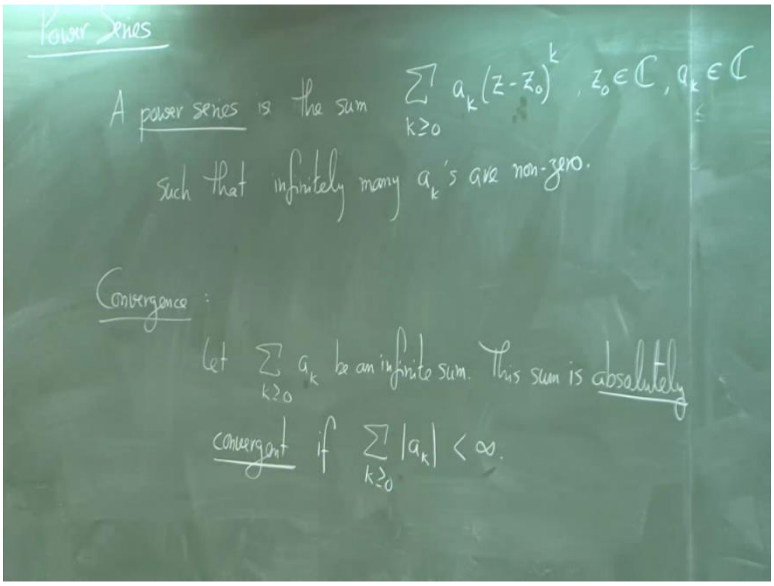

So, let us start our discussion on power series and see, which of the power series are analytic. So, definition is pretty straight forward, a power series is the sum sigma k greater than 0 a sub k z minus z naught to the k. And here I am a adopting rotation from the real analysis. So, this would be a Taylor series expansion around the point z naught. So, I am just lifting that rotation for this and as we will see this will be the right rotation for as well as. So, z naught is some particular complex number this a k are also complex numbers.

And such that infinitely many a k s are non zero, because only finitely when a k s are non 0 that we have a polynomial then we do not have a power series. So, that is a general definition of power series and that is if we just using binomial expansion expand this how. So, we can always write any such power series as sigma a prime k z to the k. So, these are equivalent definitions this makes little bit easier to understand that we are looking at power series round the certain point which z naught.

Now, the movement you have an infinite sum there are all kinds of issues which arise. We have to address the issue that, when is this convergent for any given point z whether the power series convergent at z or not is becomes an issue. So, certainly not an issue polynomial they are always convergent. So, that will be a very important notion for us when we look at power series in terms of analytic functions as well. So, let us put in some basic fact about convergent of power series before we dive into its analysis. So, let us a talk about convergence therefore.

So, there are several things we can talk about convergence, but I will just talk about the notions which are of real interest to as here. So, I will only define two types of convergence. One is a over its sequence of numbers. So, sigma k a k is just a sum of complex number, we say that this sum is absolutely convergent, if the sum of as the absolute value is of this complex number adds up to sum of number does not diverge. Now, for sums of numbers absolute convergence is kind of sanity check, because if a sum is not absolutely convergent then funny things happened this sums.

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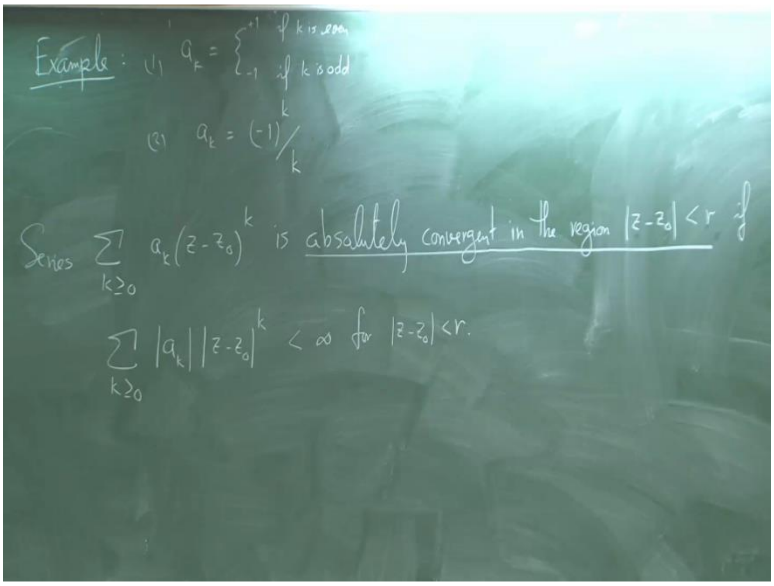

