Riemann Hypothesis and its Applications

Prof. Manindra Agrawal
Department of Computer Science and Engineering
Indian Institute of Technology, Kanpur

Lecture – 27

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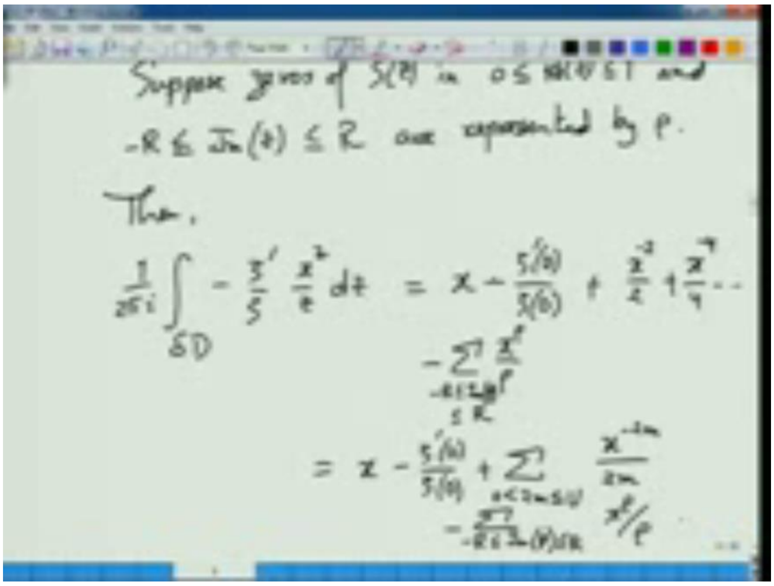

That we had this quantity, which we want estimate and we got the expressions on the right hand for this would you like with quite nice. Because you can clearly see that some of the terms we can estimate x cos x this is some constant whatever it is. And this is the function of x which is efficiently x x increase the wholes on to this will not substantially contribute to the quantity and this is a bit of an unknown. So, this is something we need to understand but even if you understand this. And we have an expression on the right hand side what we have is that this integral equals this expression on the right hand side. And if you recall what you want is that thing a part of this integral form c minus i R to c plus i R that is what is equal to psi x plus general term. So, you want to get real the remaining term here even if terms it. So, let us look at this once again.

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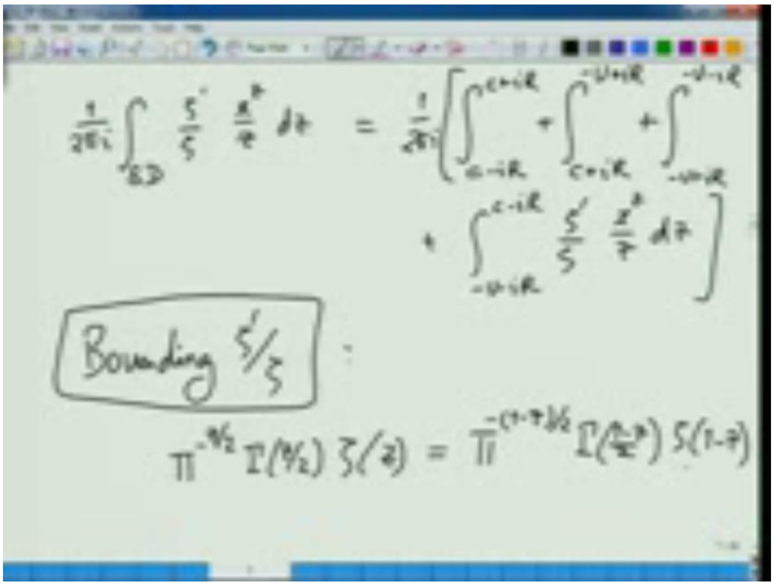

We have delta D this, if you recall was 4 consisting of 4 integrals c minus i R to c plus i R c plus i R to minus u plus i R, this is what we are interested in the first one of this. And so you want to get rate of the remaining 3 and the strategy will be same so, for choose the large number of u and R so right becomes absurd in the error term. Now, let us pick let say this one or any one of them, we last we have them in order to show that the total integral contribution is very small. The standard strategy take the absolute value which is bounded by absolute integral of the absolute values and inside the integral the integral x to the z by the z which you can easily bound using taking absolute value.

Because zeta prime was zeta is something again of an unknown quantity, we want to get hold of the absolute value of 0 phi or zeta in this ranges has in the integral varies. And you want put in upper bound on the value of 0 phi or zeta only that will gives as appropriate. So, we come back to the issue of understanding zeta prime or zeta but now this time we do have bounding. So, to bound zeta prime or zeta we are going to make use of what we have learned about it. The, we are derived some expressions for this in particular if you recall the, that functional equation that was phi to the minus z by 2 gamma z by 2 zeta z equals. So, we know that this equality whole square for all z. So, all that we need do now is take the log and differentiate, what we get?

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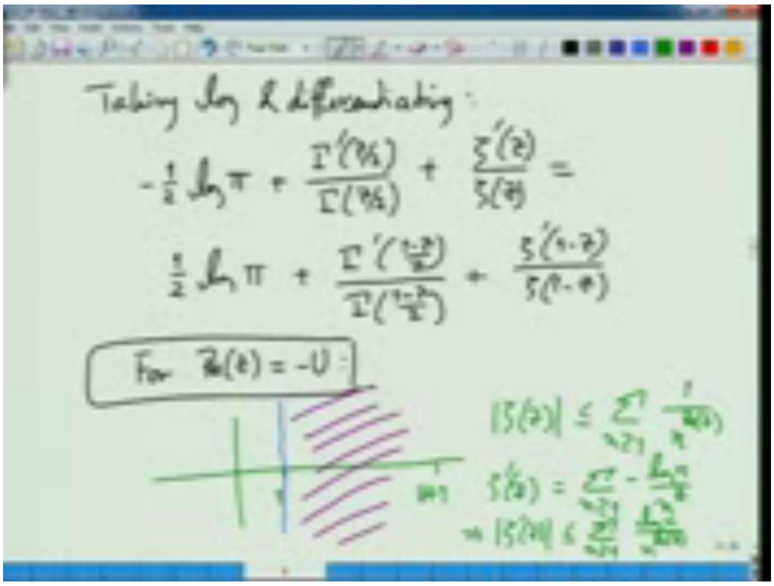

We get phi to the minus z phi to z becomes minus half log phi then we get gamma prime z by 2 R gamma z by 2 plus prime z or zeta z equals half log phi. This is that log make deferential of phi to the minus 1 minus z by 2 plus gamma prime. Now let us pick a one of this integrals lets a this one; this is the furthest both imaginary and real parts of very forward to this should be easiest to bound. So, for that so look at the real z in minus u which you can assume to be very large integrated number now for such z 1 minus z. The real part of this is going to be very large positive u plus 1 actually right now when 1 minus z is the large real number part. That means some they when we look at the zeta function at this point in somewhere far to the right of the complex x'es or imaginary x'es right.

