Riemann Hypothesis and its Application
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Lecture – 21

So, this were last time which are the big, the most important target for our course to show that this connection between psi x and the Riemann hypothesis and how the Riemann hypothesis says something about the distribution of prime numbers. Now, there are still some loose ends over here one is that we really want to know what pi x you remember what pi x is number of primes less than equal to x, but after that definition of pi x is witched over psi x. Everything since then has been done in terms of psi of x which is good, but still ideally would like to extract out some information about pi of x, so that is one loose end.

The second one is something actually it is not really loose end, but it is a very interesting observation and I mentioned this last time that the Riemann hypothesis implies that psi x equal to this. So, that is an implication and since it took us so much time and effort to prove this implication, it might appear it is a one way implication, but as it turns out that is not true. If psi x equals this, it implies Riemann hypothesis, so that is what I am going to show to you first because it is very easy proof.

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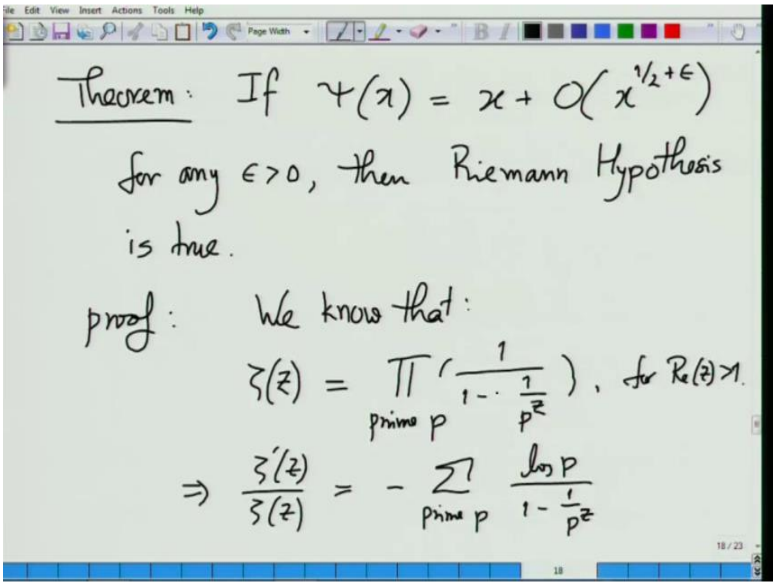

So, let us state the theorem, psi x equals x plus order, so actually we can prove something stronger here that if psi x is x plus order x to the half of epsilon for any epsilon greater than 0. I do not even have to assume it is x plus order square root x log square x, I can allow any power of log x or any to the epsilon as long for any epsilon greater than 0, then the Riemann hypothesis holds.

So, how do we prove this to prove this we just go almost all the way back and right in the beginning if you remember we derived this expression for zeta prime over zeta. Do you remember what that was, let me just refresh your memory, we know that zeta z for real z greater than 1. This is equal to this product prime p 1 over 1 minus 1 over, this is correct and now takes log and derivative this gives you zeta prime z over zeta z. This is of course for real z greater than 1, this is equal to log, we will convert this into sum and derivative will bring down some over prime p minus here and then differentiate this, you get 1 over this, then minus log p over this.

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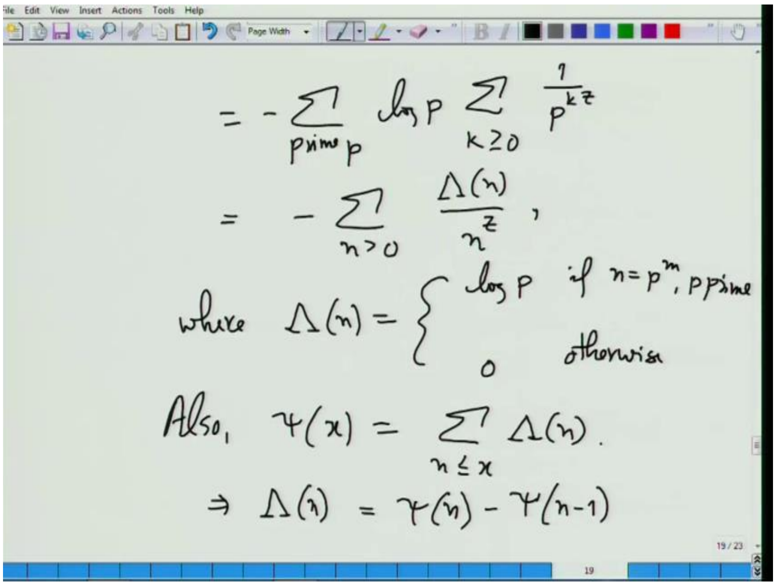

