Riemann Hypothesis and its Applications
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Lecture – 4

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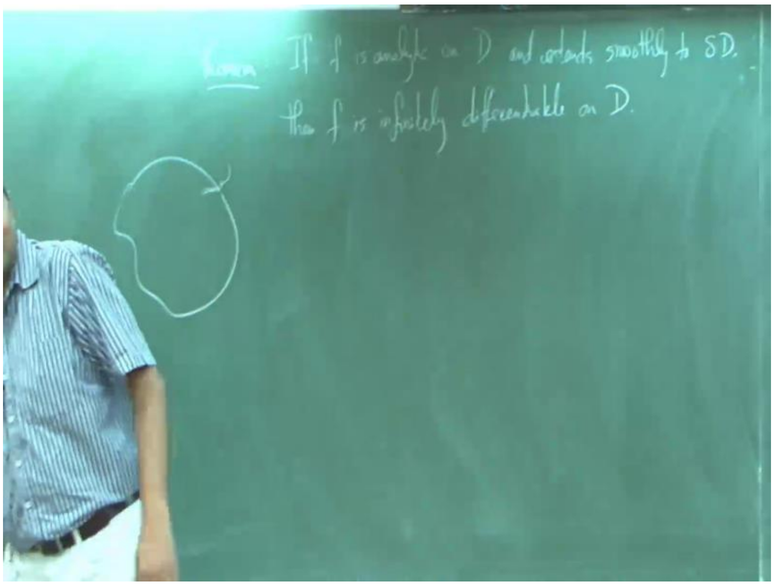

Mathematically I mean that, as you take see f is defined on delta d which on the open set and there is a boundary. So, as you take the limit of the value and approach the boundary from any direction; the value follow any path. So, essentially the continuity that you have you take any path approaching the boundary and look at the value of f along this path and look at the value of f on the boundary.

So, this limit converges to that value, it does not matter which path you follow which. So, that essentially the value of f on the boundary; does not behave strangely f is continues in d bar well that is hard to say that. Because, that would mean that if you are approaching from outside also, then it should be continues that you cannot say f is continues on d bar as long as you approach it from inside then it is fine yes. That is why I said, 1 cannot be continues on d bar, but for as to say its fine.

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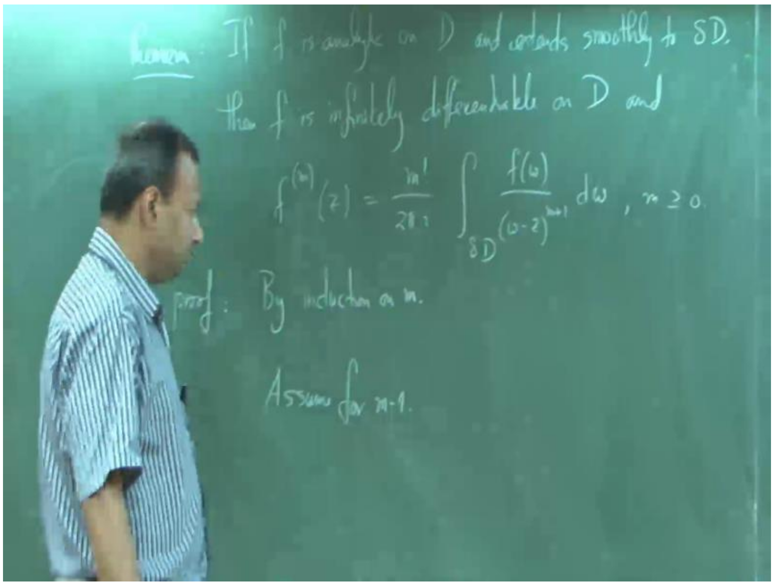

That is the theorem and proof is pretty straight forward almost directly by Cauchy's integral formula and do induction on n. So, for n equal 0 this is just Cauchy integral formula that is the starting point. So, let us assume this is holds of some value up to n minus 1 and now you want to prove it for n. So, now what do we know?