And there we do understand how there is a 0 function behave particularly zeta function there. I bounded zeta at any one of this point is it is it bounded quantity. In fact, if you look at the zeta function this is we are want to study the value of 0 function. And this is one note beyond one for any quantity beyond one when the real part of z is more than that quan more than 1. Then the zeta function in absolute value is bounded and as you in go further to the right that absolute value actually keeps on decreasing, because what you get is this an as you increase real z, the exponent of n cubes increasing and some are decreasing. So, it is the entire on the entire part here beyond this line zeta is bounded an and is bounded by some absolute constant.

Similarly, if you look at zeta prime z, what is 0 prime z again on this side we can use the, this expressions infinite. Some of some expression for zeta and zeta primes are would be what on a differentiate differential on whenever into the z which a same e to the minus z log n you get minus log n times is right. And this gives you an absolute value bound this and again if you see has for n is real z is greater than 1 this is also bounded, because the log on the numerator hardly made the difference. Because n to the 1 plus delta and has real that keeps on increasing this quantity keeps stays bounded actually keeps on decreasing. So, both zeta primes are and zeta z are bounded by some constant in this point. Therefore, in this equation this quantity whenever real z is minus u is constant this is constant this is constant so what we are left therefore, is equals.

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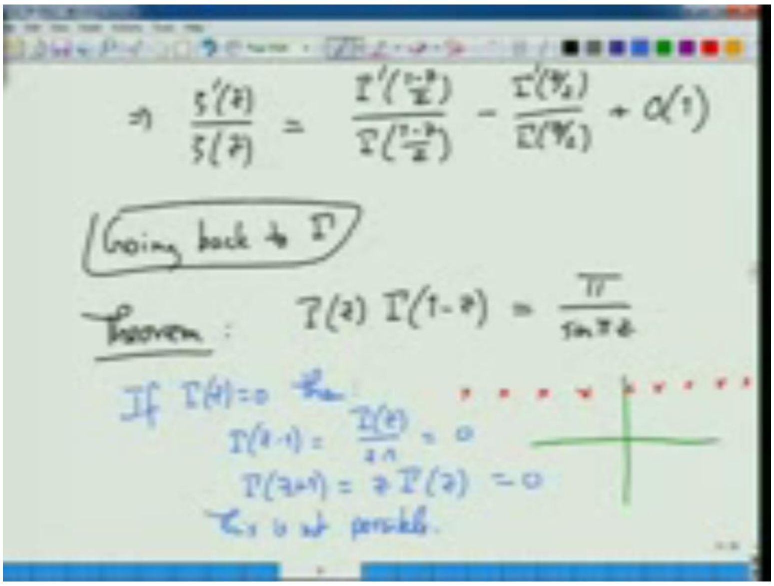

So, whistle of the for frequent bound gamma prime number gamma at all the values. Then we can get estimate bound on gamma and 0 prime for zeta for this part of intimates and gamma is a much better we have function then zeta. So, that cases they handle in getting this bound but still we need to again go back to gamma. So, we started with gamma function to derive the functional equation. Now, we again have to take into gamma function to derive an upper bound 0 prime. So, we have go back to gamma functions what would be a case of this bound. So, what, how does a gamma function behave at least on into positive integers?

Gamma n is n minus 1 factorial and we know n minus 1 factorial be use like is bounded by let us say e to the log n that give some clue of. So, you should expect gamma z to be of the form e to be z log z while if everything we will show that is in deep. In fact, there is something more interest in there we can show what rough there is what we can show. But in order to show that is not to enough to say because on in positive integer is wave like this. Therefore, on entire complex behave like this we have need to actually see behave the behaviour of gamma function on positive integers does not quite reflect behavior on the function on the entire complex. We already solve that on negative integers gamma function is actually is unbounded which is unexpected.

So, will now address in fact there also show that it cannot be everywhere bounded by u to the z log z, because z only for z equal negative integer infinity so z only not upper bounded by. So, we are to be little more careful so, let us go back to the gamma function here. I am going to prove a very interesting property now which is a sort of a functionally gamma function, gamma z twice gamma 1 minus z is exactly phi by sin phi z. This quit remarkable when can do it sorrow quick verification then this equation make since by noticing that this gamma gamma z diverges at every negative integer. What about gamma 1 minus z go to this diverges? In fact gamma z diverges there is equal to 0 also an if a look at z gamma 1 minus z when there take value 0 minus 1 minus 2. It also are take these values exactly z be equal to 1 2 3 4 5 6 should together.

The product diverges at all integers an if you look at the right hand side phi hole sin phi z this also diverge preciously at all integers whenever there is an integers sin phi z is 0. And therefore, diverges by actually something more it also show that gamma does not have a pole at any other point. In fact, this these something high hand way out in the last class when I was trying to show that the 0 free region of zeta function I said these are 0. Because of the poles of gamma function and then also said that gamma function no other poles. Therefore, zeta is no other 0 then this region are in this region all, but I did not actually prove it. Because again intuition of whatever on seems like should hole does not always hole care full.

But this theorem in this proof that the poles of gamma function at preciously at negative integers including 0, because the right hand side an no other poles only integers and this all. So, the poles at least integers except of hole of one more possibilities that gamma z may have a pole an way of gamma 1 minus z is 0 both of the same model. And they cancel each other but this we can show that gamma z does not have a 0. Let us do that in gamma z equals 0 then what happens zeta is go to the functional equation gamma has a 0 gamma z gamma z by 2 is a 0. Then zeta is to have a on by if look at this the zeta has to have a pole there. But that may be in fact we derive the zeta has no other pole using the fact that gamma is a 0.

And this actually zeta class just use this kind of recurrence for gamma function that if gamma what is gamma z minus 1 is gamma z over how was there is no z minus 1 this also 0. Now, here I am assuming that z is not an integer z is an integer I hope the gamma z has nope 0 where we clearly understood so, there gamma z minus 1 0, therefore, gamma z minus 2 0 minus 3. So, what all the way is 0 similarly, what about gamma z plus 1 that z time to gamma z that is also 0. Similarly, so gamma z plus 1 z plus 2 z plus 3 z plus 4 all of them are 0. So, that entire if say z was here and this is 0 then all this point separated by integers are 0 now is that possible?

That is not possible why I just leave this bit of an exercise. What one can show is that the one can show is that gamma z at large enough point is going to be at least as large as gamma at one of the integer point below it. There is if the real part of this is very really large.Then this value gamma at this value in absolute terms is going to be at least as much as gamma have to one of this smaller integer points and which we know is not save k. So, I will not do that simple manipulation just use a definition of gamma function to derive it to this not possible. So, hence gamma has no 0, so it does not have certain number of poles. So, let us no proof this lecture.

