Riemann Hypothesis and it is Applications
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Lecture – 3

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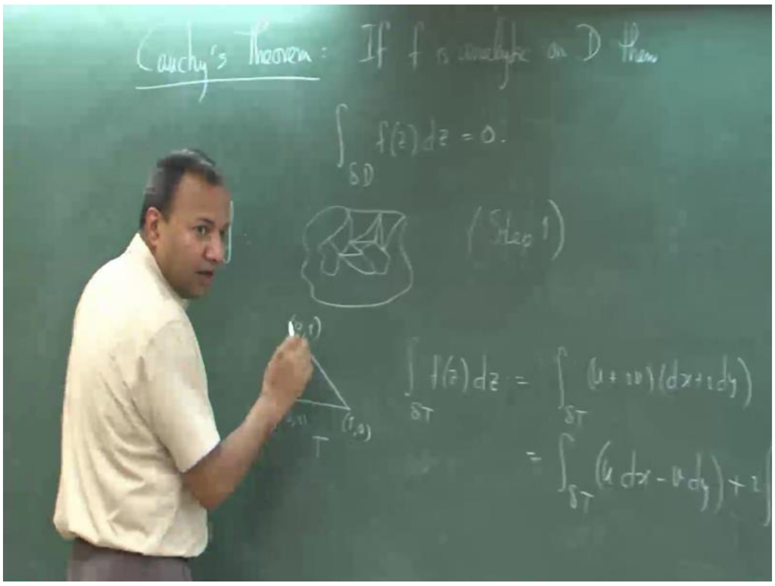

Cauchy’s theorem and we split this proof into three parts; part one to divide the domain. Of course, this is the domain into the generalized triangle not necessarily straight line, but smooth curves that is important the three sides must be smooth curves, so one can for example can do it something like this, and so on you can see that.

Student: ((Refer Time: 01:44)) it is domain is the pending curve closed integral, problem inside the domain then integral 0, because when Cauchy Riemann equation boundary then ((Refer Time: 02:01)) you taking as boundary should be smoothing.

Yes it should extend to the boundary smoothly at boundary.

Student: ((Refer Time: 02:10)) I will prove it, but it may ((Refer Time: 02:13)) since, you have asking that Cauchy Riemann equation ((Refer Time: 02:18)). Now, boundary can be analytic, because once you take the ((Refer Time: 02:26))

Boundary may not be analytical.

Student: ((Refer Time: 02:34))

So, let us prove it carefully then fine, so first step you agree that we can split it in this fashion three shapes and integrate over each one of them and add them up, so whatever is the summation of integral is the final integral. So, the problem boils down to integrating over a three this shape triangle shape and step two is the step one, step two is to look at a specific triangle a real triangle this and look at the let us call this triangle T look at the integral over delta T of z. And this we can split as integral over delta T u plus i v d x plus i d y, which is integral over delta T u d x minus v d y plus i integral over delta T of similar quantity and plus focus on this.

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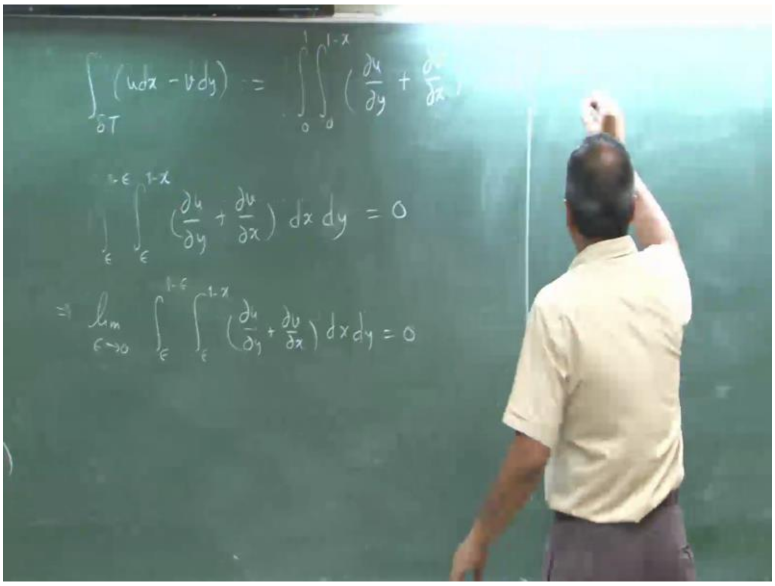

We can write integral over T boundary of T u d x plus minus v d y as integral over a double integral, actually of 0 to 1 and 0 to 1 minus x let us say del u by del y plus del v by del x d x by d y, this should be plus. And now we claim that, because of Cauchy Riemann this is this is equal to negative of this and this is 0, however has you point out this integral is going from not only through the triangle also hitting the boundary of the triangle fine.