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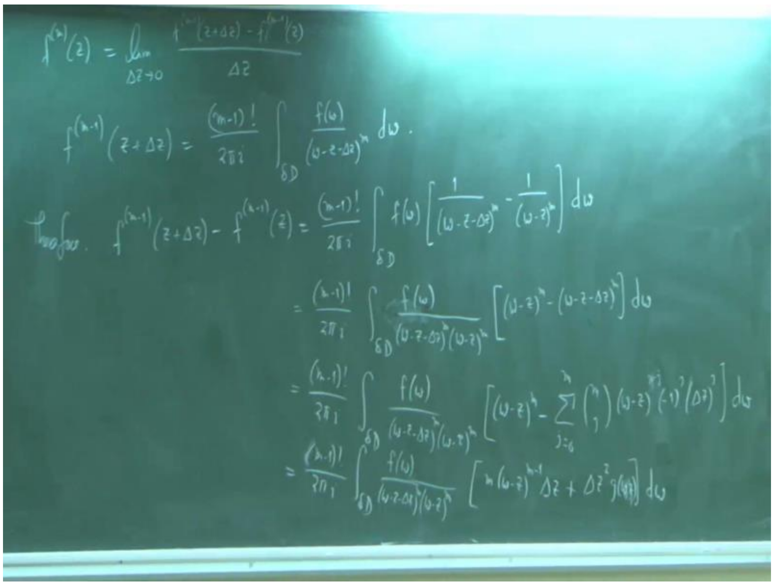

We know that M is derivative of z is defined as right, and of course this limit should exist and should be all things be around only then this is. So, let us because we know that n minus 1 is derivative, not only exits given by that formula. So, let us use it what is n minus 1 z plus delta z that is equal to m minus 1 factorial over 2 pie i integral over delta d fw by w minus z minus delta dw.

Similarly, for f power of m minus 1 and let us subtract the 2 and now let us use, binomial theorem to expand this to we will expand this out. As splitting it as w minus z and delta minus z and expanding this out, we get j equals 0 to m; m to j w minus z vj let me write it as m minus j and minus 1 to j and delta 1 to the j. And if you look at this the first term for j equal 0 is w minus z to the m that cancels the denominator. What is the second term? m choose m which is m, w minus z to the m minus 1 times minus delta z.

And second and higher terms will have delta z square or hirer. So, I will write it as m minus 1 factor m pie i delta d f n w plus. So, this is the first term that scurvies and the rest I will just take as delta z squared times something some expression in w n z and that is binomial expression. So, if we just split 2 integral out what we get is, take this sum as to different integrals or sum of first 2 integrals.

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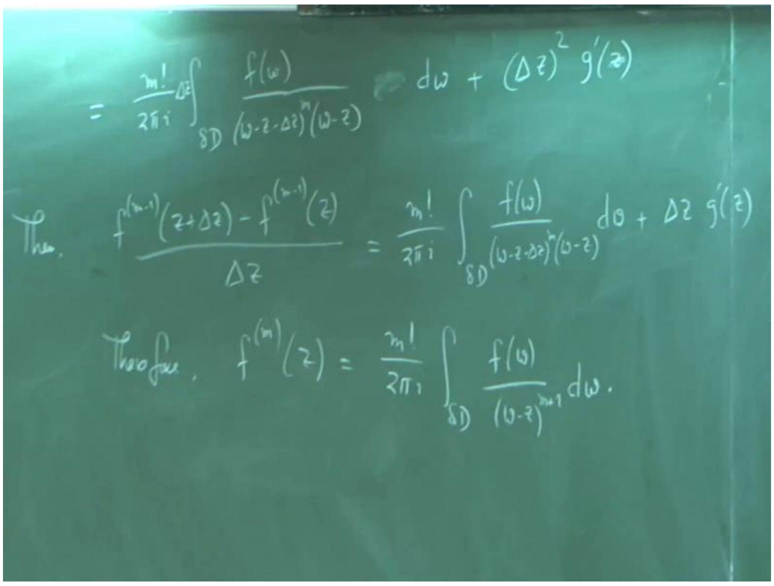

We get the first term would be, M factor l over to pie i delta d and w over w minus z minus delta z at the m and w minus z delta z d w that is what happens o the first term. And second term is in fact, delta z is uncovered because, the integral is over w and for this second term just notice that, its whatever it is it is some delta z squared times something.

Now, knowing that this is the quantity we are looking at difference at the delta z lets divide the right hand side by the delta z and take the limit. As you take the limit delta z going to 0 this vanishes and what happens to this? Again, delta z is independent on integral, so delta goes to 0 you get f to the m z and m to i this goes to the w minus z and so it is pretty straight forward.

So, these are all follows from this Cauchy's integral formula directly essentially, which interns follows the immediately; almost immediately from the Cauchy's theorem about the contour integral. So, elentic function has this really nice property of being infinity differentiable. So, there are no discontinuities or any kind anywhere as long as, the function is analytic and domain of course, it is quite possible that the function is not analytic. But at some point they will get discontinuity, you were at the basic level.