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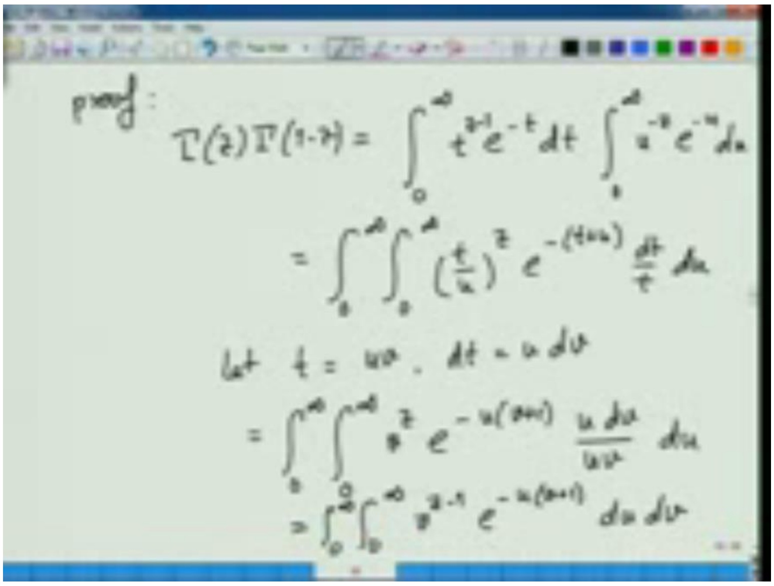

Will use the definition gamma z 2 times gamma z minus 1 equals what is gamma z t to the that the right definition no there is z minus 1 here. And now listen to the variable substitution hmmm t equals u v assuming here u to be a constant D t is u D v. So that way, integral happening inside this part. So, that D u is outside so u is variable outside so as t varies from 0 to infinity v will also go from 0 to infinity and t y u is v. So, we get v to z e to the minus u v plus 1 D t is u D v and t is u v du. So, u gets cancelled out here and what we get is u to the z minus 1 e to the minus u D v now this playing around has the idea is that now u can like do one integral easily. If you look at the variable u, the only occurrence of this in the integral and the e to the minus u v plus 1 and we can integrate that. Of course, we could exchange the order of integration of that we do there are standard theorem we should guarantee that u can do this.

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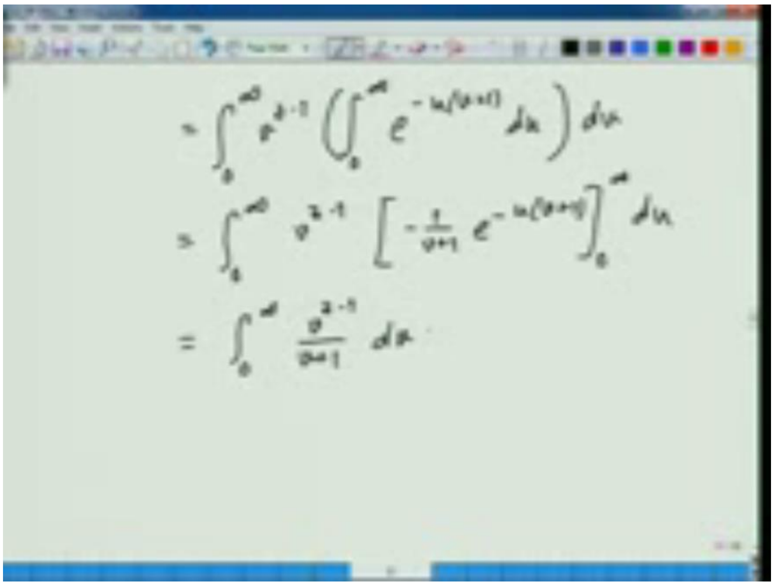

And we get v to z minus 1 what is the result of the integral e to the minus u time v plus 1 that is going to be minus 1 over v plus 1 e to the minus u v plus 1 right. And when u 0 this is when use infinity this is 0, 0 then this is minus 1 would be plus 1 then there is a negative sign. So, that x is cancelled and we are end up now we have to carry out this integrate here v to the z minus 1 here z is is a complex number v goes v is transfer into real from 0 to infinity.

So, this looks like bit of a integral because there is if only this was only not v plus 1 it was it was just v. Then put become trivial where since here is v plus 1 we can either do it through trigonometric functions So, you re substitute for v is something like tan square theta 1 plus tan square theta gives you something. And you will get from something and simplify this gets being clever. We know a non clever way out doing this integral which is again through complex analysis. So, do evaluate this integral what we do is.

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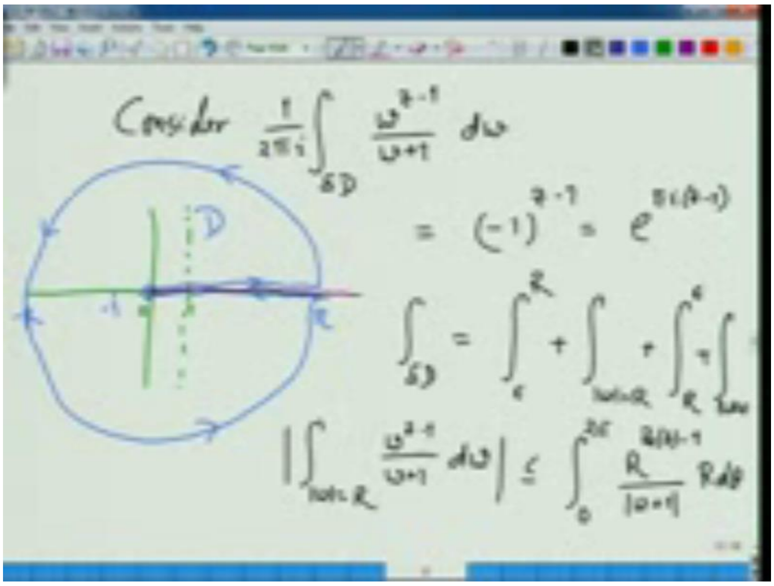

We consider this over a propriety chosen domain now what is the domain that I need to take that question depends on integral as well we are interested in this going from 0 to infinity. But if you look at the integrand is it analytic over the entire complex may z being a fix complex number onwards. If z is a interior than s it is analytic over the entire complex then except for a pole at w poles but z is not an integer. Of course, if z is negative integer an another pole at w called 0 but otherwise z is not an integer or example suppose there is 3 by 2. Then in the numerator we have square root of w the square root of w is not an analytic over the entire complex may we notice this long ago.

Because the phase change as move around in a circle and end up of with sort of move up to that their ugly with associate remount surface which goes up. And then square root it one more round will take you back to the original remount surface. But here it can be a outlawry complex, but in general is not going to be a analytic. So, there is a standard way of analyzing this which is again putting a cut on the complex plane. So, here the cut that we put is here starting from 0 to the entire real positive line we cut out now on this domain this function becomes analytic. Because now there is no problem of 2, 2 different values at this point incrementally easily define the angles or arguments around this and without any problem of getting.

Of course, as increase from here and go around here the argument will keep on increasing, but it will never will not get 2 argument on some time because this line will cut out. So, with this domain we know that the integrate is analytic on this but we need to choose a domain specifically for this over which we want to integrate. And that we do the following once we what is the combine square root of value what is cut? This cut works for any search w to z whenever you have this duplication of values. You should take out one cut starting from origin all the way along any direction actually.