So, let us we need to be careful about the value of this double integral let us do the following, let us integrate for x going from Epsilon to 1 minus Epsilon and y going from again Epsilon to 1 minus x. How is 1 minus x little to be careful here x minus Epsilon 1 minus Epsilon and this is Epsilon this is fine this is 0, because this is entirely inside the triangle where the Cauchy Riemann holds.

Student: ((Refer Time: 06:22)) the boundary time holds whose make a ((Refer Time: 06:28)), because that time ((Refer Time: 06:30)).

Within the boundary.

Student: ((Refer Time: 06:33)) inside.

Yes.

Student: ((Refer Time: 06:36))

Right.

Student: Boundary triangles one of the side is completely ((Refer Time: 06:40)).

Yes I agree, so then there will be problem, but this is not a problem this is inside, so now, this holds for every Epsilon greater than zero, so now, take the limit Epsilon going to 0 these functions are continuous as you, because of the smoothness of f on the boundary the extension. So, all we need is continuity to show that this integral when we take Epsilon to 0 remains 0, see as long as we think inside is continuous and we take a limit going to a certain point.

So, as…

Student: ((Refer Time: 07:29))

Yes.

Student: ((Refer Time: 07:36))

Yes.

Student: ((Refer Time: 07:37)) that time ((Refer Time: 07:39))

Yes.

Student: ((Refer Time: 07:42))

Sure.

Student: ((Refer Time: 07:44))

That is inside.

Student: ((Refer Time: 07:56))

So, no let us take extreme case, let us take this triangle and let us assume f is analytic inside, but not at the boundary, but only thing is we can assume is f extends smoothly to the boundary. So, the same objection that you raised are here for all three corners of this yet, because of continuity of these two functions at the boundary this limit holds good, that completes step two, because this other integral you can do the same thing you will get the similar think, but other Cauchy Riemann equation will come into picture that completes step two.

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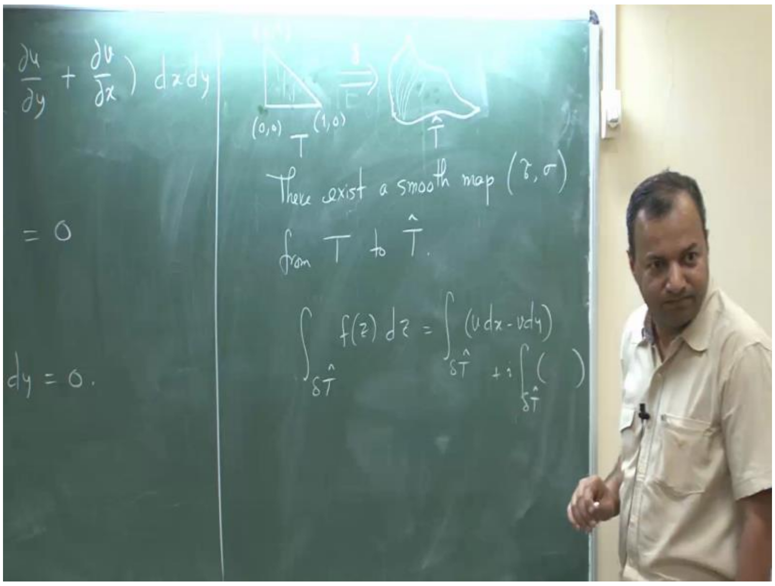

Now, let us final step is that we have this triangle 0 0 1 0 0 1, let us take arbitrary general triangle, which has three corner point connected by smooth curves and what I want to do is to translate this result to this boundary. So, one thing you can immediately observe is there exists, so there is clearly a map from points on this triangle to points on this shape. And this map can be made continuous as well as differential, because everything boundaries are smooth that is important and there are only three boundaries.