For example, if we look at a to be plus 1, if k is even and minus 1 is k is odd and this sum is not absolutely convergent. Or if you add it up you can start with a 0 a 1, which cancel each other, a 2 a 3 cancel each other. So, you can conclude that is sum is actually zero.

But if you take a 0 out and cancel a 1 with a 2 then a 2 with a 3 and a 3 with a 4, a 5 with a 6 then the sum would be plus 1. So, you can depending on how you bracket the sum, you will get different values that is really nor very nice summations. So, we do not want this. Well of course, if never want this to happened, but it does happened something, but will not want to get it to there. So, we will only try to look at absolute convergence of a series, ok.

Sir.

Yes.

Which is things unlike does range we have occur for all the infinite some feature absolute k convergent?

Not necessarily, yes, not necessary there are, I guess if you look at this sum then absolutely, in absolute term this diverges, but if you add it up I think it actually convergence to a sensible value, irrespective of how you bracket, but I do not take my word for it, convince yourself with there is a guess. So, there are cases actually, when you do not really necessarily need absolute convergence, but that is like, when things are not absolutely convergent then things can be very massive. So, it is pretty safe to look at absolute convergence then the nice things happened ok.

But that is only sum of numbers, what about sum of? Well these are sum of powers of z actually right. So, these are like not just numbers, but these are a whole actually these represents infinitely when you such sums one for each value of z. And we want to say something together about this whole z of sums. We can for example, talk about absolute convergence of such is series are well and which we can define easily.

So, we said that, this sum or this series is absolutely convergent and here we have necessarily have to talk about a particular region in which the value of z lies. And that is typically or the natural way of defining the region and we will come back to this later on to justify this choice, ways to say that z minus z 0 the absolute value is smaller than some number and this is smaller. So, if it is absolutely convergent and this region if whenever you take any such z there is the corresponding sum of numbers absolutely convergence. That is a natural generalization of absolute convergence there to the power series.

Unfortunately, absolutely convergent power series is not strong enough for us, because absolutely convergent power series can times I have behavior which is not very nice. Yes, for every, any z is in this region is sum convergence absolutely. Show you an example of an absolutely convergent power series which is not very nice.

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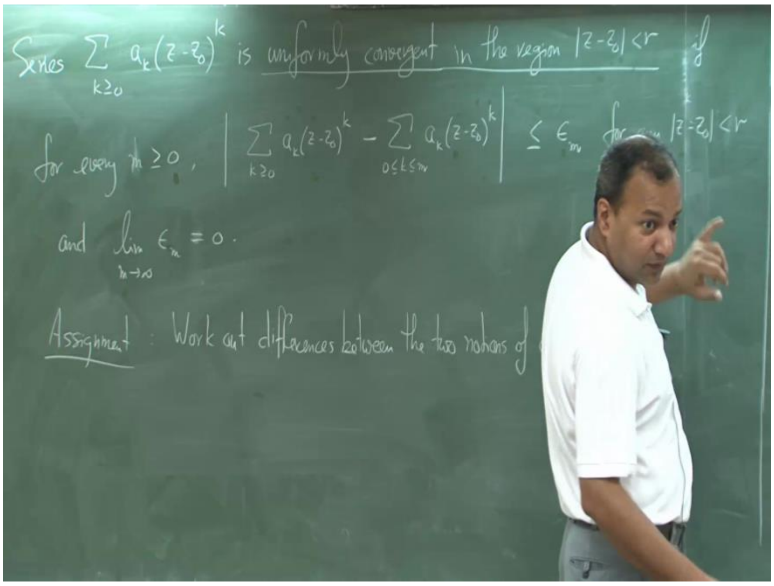