This is equal to in fact this is precisely the expression we used in that integral to replace this sum by zeta prime over zeta and that is how we got psi related to zeta prime over zeta. Now, what is lambda n lambda n equals is log of p if n is p to the m 0, otherwise that is a differential lambda and what is psi x, psi x equals n less than equal to x summation of lambda n, so in other words I can write lambda n as psi of n.

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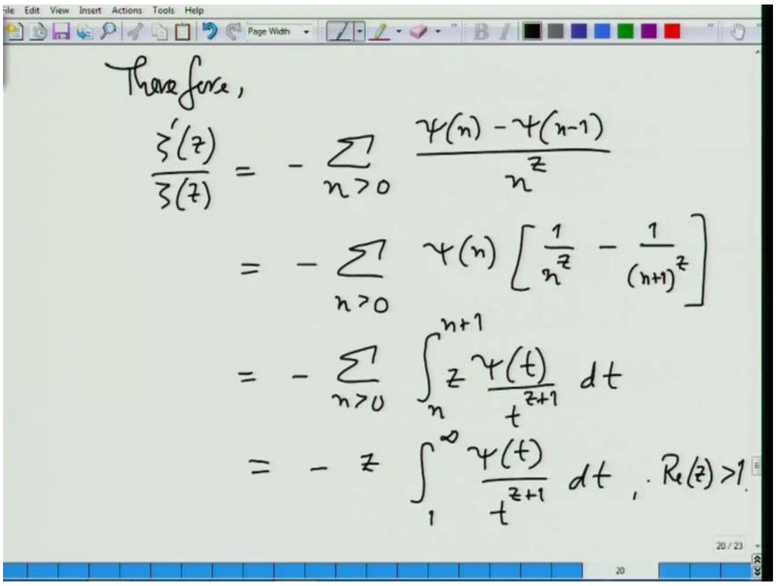

So, let us plug this value in of lambda in the above expression, the next step is to convert this into an integral. Now, rearrange this collect psi and psi n minus 1 together, so what is coefficient or multiplier to psi n, you get 1 over n to the z. Here, you get one more what is that 1 by n to the z minus 1 by n plus 1 to the z anything missing n equals 1, you have psi 0, psi zero is trivial. Now, look at this, so this is an integral of n to n plus 1 psi t by t to the z plus 1 d t times z.

In interval n to n plus 1, psi is constant, so the psi comes out, so you get d t over t to the power z plus 1, it is integral is minus t to the z over z, so that z, z cancels out. So, you get 1 minus 1 over t to the z and then n plus 1 to n, so this minus goes away or does it not go away? Now, this is good because we can take this z out and make this from 1 to infinity, now keep in mind the real part of z should be greater than one.

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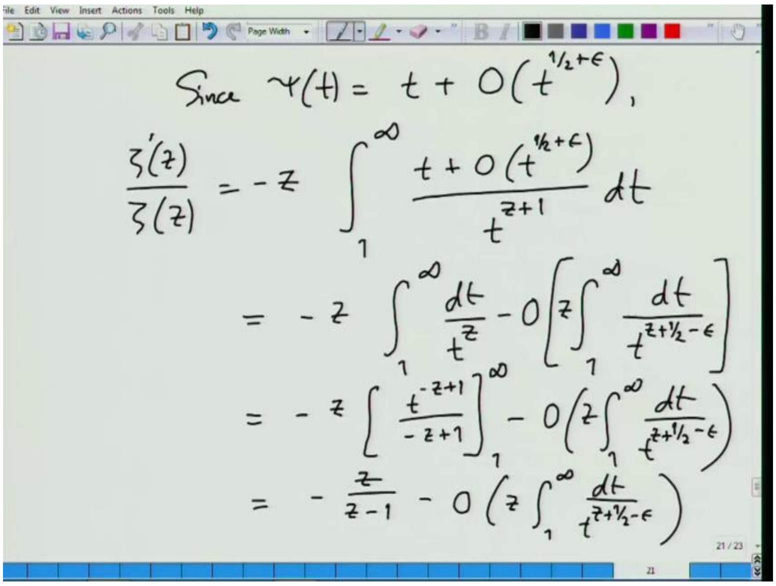

Now, since we have psi t equals to t plus order t to the half epsilon, we get zeta prime z over zeta z to be z minus z 1 to infinity t plus order t to the z plus 1 d t. so, the first term this gives you minus z 1 to infinity d t over t to the z and second term this. Now, what does first term real z is greater than 1, so first of all, we will converge what does it converge to well it converges to this. So, this is basically minus z, this is 1 over z minus 1 p to the minus t to the correct when t is infinity and real z is greater than 1 that vanishes.