So, map this point to point, this point to this point, this point to this point, these are only three non differentiable points, which you may to map to three non differentiable points. And all the other this edge is mapped to this edge and there is a smooth way of mapping this and all inside is inside mapped this edge is mapped to this edge, this edge is mapped to this edge, this edge is mapped to this edge.

So, exists therefore, a smooth map and let us call this as smooth map this is actually a pair of functions, tau sigma from this triangle T and let us call this has T hat T to T hat, how do you prove this, you can formally prove this, but right now I am for only appeal to your intuition. You can just look at real analysis basic chapter of real analysis you can prove this just by using differentiability and continuity of between this and this of these regions, intuitively what you can do is you take this point map to this and just move along, mapping these points along this edge.

And you can stretch this length may not be same as this length you can stretch the map a bit and also keep turning around according to this curve. So, the key thing is neighboring points get mapped to neighboring points and tangents are also moved very smoothly, because this is smooth, if you look at neighboring tangents they are also there values are close to each other, so that shows that at least mapping from this edge to this is continues and differentiable.

Now, once you have fix this then look at points here very close to this, just map the point very close to this similarly, look at this and map it to this you can now look at vertical lines here and map them to lines like this or curves like this. That is basically the intuition here.

Student: ((Refer Time: 12:46))

Is whole domain is mapped to the whole domain they are very smooth.

Student: ((Refer Time: 12:52))

This is the x coordinate this gives you the y coordinate, this is two dimensional map that is all. I am going to say for this map I am sure all of are convinced that there exist some map like this, if you really want to get formal proof go and read some book on real. So, now with this in place let us look at this integral, I want to show that this is you assuming of course, at f is analytic inside T hat. Now, again I will use that to write it as del u d x minus v d y plus I delta T hat and sum, similar expression I will only show that this is 0 and correspondingly one can show this is also 0.

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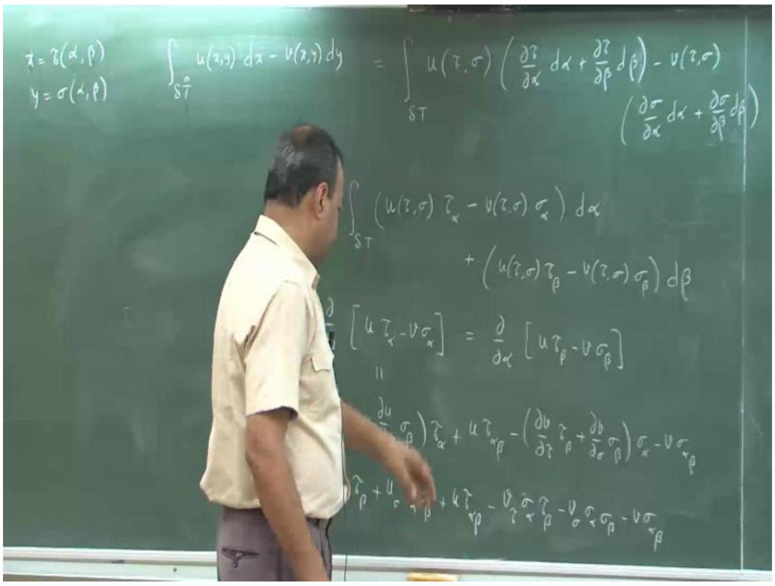

So, let us look at this integral and since going through boundary of T hat is same as going through the boundary of T and as you move along map everything using tau sigma. So, I can rewrite this integral as running through the boundary of t, but before that let us write here u is tau. And let us have two more coordinates, alpha beta x, so here I assuming that there is one coordinate system given by alpha and beta on that coordinate system, the triangle T exit is and I am mapping using tau and sigma to the x y coordinate system where T hat exists.

So, this can be rewritten as u tau sigma d x is del tau by del alpha d alpha this is standard differential analysis minus v tau sigma del sigma by del alpha d alpha, yes.