Let me give you the other definition then we will try to see the difference in the two. So, we call as power series to be uniformly convergent in a region, if the following property holds, that for every end greater than equal to 0, if you look at the difference between the power series and the truncation of power series up to the first m plus 1 terms. So, this difference is bounded by in absolute value, this difference is bounded by epsilon m, whenever z lies in this region. So, epsilon m is independent of the actual value that z takes here.

And further, as m tends infinity this epsilon goes to 0. So, in other words what this is saying is at this truncation of power series becomes a better and better approximation of the full power series and this range. Now, I claim that this is the stronger notion on that, do you believe it? Any show that it uniformly convergence series is also absolutely convergent?

That value of epsilon now, how we work there. Let me give this as home exercise instead of now spending too much time on that. So, this is a general statement what I am looking for here is any implication, if you can show that this implies that, that is one part of it. And that this is stronger so; that means one example which is absolutely convergent, but not uniformly convergent. Or if this does not implies this then one example, which is absolutely convergent, but not uniformly convergent. So, just make sure that all bases are count. Good. So, this is the key notion of convergence for us.

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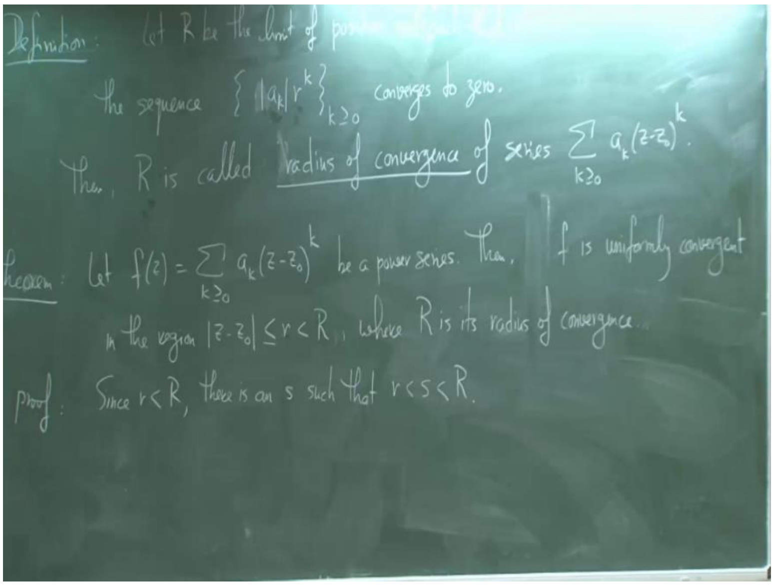

Now, here is a very useful theorem about power series before that lets define. So, let r be the largest positive real number such that this sequence absolute value of a k times r to the k, you just think of these are a sequence of numbers and see where these sequence of numbers going. And if you naturally, if you choose a very small r there is more likely hold and this sequence will actually converges to 0 right. This smaller r you choose more likely you will have that these sequence convergence to 0. So, let r be the largest positive real like that this happens sequence convergence to 0.

Then sometimes it possible that there is no such largest, thing we can always get slightly larger 1. So, I should not say the largest positive real. So, let us say, let R be the limit of we may have to lake a limit, when R is called. So, this tumble is well defined it can be 0, if the a k are grow so rapidly that, quite for example, if a k is a k to the k power z, then no R can send this to 0. This sequence will actually keep no divergent. So, then when then capital R is well defined it is 0. So, small r being 0 will certainly make this example.

If these grow very slowly then you can get away with the may be a very large R also, for example, again if a k is on the other side is 1 over k to the k then you can get away with a every r. So, then there is a radius of convergence is infinite.

Yes.

Why doing one sequential convergence 0 only want a very (( ))?