So, we will look at that aspect as well that function being analytic at most of the places, but there will be point where it will not meet. In fact, that is the kind of function we will mostly will be interested in because zeta function is one of that kind. But before we try to understand that let us, you know clean up the basic of analytic functions. So fine, analytic functions are infinity are differentiable.

Then, they have this property that in any domain where, the functions its analytic its value un sided point domain is determined by the integral along the boundary of the domain. And this is Cauchy's formulae theorem that, integration along the boundary of the domain of an analytic function is 0. What 1 can may ask what about the converse the analytic function has this property of Cauchy's theorem; what if a function is 0 when integrated along a boundary of domain? Then, is it an analytic that would be the converse of the theorem; converse of Cauchy's theorem and actually a very strong form of converse actually which is Mores theorem.

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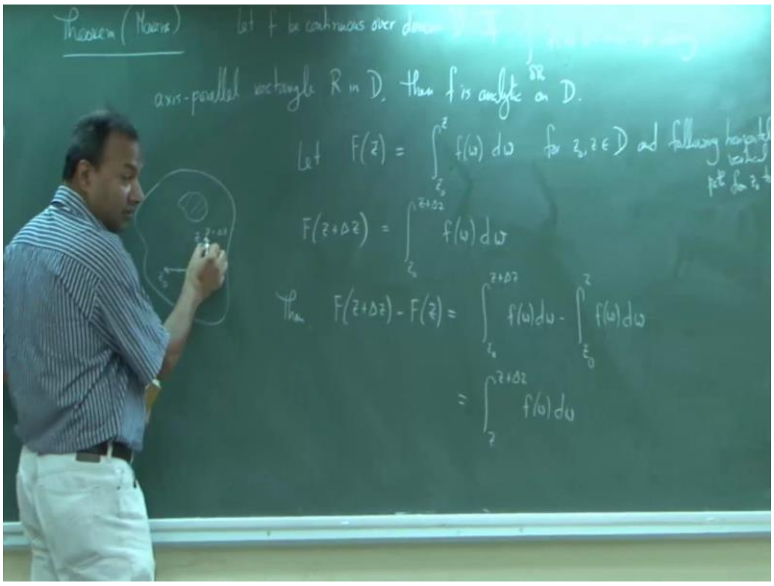

This is very useful theorem, when we try to prove some certain function analytic. So, this cord continues function on a domain that is a minimal we need any way and if this function, the integral of this function along the boundary of an axes parallel rectangle. So, R is in axes parallel rectangle, so the sides are either parallel to x axes or y axes. See for every such x axes parallel rectangle inside the domain D integral this integral is 0 then F is analytic quantity.

It is a strong converse because, if f is analytic on d then for any closed curve inside the domain d this integral is 0; where is here we view only the quads 0 on the along the x is parallel rectangle. Is that clear? Why this is a strong converse? It do not even needed the assumption that f is 0 along the integrated loop. So, and the proof is quite simple, but this is neat trick.

So, what we do is suppose this is a domain D; pick up a point any point in that domain call it z and let us call, it starting point at 0. And pick up any another point here le us, call it z s and pick up yet another point which is very close to z call it z plus delta z fine; delta z will as the expression says, eventually say that z goes send it to 0. Now let us, define a way of integrating between 2 points say between z0 and z.

So, we will how do we integrate between this how do we integrated between this? I am going to see f is given as integrate f from z 0 to z, but whenever we try to do it we have to define a path of traversal from 0 to z. And the path we will choose is, very simple probably the simplest possible; move around move parallel to x axes from z0 until the point we reach just below exactly below the 0 z and move vertically.

So, that is the path we will take if I have to integrate from z 0 to z plus delta z I will do this I will go up there and go up. So, whenever I am going to integrate from z0 to z1 where 2 point z0 and z1 are given it will be this kind of a path you followed. And let's, define F of z to be this integral from z0 to z of F w d. So, this integral well defined since F is continues its finite integral it's well defined.

Now similarly, F z plus delta z is the integral z0 to z plus delta z f w d w and now the path followed is this 1. Then, what is F z plus delta z minus z z this integral or the difference of these 2 integrals where this is following this path and this is following this path. Now, we use the assumption is complete this; the second 1 is negative which means, that the first we are going from here to here right.