Student: ((Refer Time: 16:43))

No these are two dimensional maps, so you x both x and y vary for a point, if point is specified by both x and y coordinate and for if point tau gives x coordinate of corresponding fine and sigma y gives coordinate of corresponding, so they both integral depend on both the coordinates. So, have this expression rewrite a bit delta T collect everything d alpha u tau sigma and I will short hand this by tau alpha minus T tau sigma and this is sigma alpha d alpha plus u tau sigma tau beta minus sigma beta.

And now let us recall this integral, u d x minus v d y integrated over boundary of T is 0, as long as u and v satisfy Cauchy Riemann, so the coordinate system has changed instead of x y, we now have alpha beta. So, you have this as your new u this as your new v except that the sign is negative of this is new v and if these to two satisfy Cauchy Riemann, with respect to alpha and beta then the integral will be 0.

So, for Cauchy Riemann the kind of requirement is del u by del y should be negative of del u by del x, so this is my u del u by y beta del beta and need to calculate u tau alpha mu sigma. And need to show this is equal to this del by del alpha u tau beta and if I can show this integral is 0 and everything works out. So, what is this left hand side del by del beta, this is equal to del u by del beta what is del u by del beta, u is over sigma and tau and tau and sigma.

So, I can rewrite del u by del beta as del u by del tau del tau which is tau beta plus del u by del sigma, sigma beta.

Student: ((Refer Time: 20:47))

Yes, I need to show this is equal to equal this. So, here I am expanding left hand side

Student: ((Refer Time: 21:00))

Negative no this already this is v is minus of this and then what I need to show is minus of this, so 2 minus cancel out.

So, it is plus tau alpha plus u tau alpha beta tau alpha is differential beta is tau alpha beta similarly this would be del v by del tau, tau beta. Let us short hand this further u tau this is u tau, tau alpha tau beta m u sigma tau alpha plus u ((Refer Time: 22:32)).

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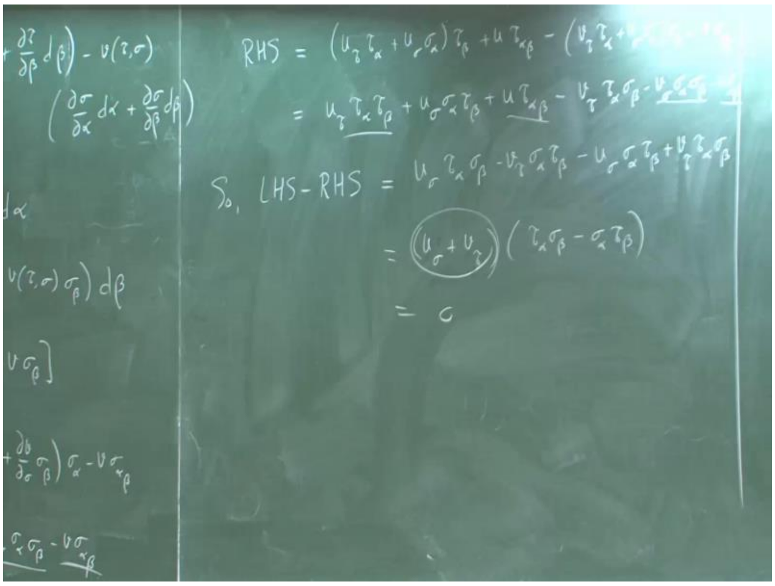

What about the RHS using the same expansion, you get u tau, tau alpha plus u sigma, sigma alpha tau beta plus u tau alpha beta minus tau alpha ((Refer Time: 23:38)). So, let us look at difference between LHS and RHS and see what term cancel out, this cancel this the second term that will cancel, no hat this cancel with this, first term no fifth both side cancel. So the difference u sigma, tau alpha sigma beta minus v tau sigma alpha beta minus u sigma, sigma alpha tau beta plus v tau, tau alpha sigma beta ((Refer Time: 25:55)).

Now, apply for Cauchy Riemann theorem, so this is u tau sigma, so just change the coordinates v tau sigma this differential respect to sigma this differential respect tau ((Refer Time: 26:29)). That complete is, now Cauchy’s theorem is the most fundamental or most important theorem in complex analysis everything that we are going to prove almost everything is going to flow out of this seems very simple theorem. Of course, we can see that it is quite interesting that integral around any closed contour is 0 for any analytic function, but it is applications are quite remarkable and we will see a sequence of such example.