Then basically a over to cut out is all the rotations around the origin which is the path causes this duplicate values besides. These points you can now associate the unique values same thing with log z. Then if you cut out one line then you can uniquely define log z and are not uniquely define then they still exist many definition. But at least there is its with every definition is analytic on the they deduced to domain we choose is oh this is minus 1. And we pick a large number R here and corresponding minus R here and take a circle here on this of course, this circle will not complete because this line is cut out.

So, just before this line this circle is start and makes a tiny circle around the origin, because origin is not also, not available and then comes. So, that is the domain what is the value of this integrate inside this domain this given by the cassis's formula there is only exactly one pole assuming odd is large enough. So, they enclose is minus 1 there is exactly one pole of the integral in this domain. And so, they this equals residue at that pole and the residue is will multiply with w plus 1 and take the limit w going to minus 1 minus 1 to the minus 1 which is e to the phi i z minus 1.

Good so, we know the integral and know we spilt this using all standard method into this integral into 4 parts epsilon to R. That is this part this line plus this integral around the circle w is equal to R. Of course, this is taking slightly different way, that do not complete the circle you just stop you did before. Then you go from R to epsilon again these integrals are taking to mean I am omitting this slide here because you do not really have the real line available to you. So, the numbers cannot be exactly epsilon and R it has to be epsilon plus a tiny imaginary amount to R plus a tinary imaginary amount right. And similarly, this circle will start from that and then come back, but stop just little before R so R minus tiny imaginary amount epsilon minus tiny imaginary amount. And finally, around this but this is no counter clockwise sorry, this is clockwise these are counter clock wise.

Now, let us bound the ways some of these integrals so that this integral around the circle look at the absolute value this would be absolute value w is R is there. So, just take absolute value the, everything inside first doing the substitution the w is R e to the i D theta. So, the integral goes from 0 to phi again little more than 0 little less than 2 phi. And this is the w to the z minus 1 the absolute value of this is bounded by this quantity R to the real z minus 1. And could this absolute value here and D w is R D theta and when take here take out the absolute value so we get this.

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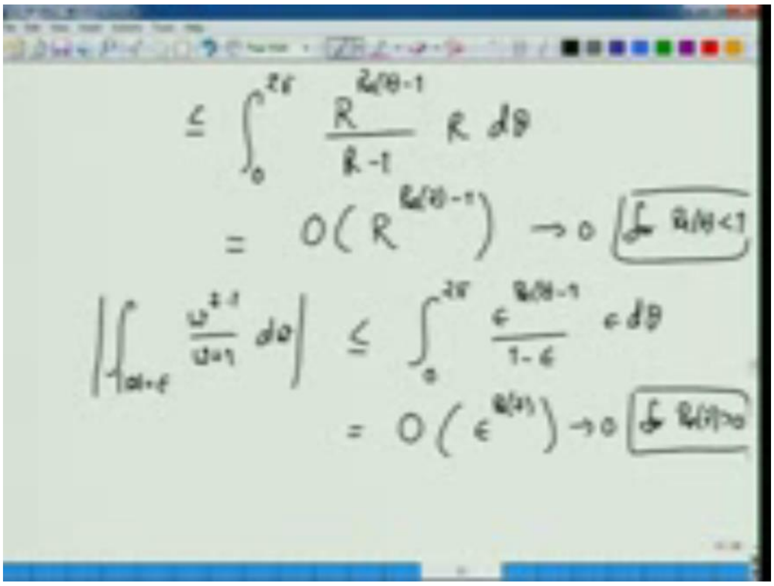

And this is less than equal to 0 to 2 phi R to the real z minus 1. Now, look at this denominator and things we are looking at upper bound, what is this smallest value? smallest value has to be at least R minus 1 right it has to be at least this much. So, this is upper this is all when R is large R R minus 1 take care of themselves R to the real z minus 1 and this goes to 0 provided for real z less than 1. So, we will choose our z when integrating this so that real z is less than 1. And this is something we need because we are not interested in this particular integrating. Similarly, if you look at the fourth one and absolute value of this less than equal to again 0 to 2 theta epsilon to the real z.

So, we have epsilon to i theta again absolute value of w plus 1 now what is the smallest value of w plus 1 now again 1 minus epsilon in the same. And then epsilon D theta again following the same structure until now we take epsilon to be very small. So, this is actually order epsilon to the R e z real z, because 1 minus epsilon is close to 1. So, just multiply epsilon to this epsilon to the real z this also I want to send to 0 and this will go to 0 for real z greater than 0. So, which means that here so that z we pick in this strip any z in this strip is fine. Now, let us look at the remaining to 2 integrals the first one was epsilon to R second was epsilon to R to epsilon.

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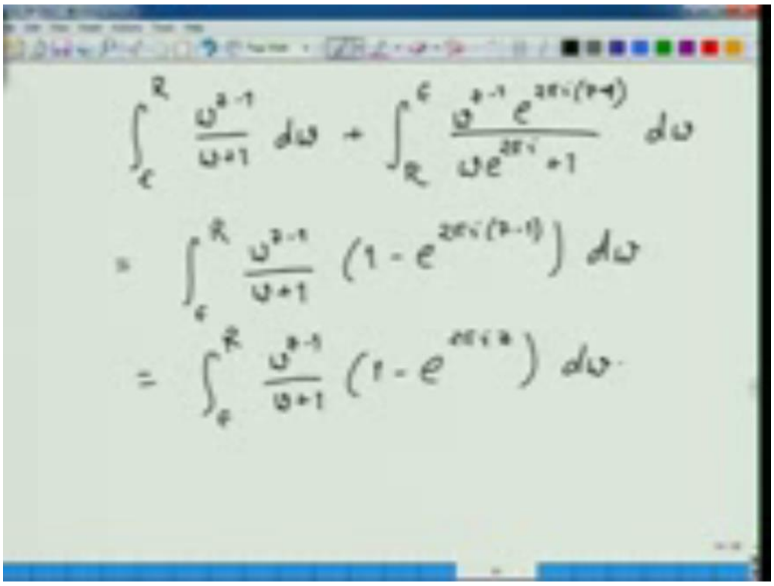

Plus R to epsilon now, here I going to be little careful analysis here, because what happens, what is happening here is here you see that w is starting from roughly 0 epsilon. I am going to R as you were expect normally, but when you come down here. And w is going from R to epsilon its actually not R to epsilon its R e to the 2 phi i little less than that actually to epsilon times e to the 2 phi i little less that is the distinction I am going to make. So, when i stick that in basically I will get of course, we end this is taken with the little bit of conservation here But that conservation, we can make very very small essentially make it balance so that we need not worry too much about that.