In the second 1, since it is negative, so the sign the directions of the traversal is reversed. So, there we are coming down from here to here and then here. So, I can split the integrals that is easy just split this integrals first 1 as going from here to here, then here to here and then going up there. And the second 1 is coming down from here to here and then going from here to here. So, these 2 parts cancel each other out right. So, what is left is coming down from here to here then going from here to here and the going up.

Now, we use the assumption that the long the any closed long parallel axes rectangle the integral of F is 0. So, look at this rectangle the integral of f on the boundary of the rectangle is 0 in the integral there we already have covering 3 sides of this rectangle fine. So, let us add the fourth side also add and subtract fourth side; the fourth side would be addition would be in going in this direction and if you add that then this whole thing is 0 that part.

Then if we subtract it which means, integrate from here to here that survives and then this from this point to this point also survives. So, that is all the survives is which means that these expression the right hand side is equal to z to z plus delta z f w b to the d. And remember this important thing is that, whenever I write integral from 0.1 to 0.2 it is in that mode of traversal horizontal and then vertical; which is in d is followed from z to z from delta z i first go horizontal then vertical.

So, all these things cancel out then ensure that we get this integral.

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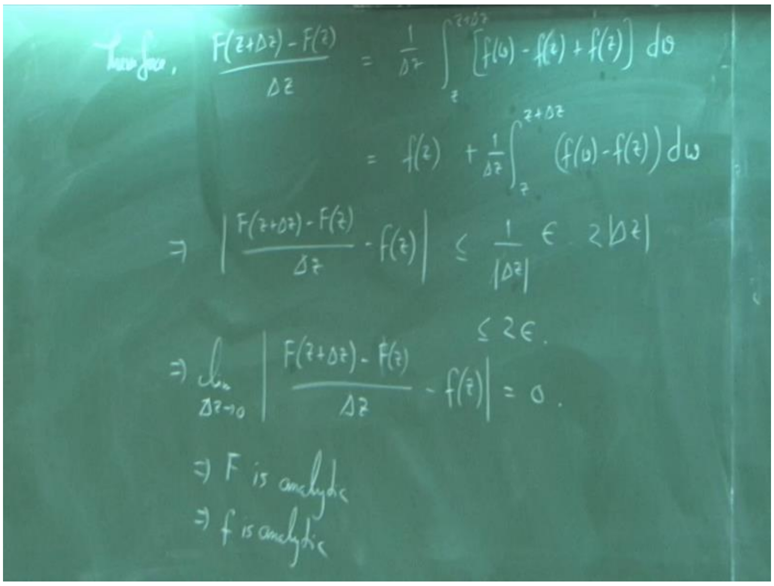

Now, let us divide by delta z what is this equal to see, if we just take this as z and integrate from z to delta z of d w that will only give me delta z I will take that later. So, I am just writing it in this fashion and I then just integrate this part. If you just look at this part, as I have f d w integrated from z to z plus delta that will give you delta z f z delta that will get divided in this f z.

Then, you get plus then take this 1 on the left hand side and take the absolute value now, this I continuity this values bounded by a small amount right. So, let us say a for a small amount of delta z this is bounded by epsilon, where epsilon will goes to 0 as delta z approaches 0. So, this is then epsilon and then this integral now this absolute sense that delta z will have at most 2 times absolute value delta z.

What is that? This right, this is less than absolute what I am just saying the rest of the this will come out and then the just the integral z to z plus delta z of d w. Because, the signs inside the modular’s, so I cannot say that it is exactly called you delta there, but it is bounded by 2 times delta z correct and there for this is less than equal to epsilon. And now, take the limit so; the derivative of F is f that was whole idea of showing this.

So, F is the function which is differentiable on D its straight forward it is there also it is continues on d by its definition actually. Since f is continues on d it just translates this property into the integral which is F. So, F is both continue and differentiable on the domain D therefore, it is analytic on domain D. Since F is analytic it's definably differentiable. So, its first derivative which is f is also both continues and differentiable and therefore, f is analytic.

So, now we get this correct right sum a analytic function in terms of their contort integrals on a domain z where you point on domain D. Because, it is a domain and z not I can choose and which is a let us say it is an open set D. So, any point z I can always by it is a open set there is a disk; which is entirely containing D and within this I can pick any z not by z plus delta z.

Because, anyway delta z has to be very tiny good, and then let me prove 1 more very interesting theorem for analytic function today which will which is again quite remarkable it allows us to extend definition of functions of real to complex numbers. Let us take few examples for analyze functions.