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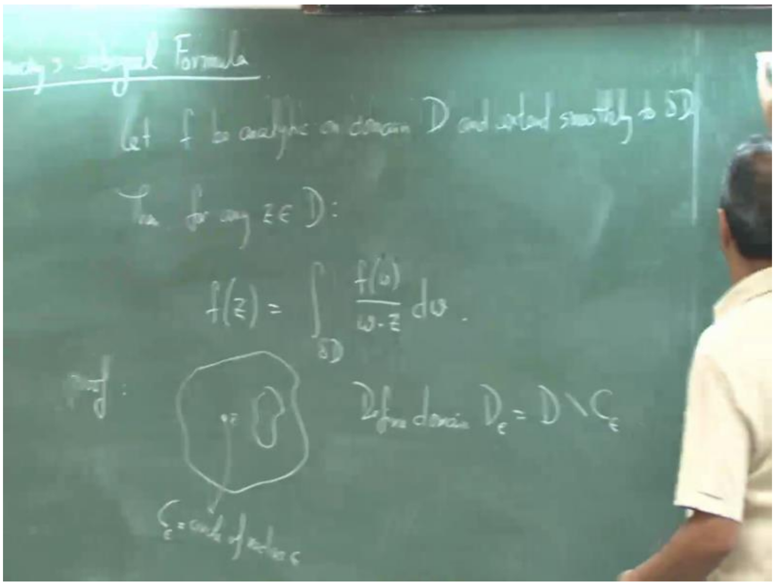

I can show you one immediately the Cauchy's integral formula, so if f z is analytical on domain d and extend smoothly to the boundary this is very standard assumption that we are always going to make, then for every point inside the domain T f and z equal this integral this is integral over the boundary of f of w divide w minus z d w. It is very interesting that integral over the boundary is determining the value inside the boundary, in a very simple fashion all you need to do is divide f by w minus z.

The proof of this is already remarked, almost immediately flows out of Cauchy's theorem, let us see that this function is not analytic inside the domain because of this z is inside the domain when w takes the value z this diverges. So, it is no longer analytic we need handle this except for point z this function is analytic at all the process. So, what we do is this is the domain d and this is point z, so let us define new domain out of it which is formed by striking out a very tiny circle around this point.

So, just cut this out from the domain you get a new domain, let us call the new domain and this circle of radius Epsilon and let us give it a name c Epsilon take out the circle c Epsilon from the domain. And define a new domain d Epsilon, now in the domain d Epsilon this function is analytic.

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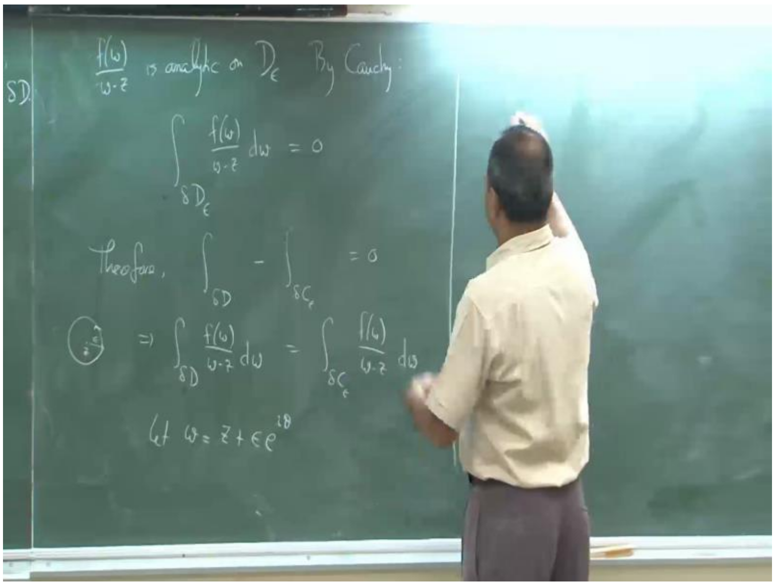

Therefore, by Cauchy if you integrate alone the boundary of this domain you get 0, now let us keep this integral which is along the boundary of d Epsilon in what way does do the boundary of d Epsilon and d differ, the boundaries are identical except for this additional circle, which is present in d Epsilon and the since it is inside. So, I think I remarked this already that the positive, whenever we talk about integral over a boundary we need to fix a orientation to traverse a boundary.