So, that is the some of the 2 integrals that we are left with and if you just changed the order and add the 2 things of 1 minus e to the only thing that survives has that should as you would expect is that this because when if z was an integer. Then this will not survive this will also go away within this as normal, but if z is not an integer then this will not go away this will give a h shift. And now take the limit R going to infinity an epsilon going to 0 where there e 2 e 2 phi I is 1 w repairs w by e to the w time p to the 2 phi d w is d w time e to the 2 phi e to the e phi i is one that is good what? In fact, one part we can simplify is the little bit of course, if the minus 1 is goes away. And now taking limit epsilon going to 0 R going to infinity we get the integral that we want and this is equal to.

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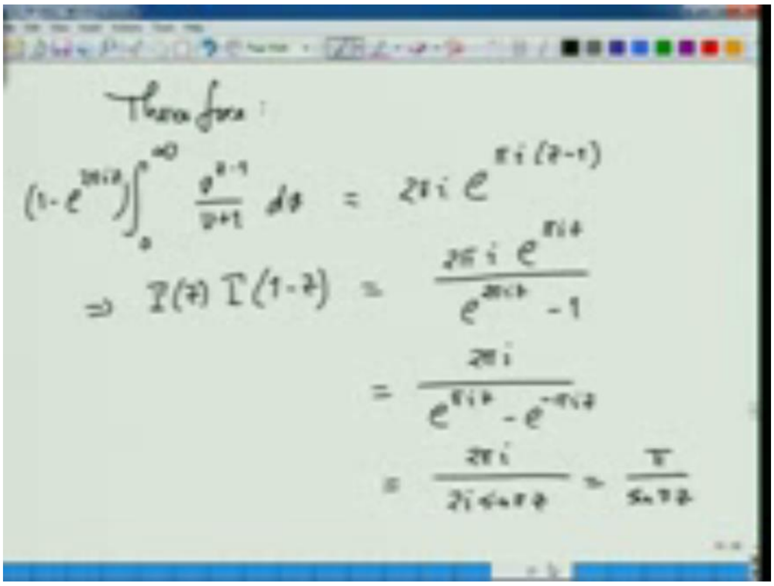

That we wanted was 1 minus e to the power 2 phi z 0 to infinity this is equal to, because the left hand side there is precisely equal to the integral when w goes from R a epsilon to R then there is real actually. So, we can just replace w by v this equals we just drive this formula at this is equal to this integral and this part was equal to that integral. So, 2 phi i times this what we are looking for so what we get is thou z gamma z gamma 1 minus z equals 2 phi i phi i z. And this is equal to e to the phi i z minus e this we know this is mistake the corresponding cos in sin terms we get sin phi z minus minus 2 i sin phi z.

And therefore, but notice it this whole thing the entired analysis required z to be in the strip. So, what we have so far so an z whenever z in that strip between the 0 and one. Then this equality holes now in your left hand side is the analytic code their entire complex frame expect for pole set at negative integrals. The right side is analytic at all and entire complex when expect for pool set integrals. And they agree on the steps now you will your uniqueness of analytic continuation they must agree away. So, we have now this question and we want to now coming back to our original problem which was of try to get a estimate of gamma prime over gamma.

We want to use this to get this estimate hole it is not quit-sufficient to get there estimate yet, because we have only a product of gamma 2 gamma values giving this. And we want in fact, getting a single gamma because you can take the log and differentiate. But that will you gamma prime over gamma of z plus gamma prime over gamma 1 minus z. And there combine some is bounded by some quantity this is this we know how this bounded it right? That is not quite-sufficient, because it still observe that may be still diverge which might give as some strange things.

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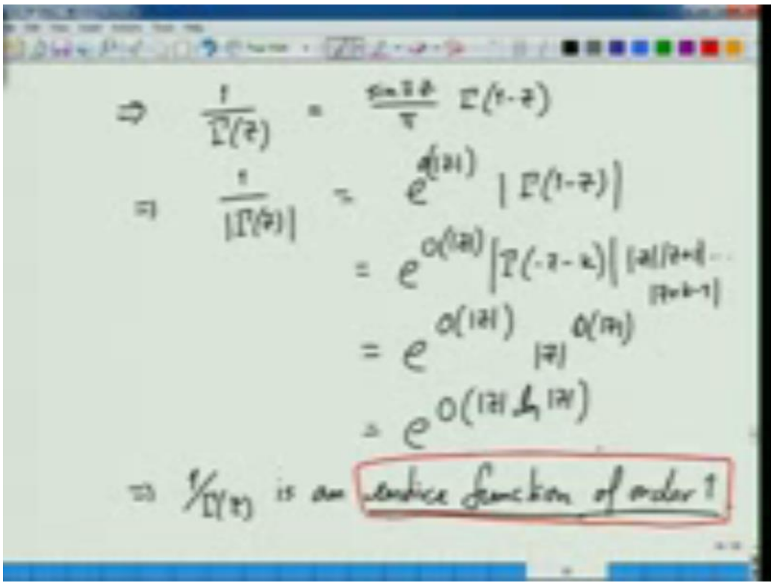

However, what we can conclude from this is this and this gives to look at the absolute value. What is the abs sin phi z is absolute value sin phi z is the difference of 2 exponential form e to the i phi z and e to the minus i phi. Then we take absolute value they are grow like e to the odd of z right hence gamma 1 minus z. How does gamma 1 minus z go there is upper bound to this well we can do that the following way you one as look at the real part of this. So, if 1 minus z have a real part which is positive than you bring it below using the recurrence equation suppose positive. So, then 1 minus z is positive we want to keep on subtracting at once to from this.

So, we want to go minus z minus z minus 1 and so on something like this at once suppose k is such that the real part becomes very small. Then we can again work that that property of gamma function which I just ask you to that gamma of z for and a real z is positive is sand wedge between the 2 gamma values which are integers start to for away.

So, when z the real part this is very small and then it is small number and absolute values. So, it can this write this as e to the odd z and ignore the that part and how much is this z z plus 1 upto z plus k minus 1. This is product of how many of z's k of them and k is what k is bounded by absolute value of z right?

This is something like z to the odd z. Now this is interesting because this steps at one over gamma z grows is upper bounds to go always upper bounded by this which is something you can completely believe in. Because gamma has no zeros it has poles, but 1 over gamma therefore, as 0 as bound. So, there is its always upper bounded by some quantity and off course it has no poles. So, this means one over gamma z the entire function of order 1 I did define a entire function. You remember what are entire function is entire function are analytic over the entire complex no poles those are entire functions and order 1 is something I am not define which I will define now.

So, now off course, we must realize that I never mean job which was to no say something about the prime counting that keeps getting interrupted I do something. Then interrupt go into take diversion we can come back do something more interrupt another diversion. Now, we are in another diversion which I analyzing gamma function and now this is the am going now I am going to the second round of diversion. There is interrupt the analysis of gamma function and gridy diversion into the theory of entire functions put I limit myself to entire functions of order 1. One can easily generalize it to entire functions of any order an if you seen inception. So, it is bit like that not quite as complicate, but still like that you pass level and go downer level and then do things that. So, we come back I promise that I will come back all the levels and closer by the end of the. So, what are entire function? We know entire functions.