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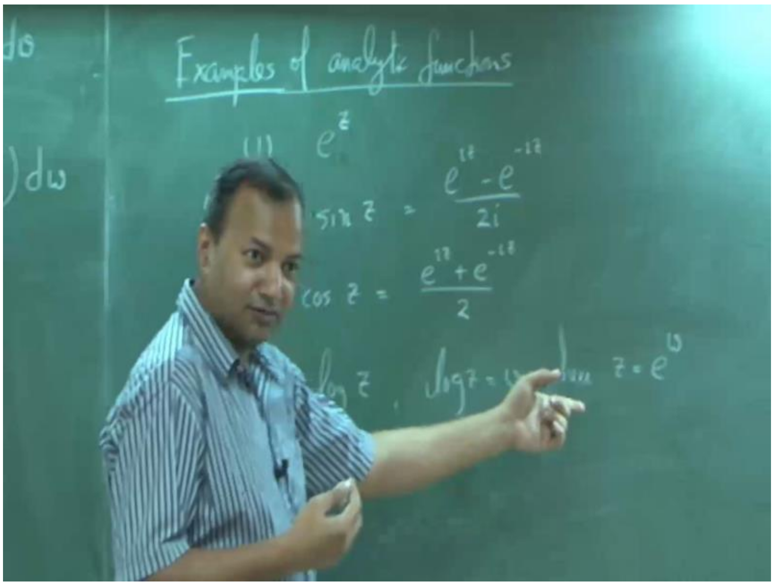

Think we already saw all the polynomials are analytic by the nature e to the z is analytic. Why? Well its continues and differentiable we can define of course, you do it by first phase full, we cannot just extend the lift it definition of differentiability you or real's and plug let us assume that differentiable here. So, you have to just look at the e to the z plus delta z minus e to the z take the relevant of delta going to 0 sign z.

What is the of definition of the sign z? z is a complex number right, that is the way to define e to the iz minus iz divide by 2i. Similarly, cos z e to the iz plus e to the minus iz divide by 2; there the fact that their analytic follows essentially from the fact that this is analytic e to the iz is same as the e to the z its x are different it’s only a. In fact, exact answer is interesting point to note that look at the plot of function e to the z it is a actually a 4 dimensional plot.

Because, z has to parameter z and y and this will also parameter x being this is a complex it is a 4 dimensional plot, but not worrying about the values of this. If we just look at the z plane the xy plane which defines the complex number z and look at the corresponding e to the iz this lot of this is 90 degree rotated from this. Because, z is x plus iy iz is ix minus y.

So, x goes to minus y y goes to x so that, is the just the coordinate shift that happen between these 2. So, which clearly shows that all continuity differentiable these are all maintained is just a simple rotation as signs and cos said therefore, all naturally defined. And differentiable and therefore, analytic this is going to be a very interesting function for us log z. How do we define log z? Well in the real world in the verse of exponential function.

So, can we define it as a log z is w where z equal e to the w that will be a natural definition for log z. But there is a problem in this definition, it is a 1 to many map log z if we define this. Why? Because, e to the z is not a 1 to 1 map which many to 1 map.

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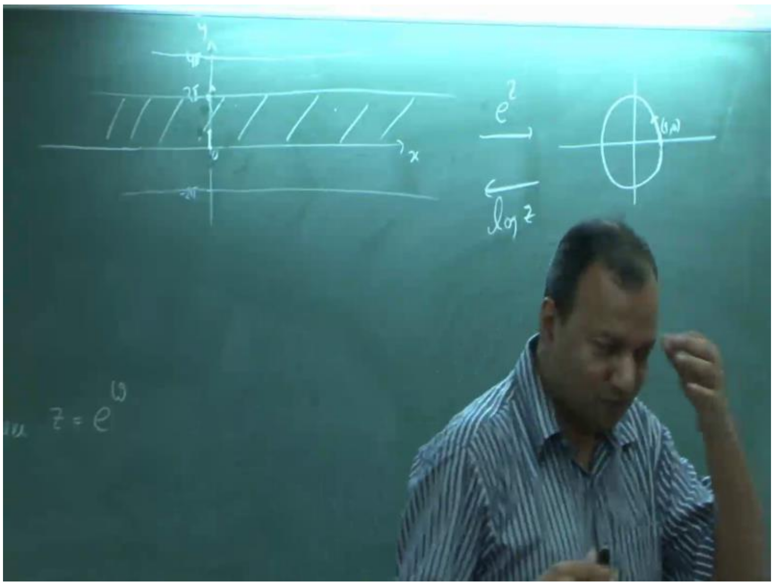

If you look at the xy plane and see what for any z on this plane what is e to the z. Let me, mark out 2 pie 4 pi center here. So, in this trip between 0 and 2 pi e to the z is 1 to 1; however, in this trip which 2 pi and 4 pi e to the z values are identical to this trip. Similarly, in this trip the values are identical because e to the z is e to the x plus iy and as y takes the value of y from goes from 0 to pi is the cycle back right.