So, for us the positive orientation would be clock wise on the outside, so the intuition is as you traverse the boundary the points inside the domain should be on your left, so that is the positive ways of traverse. So, whenever we say that integrate over boundary of a domain is the integrate over all the curves, which defines the boundary in this fashion that the traverse this curves to ensure the points inside the domain are only left.

For example, this traversal will be this direction the traversal here would be in opposite direction clockwise similar the traversal here will be clockwise, so when you traverse the boundary of d Epsilon that is the direction of traversal of each one of this defined curves. So, I can write the traversal of d Epsilon has traversal of d minus traversal of c Epsilon, because when I say the traverse of c Epsilon that is in counter clock wise, where as in d Epsilon we traverse c Epsilon clock wise that is the negation of the integral and this is 0, notice that I can split it like this, because on this all this boundaries this function is well defined does not diverge only diverges at z.

So, this implies that integral of f d over w minus z w on the boundary of d equals integral of s d over d w on the boundary of the circle c Epsilon, now what is this boundary of circle c Epsilon, c Epsilon is the circle which center is point z radius is Epsilon and it traverses this in counter clockwise. So, and distance between and d is follows this circle, so w is therefore, exactly at distance Epsilon from z always. So, let us whenever you traverse a circles it is always good to change the coordinate system and move into polar coordinates, so we let w be z plus Epsilon p to the i b. That precisely captures traverses the points along the circle w plus Epsilon is the radius from w and into I theta as theta goes from 2 pi and traverses the circle.

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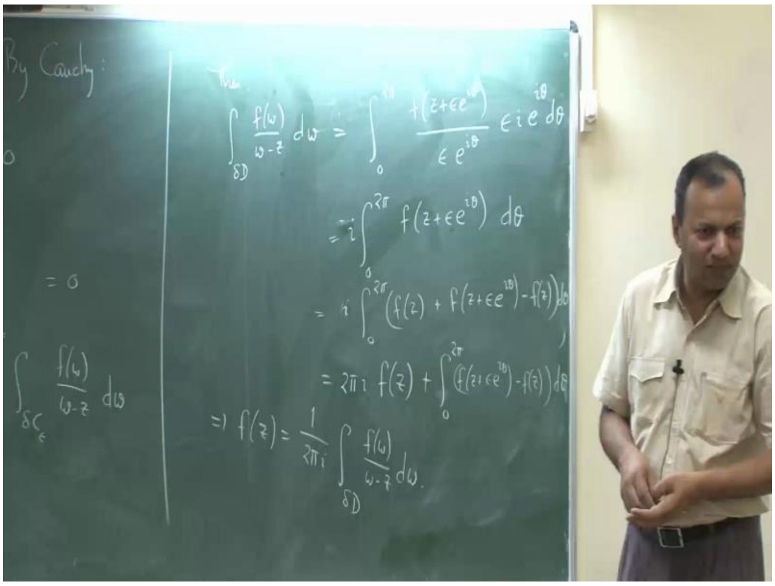

So, then this equals 0 to 2 pi f z plus Epsilon p to the i theta, what is w minus z Epsilon into the i theta this is wrong, I will fix it what is d w Epsilon i p to the i theta this all cancels out get an high outside now, I think you can see the proof. Now, we send Epsilon to 0 as Epsilon tends toward 0, where does this quantity tend tends towards f z when do Epsilon very close to 0 this is more or less f z, so actually let me be more specific lets write it as 0 to 2 pi f z plus f ((Refer Time: 37:25)) Epsilon minus z into theta when Epsilon very close to 0 this is very close to 0.

You can put an upper bond if required upper bond which tends to 0 as Epsilon tends to 0, so this whole integral will therefore, tends towards 0 as Epsilon tends towards 0, because quantity inside this tends towards 0 and integral is finite d theta, this is d theta this is d theta at most is integral value is 2 pie. So, this whole integral will tend towards 0, so and we can send Epsilon towards without effecting integral on the left, so what is left out, therefore is f z is 1 over 2 pi I integral around this needs to be changed 1 over 2 pie.