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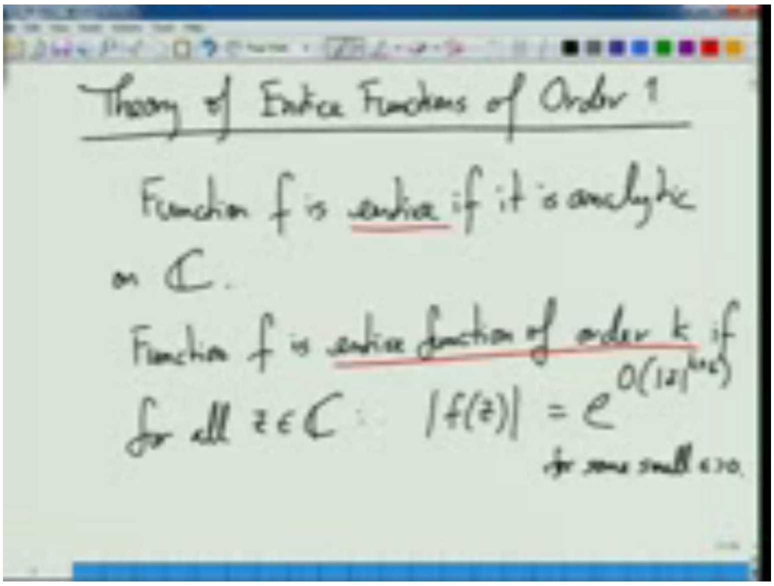

So, f is entire if it is analytic on the entire complex, will say f is a entire of order k. If absolute value of f on z is bounded always by e to the order mod z to the power k allow little more than k k plus 0.1 something. So, this obviously means its entire, because its head of v point on the complex plane its value is bounded. Now, off course, which means the entire function what are one R when the case 1 this functions are the most well behaved analytic functions. And therefore, we can say a lot among them this simplest one of these are entire functions what are assume those one can show.

And the when I to the analysis for order 1 you can use those tools to show for order 0 that this functions are polynomials grow as a polynomial of degree D will grow as about z to the D modulate to the D which is in the exponential. It is exponential it is e to the D log z an log z is bounded by n z to the epsilon. But we are not interested in that we are interested in that function now what we are target is to derive an expression for entire function of order 1. And will see that entire function of order 1 have a varying fixed expression then practically write on what the function does like the before I prove that let me further specialize.

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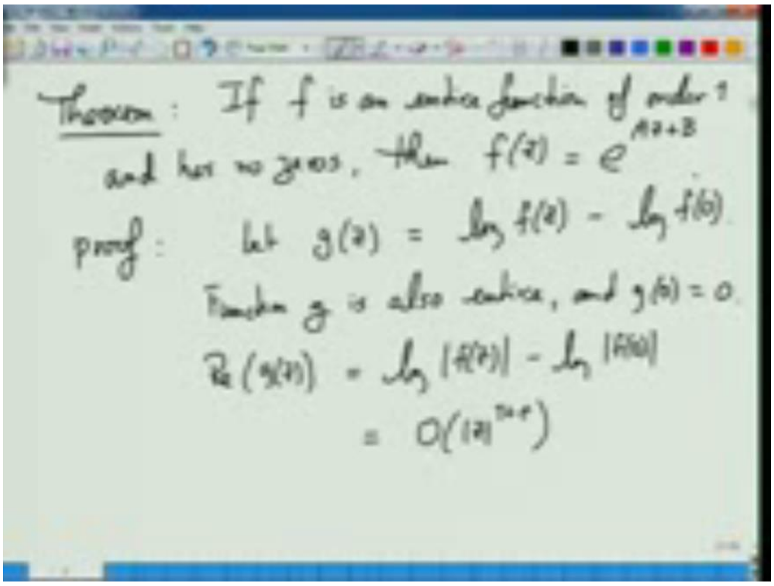

If f is an entire function of order 1 and has no zeros then f can have where f will have precisely this form e to the A z plus B for some A z A and B B complex number complex number. Clearly this is an entire function of order 1 e to the a z plus b and it clearly does not have any zeros. And this theorem says that that is all infinite R function is order 1 and has an all zeros this is the only possibility. And the proof is quite interesting is not straight forward and define the function g as log of f minus log f log of z minus log of 0. Now, whenever we take log with some start ranging because log is not such a symbol function here.

So, but it turns out because of the fact that f has no zeros any where the log is define very nicely see what, why because log causing a problem. For example, log z log z was causing a problem there if it travels around the circle around the circle around 0. You do not end up at the same point; you go level high if u think of the it in terms of lower surface and keep it higher. That was if u think about it carefully with I will not explained it too much. But if you think about it you realize that is happening. Because we are travelling around 0 which was 0 where z is 0 that is we are looking at log of f z. And we are travelling around a 0 of f z and there is when all the funny things are happen if we took log z function log z itself.

And to get made the circle around z equals 1 the small circle around z equals 1 everything will be perfectly defined and take these. Because again if you think about the in terms of even surface if u just travelling around the z equal 1 you will not change the level you will came back to the same level you only change the level when you travel around z equals 0. And that happens because z is 0 at that point. Now, because f has no zeros the log is very clearly defined these are no funny things happening here.

So, we are actually we do not to actually assume some funny rumens surface for defining this function we can defining the complex can itself. And conclude that it is still analytic everywhere because this no 0 g of z is defined everywhere perfectly. And because of I am taking difference of 2 log i can take any definition of log the difference will uniquely define. So, function g is also entire and g 0 is 0 again force by the definition. Now, the fact that f has order 1 this means that the absolute value of f of z is bounded by e to the z to the 1 plus epsilon what is the real part of g z and this f of z bounded by e to the z to the something.

So, we can say that this is bounded by k this is a bit of what would idea like that an absolute bound on absolute value of g z in terms of the absolute value of z the reason will be clear very soon. But what we get the directly out of the definition that we real part of g has this bound what now we use the fact that. So, what we want the is to get derive a bound absolute value of g. And we will use the fact that g is analytic everywhere and this, the general property of analytic function. Actually, if the function analytic everywhere, which is entire function. Then, if you can bound it real part graphing, the same bound hold on it same absolute value also.