No, you want sequence each individual number with keep on shrinking and go towards zero.

Well, see this theorem is the motivation. That is the, this number defines the region of convergence for the power series. So, let f z p a power series then for, not for. That f is uniformly convergent and this region, where R is radius of convergent. So, yes it is look a little odd here with by I am just looking at sequence of numbers and seeing where that converge that sequence convergence, but the connection will be fairly straightforward proof is regionally easy here. So, let us show that, this is the radius of convergence so; I should have added one more thing, we will do that may be in the next theorem.

So, I need to show that for any small r less than capital R, if you look at this region the series convergence uniformly right. So, pick a number s between small r and capital R fine.

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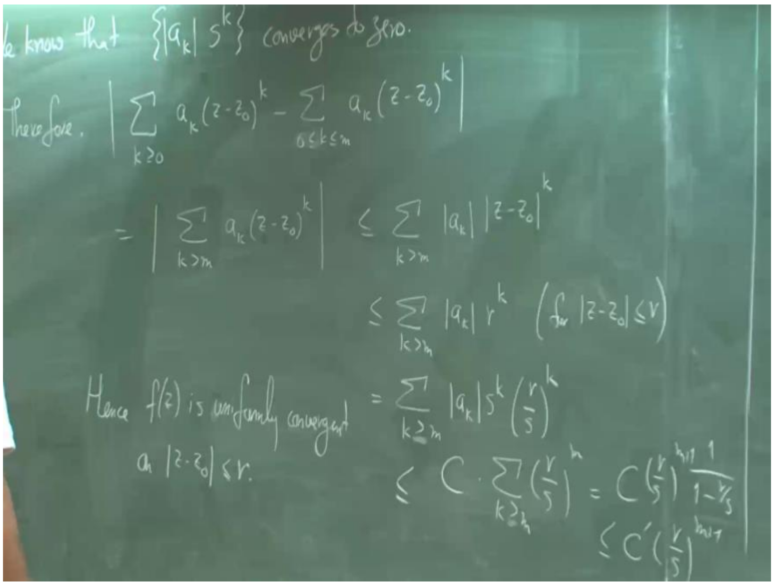

Now, since s is less than capital R by definition we know that this sequence convergence to 0 right as by definition. That is a definition is less than capital R. So, this sequence by definition R is the limit of all positive reals R. So, that this sequence converges to zero.

That is a definition this is I am saying R is, definition R is radius of convergence is of power series there is defined to be the largest number or the limit of the limsup of all the numbers for which this quantity converges to zero. So, this is the definition.

If it is larger, no, it what see if it converges to any positive number instead of 0, they when you add up you can infinite right. You have to take a big sum here. So, you need a to go to 0, not the sum, not the sum, it just a sequence individual numbers. It is looking at limit of the individual numbers ok.

Now so this sequence converges to 0 we know that fine. Then therefore, if you look at this quantity of course, same as this quantity is less than equal to this quantity now; this is less than equal to ok.

Now, went definition this converges to 0, this quantity is a clearly as an upper bound for all k, I can write an absolute upper bound on this quantity. So, this is less than equal to sum C times k greater than equal to m and what is this? This is just a geometric series 1 over this is a fixed number non zero. So, this is sum C prime times r by s to the m. So, this is your epsilon m, which is independent of the actual value the z takes inside that region, that is property one. And property two was that as m goes to infinity epsilon m should go to 0, which also happens, because r by s is a number less than 1. So, as m goes to infinity this goes to zero.

So, that is proof of the theorem yes. How this? So, this converges to zero. So, this clearly means that, there is an upper bound, this quantity for this series is upper bonded by a some number it does not diverge right, it is converging to 0 or converging some number, which means that there is an upper bound to the series. So, let see with add upper bound in absolute value. So, I can write each one of this as less than equal to C therefore, this holds and then we have done, which one? Oh right, right, it should be greater than m you are right and this is m plus 1. This convergence reasonably straight forward fashion uniformly and that which...