Yes.

Student: ((Refer Time: 38:59)) the integration of the boundary ((Refer Time: 39:03)).

Yes.

Student: ((Refer Time: 39:06))

No, as you integrate along the boundary this not w is never equal to z however, to apply Cauchy's theorem if you are integrating over a boundary of a domain this function the integrant has to be analytic inside the domain on all point of domain which it is not.

Student: ((Refer Time: 39:40))

How do you formally put a bond, so we use basically use continuity this is continues, so as Epsilon tends towards 0 there is some delta which bounds this value simple continuity delta will also tends towards zero. So, this whole integral in absolute value bounded by 2 pi delta and when you send Epsilon to 0 delta tends towards 0 and therefore, it is closed. A math one on one those of you did this how have done this course one of two, I thought it is continuity Epsilon delta it is in one on one and quite a use to be a very famous course.

Because, the first year students first class they are hit by this continuity and Epsilon delta and half the class does not understand, I do not know how it happens now probably the same. But, this is actually very useful you can see continuity is very useful in analyzing functions.

Student: ((Refer Time: 40:49))

F is analytical around f analytical on z and around z eventually I am going to show that any analytic function on a domain can be written as a tailor series that is going to be a very key theorem on analytic function, but to prove that you need all this.

Student: ((Refer Time: 41:16))

Putting limit inside integral is also difficult

Student: ((Refer Time: 41:25))

Yes, There are things one has to be careful when moving thing inside or outside integral or infinite sums and I am sort of not worrying of them too much, because otherwise just get stuck in that. So, whatever appeals intuitively will just go along there are some points we need to careful about and I will highlight those points. And one point that you raised that was very useful that even though on boundary function is not analytic, one can still extend the integral all the way up to the boundary and prove that good.

So, what is next, so you might say what is what is the interesting part of this theorem, it says that you can get a value of f on any point inside the domain by integrating along the boundary. But, that in itself does not sound exciting result, but what following is really exciting.

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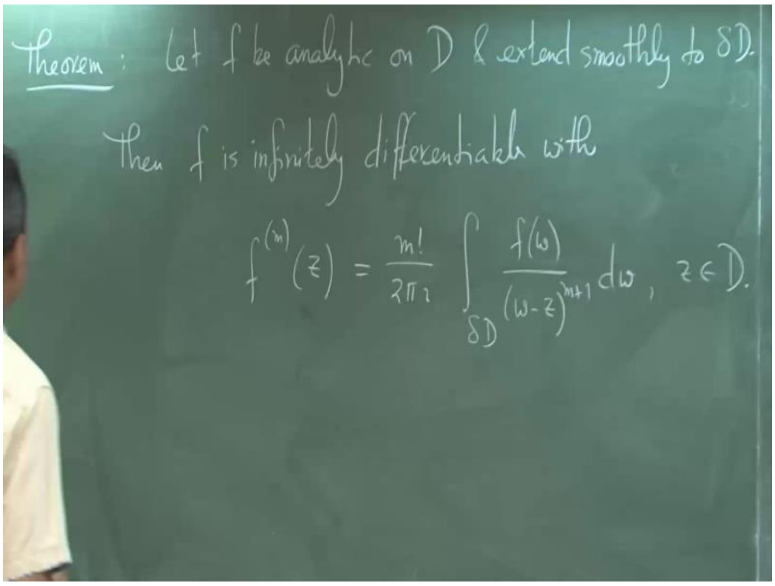

If f is an analytic then x is infinitely differentiable and math differential is given by this integral form again these constants are little suspect, but as we derive this will figure out the constants. This is remarkable, because it is completely different from real analysis differentiable function, which is what an analytical function is does not have to be second order third order function, forth order differentiable here if it is first order differentiable is differentiable all the way.

And there is a nice formula for calculating that as well, nice correspondence between differentiation and integration here as well. That you can calculate the differential using integral or vise versa, actually calculate integral using differential as well as well, so will prove it in a next class.