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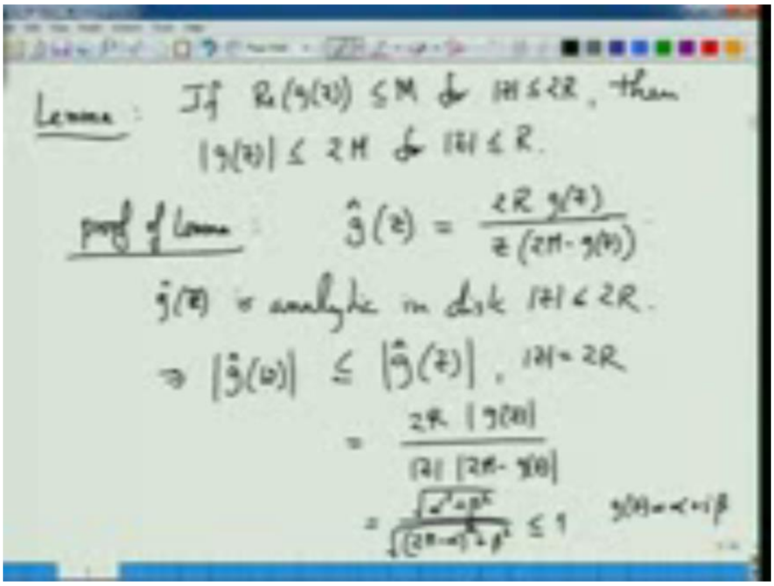

Let me just collect this fact lemma not for f of z the f of z make sense that to the complex part the other the complex part the argument could be large. Of course, you can reduce it then u lose the analyticity the continuity is broken u shrink it the absolute is growing in continuous way. So, its argument part is also growing if you use the fact that took it subtract 2 phi I if it goes to phi i it lose the continuity and then g is remain nothing will work. So, that is you have good point one has to keep issues in mind So, if we can put bound on the real value of analytic function g.

So, this is the function g has an analytic function only in this disk z less than equals to R. In that case the side is smaller disk the absolute value of g is not too much be there and this is need trick is learn basically proof for the case learn trick. So, what we do is let be see this I can figure this out. To find this g z now, g at analytic in fact g at we can conclude this why we only have to worry about the poles. What the poles of g are at is z equals 0 pole no g z also 0 g z g 0 is 0 so z equals 0 not a pole. So, that pole is taking care of, but what about the other one when g z is 2 m then it is certainly a pole right. But we assumption of the lemma z the real part of g z is inside this disc the g z is never 2 m.

Therefore, and there is also inside this disc as the that pole so inside the g z g had is analytic. So, there is a lemma of question dual form which says that for analytic function define on a disc. The value of the function the absolute value of function inside the disc is at most the value of maximum of the function value around the disc around the edges of the disc. That is the function attains it is maximum only at the circumference of the disc. And that is easily proven if it is attains it is maximum somewhere inside then you look at that point and take it angle circle around us within the disc now the that circle. Now, apply cos integral formula and that we you can express this value which is inside the disc which is maximum as a integral of the values around at the at the circumference which are all smaller and that will tell you that there is not possible.

So, if you just think of cos integral formula taking the average of the values around the circumference. Then it will you can immediately realize that the value inside is only the is bounded by the values around the circumference. So, I will just invoke that G at w is less than equal to g absolute value of g z when there is 2 and what is this absolute value of z is 2 R. So, this consists what is absolute value of g z all that actually equal to square root of alpha square plus beta square where assume that g z is alpha plus i beta divide by 2 m minus alpha whole square plus beta square root.

Now, notice that use the fact the real z is at most m real g z is at most m. So, alpha is at most m which means 2 m minus alpha is at least m and off course theta is a same. So, that means numerator is at most is less than equal to the denominator, because this quantity is at most time this quantity is at least time this is less than equal to 1. So, this says that g at everywhere inside the disc is at most one now, let us redo the look at this expression right g in terms of g at.

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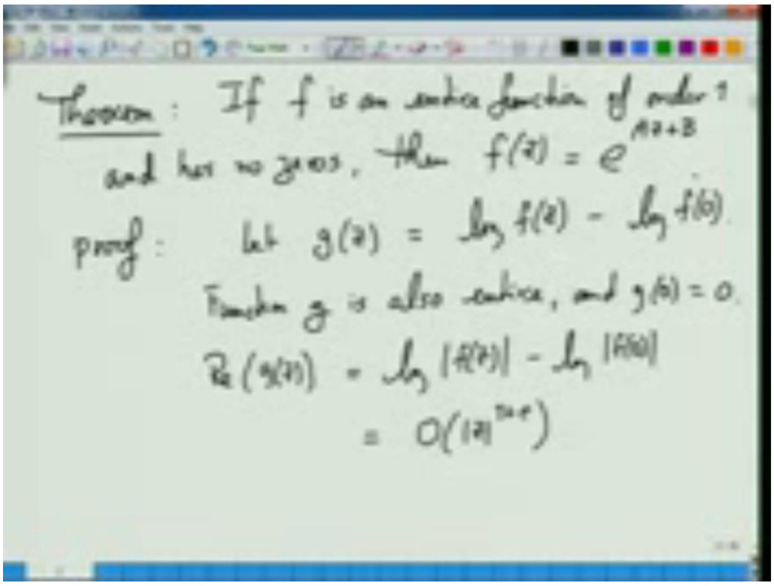

G z equals and z this will 2 R g z equals g at z dot z 2 m minus… Now, for absolute value of z equals R what is absolute value of g z absolute value of z is R. So, this 2 cancel out absolute value of g z is this is at most one; this 2 cancel out so get m divide by 1 plus 2 R g at z now in the denominator there is 1 plus something. So, how small this quantity can be so which is a certainly if you can say that if you look at the maximum value of the second part and if you can bound that to be less than 1. So, this quantity the denominator is 1 minus the maximize. So, let us look at this for absolute value z equals R this is R so, this is half R an absolute value of g z is at most one. So, absolute value of this second quantity at most half so absolute value of this is at most half. So, this means that going back to our function because a real g real part of g is bounded by this use using the next lemma.

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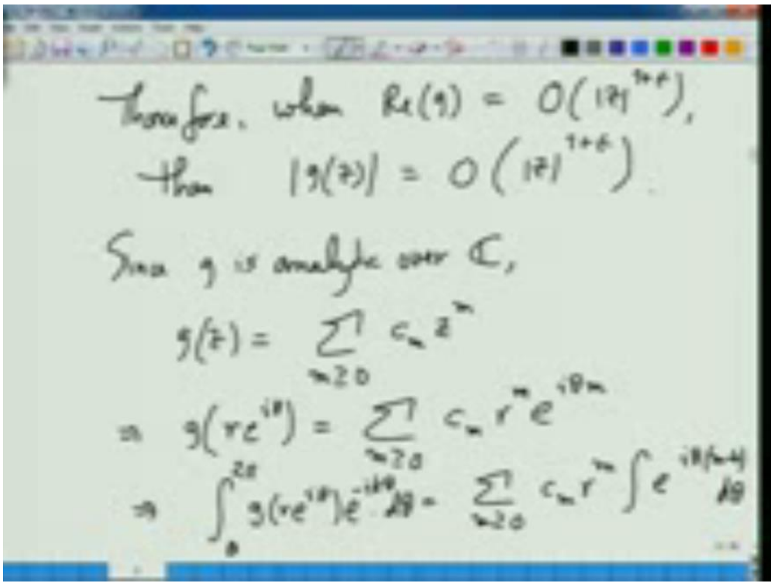

We conclude that equals an absolute value of g is also ordered. So, we get the bound one absolute value of g z. And now the life is very simple just using this we can easily derive the g must be of the form a z plus b not get even a z plus b g must be precisely a times z why g is analytic function over the entire complex? Therefore, entire polar used presentation which is valid. So, this polar is expression for g is valid over the entire complex taking around 0. And now, let us look at theta write it in terms of polar coordinates [no audio: 75:02 to 75:48]. Just multiply this thing quantity by e D with minus i k theta integrate from 0 to phi this what you say about right hand side. This integral when m is not equal to k this is 0. So, this is non zero even only.