So, e to the z function itself behaves is differently exponential function on real in real world is just assume its keeps on going, that does not happen here. In fact, exponential function in real world never take negative value, here it can take negative values. If value of y between pi and 2 pi it is would be negative. In fact, if we think about for a minute e to the z maps this strip to the entire complex plane, except 0 I think it will never hit 0.

But every other value is mapped to by this strip and to this is map to the this again map it repeats the same set of values in maps 2. Again see, its e to the z is a map which is to many maps infinitely when you points to the same complex number. Now, we divide the inverse of the which is log z the question is there are infinity many w's which are for which this holds.

So, which w do we assign this to well you can say that there are infinity when we log functions 1 for each particulars strip right, you can say that first log s when z is mapped to w where the y coordinate the w is between 0 and 2 pi. The second log is when it is mapped to y coordinate between 2 pi and 4 pi third log and there are of course, to do something about negative side also here.

So, which is not a very nicely behaved due to have infinity the log function make sense, the other alternative is to say is forget about others trips we just say that focus on 1 range of values. And this is like a the main strip between 0 and 2 pi see like means naturally we are in more in cline to think about this strip as the right strip for us. There is mathematically there is really no difference between these strips, but for us somehow between value between 0 and 2 pi. And seem more natural than the value between 2 pi and 4 pi.

So, we say that log z is always going to refer to this strip then it is 1 to 1 there is no confusion about the definition of log z. But, then this use also somewhat outer of course, and there is which is fine we can if outer we had defined function, we can define anywhere which you want. But there is another problem there that, this function is no longer very nicely behaved.

What do I mean by that? Even the continuity of this function is not present everywhere. Because, suppose you are mapping this e to the z maps this coordinate from this to coordinate this and log z being the inverse reveres that. So, let us pick up a circle of radius 1 in this coordinate system, this 1 let us start from this point and traverse this circle in canter clock wise.

Let us see what happens to the log value there when we adopt that log z is this correspond to this strip. So, what is this point? This point is 1, 0 what is the corresponding log value to this? Why would be 0 here and x would be 0. So, it is this point 0, 0 actually we cannot include both edge ends of these strips also. So, we just have to take this line and throughout this line.

Because, otherwise 0 and 2 pi will also have a conflict fine, so you start at this point and let us start traversing it quarter point as I said. Now, as it traverse the modules of the radius remains 1 which it would mean that the x coordinate will be always 0. And as we increase go this and the angle goes up which means, we start traversing up y axis. So, you rotate go keep on going, keep on going, come all the way up to here.

So, you reach somewhere here and now in this plane I can just look back here. But happens here? You from here you have to jump back here. So, a continues motion on this plane leads to a non continues motion on this plane; which is a got a nice property here. So, will see how to fix this there is very nice way of fix this and where it works is that as you move there the you keep moving up y axes.

Then, as you traverse cross this x axes here, you will move from this function to the function which maps this plane to this 1 in a very natural way. There is no you do not have to use any artificiality while moving there, that is called analytic continuation will see exactly how that happens. So, it would bring in this inherently many a different possibilities of log z, which I have already first defined.

But instead of making them outer rejoice it would tie them with certain ways of traversing, the plane here and that will bring in this all this copies of are different types of log z functions and lies together. So, but why is log z and analytic we haven't figured it out. We haven't shown that, inverse analytic function is analytic. No inverse is 1 over the function analytic, but this is inverse or e to the z.

So, that is probably the simplest we are doing it, but you have to show that this side is Cauchy's 1. But to do that also we have to substitute a z equals to I y and you then separate it out into complex and non complex parts and imaginary and real parts. You can write it in polar coordinated that is probably better idea yes, you use Cauchy's and polar coordinates you write it in polar coordinate and then check it out. So, that will show that this is an analytic which I will leave to you to verify. And as I said this function is going to be really important to us and the reason for that will see later.