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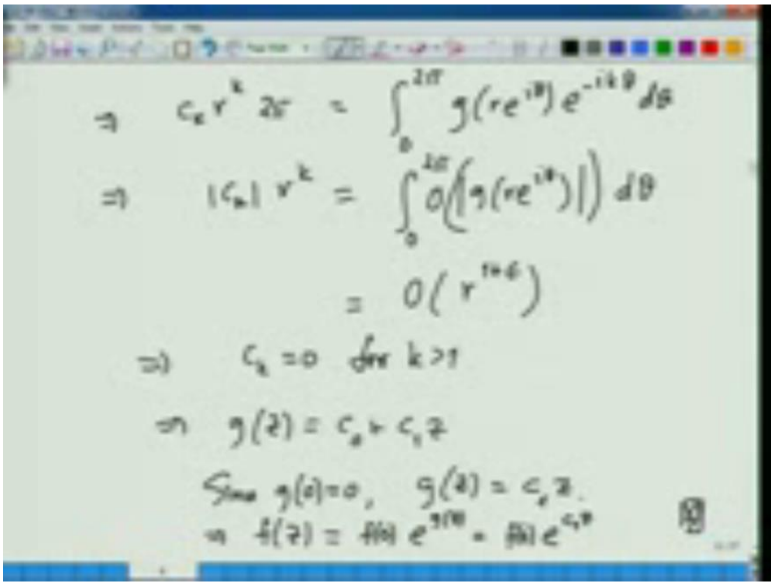

If m equals k so just writing at the reverse side when m equals k we get this. So, we get c k R to the k and then the value of this integral is I think 2 phi I 2 point that is equal to let us take the absolute value. And what is this absolute value of this is and this hold for any any value of R, what you get out of this absolute value of c k times odd to the k is of order R to the 1 plus epsilon k at most one in case 2. Then is c k c 2 absolute value of c 2 times R square is order R to the 1 plus epsilon for any value of R that is not true or actually I should say c k equals 0 for k greater than 1.

So, this means g z is c 0 plus c 0 1, but g 0 is 0 g z has to be exactly 0 types now recall the definition of g there is log of f minus log of f 0. So, this gives f z to be e to the f 0 f 0 f's into f 0 times e to the g z f 0 c I z and that is precisely. So, all entire functions without zeros and like this can be extended as a for higher order entire functions also order k. Entire function will have a degree k polynomial inside in the exponent that is honor it has to be e to the a degree k now. But what about the rest of entire function the entire function which use this, is entire function without 0 which has a very nice and as in terms of entire function with zeros also.

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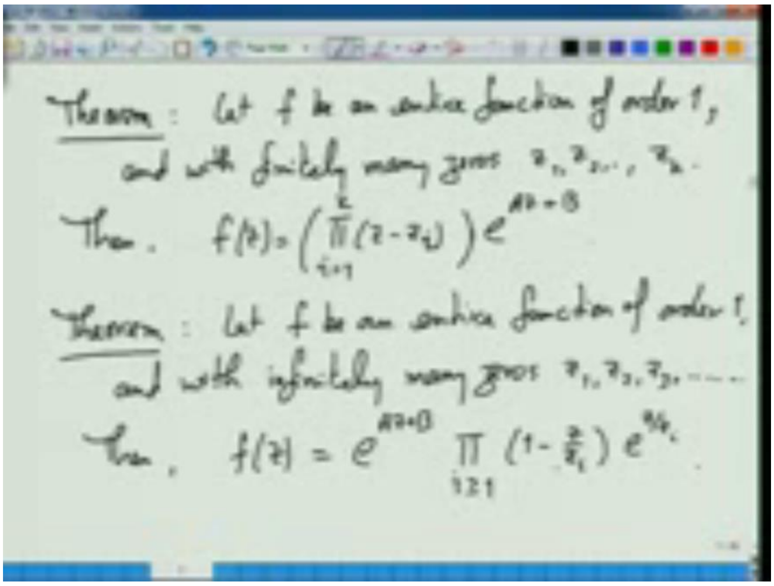

So, that is makes a 0 have use firstly to define g cannot even define g if f is a 0 we cannot define g everywhere. So, that g will not be analytical over the entire complexity than and then therefore, the power series representation for g will not whole for the entire complex. And if it recall we require it hold over the entire complexity, because this equation will fail to whole only on only when R is larger. So, there is but another point to note here is I do not needed to whole this bound on it is bound on absolute value of g. And that is a fact that will come use of use 2 as around this bound on g does not need to hole for all values of R as long as it hole for infinitely many values of R ever increasing then is good enough to derive this conclusion.

Because all we need is a large enough value of R this equality whole which will force equal to be 0. By the way if f is an entire function of order 1 with finitely many zeros then its expression is very clear. Let me just write that and with finitely an f z is simply quanti multiples this is state forward. Because if there are this are the zeros of f you divide f by this product then it becomes a function with no 0 still remains an entire function. Therefore, it has that form what is more interesting is an entire function with infinitely many zeros. Because it has no poles but f f is 0 at z. Now, if we have infinitely many zeros then what should f z that this one was easy to guess that removes all the zeros on separates of the all the zeros.

And the rest is z can we write the same thing for real can we write for example, do like this we cannot for very obviously thing can see that. Yes exactly this for most of the z is unbounded this product infinite product and just take a particular examples of suppose z I is the 1 2 3 4 goes infinity. And you take z to be 0 then this product would be one times 2 times 3 times this unbounded. So, it is does not even capture the entirety of function f. So, this is the reason you all you are all like to write something like this we cannot what no its not 0 everywhere.

There are entire functions with infinitely many zeros whenever gamma, we just wrote as entire function an entire function has many zeros at all negative integers. So, this was a beautiful inside by in late 19 century when we studied these entire functions. And they will complete classification of them and key inside was to guess what should be this form. The idea is remains the same that you want to just take the product of z minus z i and some way. But still to make it converge and what came out with the following so this is the part which takes care of all the zeros product over I greater than equal to 1 1 minus z divide z i.

So, these are this is where this part captures the zeros, but to ensure the convergence be multiplied with e to the z whole square. And then take the product for all i's So, this part and we will see the will see the proof of this theorem is an entire function with precisely 0 z z 1 z 2 z 3 etcetera. And therefore, when it divide f by this your entire function with no zeros which has arrived the form e to the a z plus b that is a very good point yes. So, I should tick that in many zeros f 0 if f 0 is 0 let us say of order k. Then we just multiply this by z to the k and the beginning why it does not make unbounded; will see next class will see.