This part vanishes when t is 1, then you get negative t is 1, so you get this one, now let us look at z prime over zeta and this expression the expression on right hand side is analytic as long as real z is greater than half actually greater than half.

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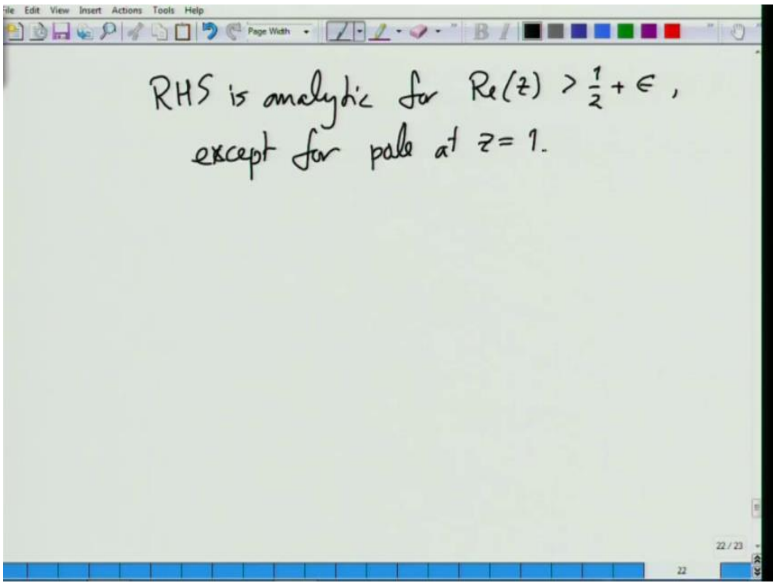

Half plus epsilon because then this part will become real part of this will become more than 1 and then this integral converges this is analytic except for z equals 1, real z equals 1.

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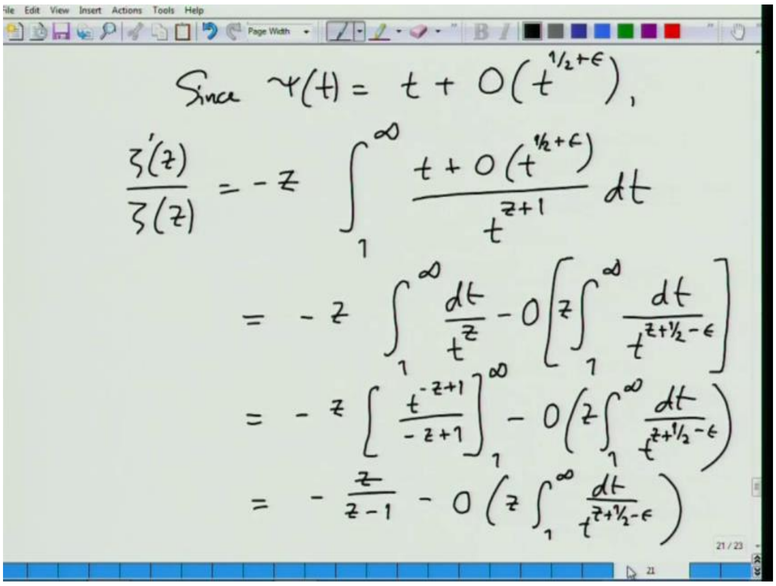

So, except for pole at z equals 1, so that is how I can write left hand side is zeta prime over zeta, we know that pi or analysis real z greater than 1 and left hand side equals right hand side. Now, by again analytic continuation zeta prime over zeta is equal to right hand side for real z greater than half plus epsilon.

On the right hand side, except for pole, I am saying this except for pole at z equals 1 is analytic everywhere on real z greater than half plus epsilon. This is true for any epsilon because my assumption was any epsilon psi of x is order x to the half plus epsilon.

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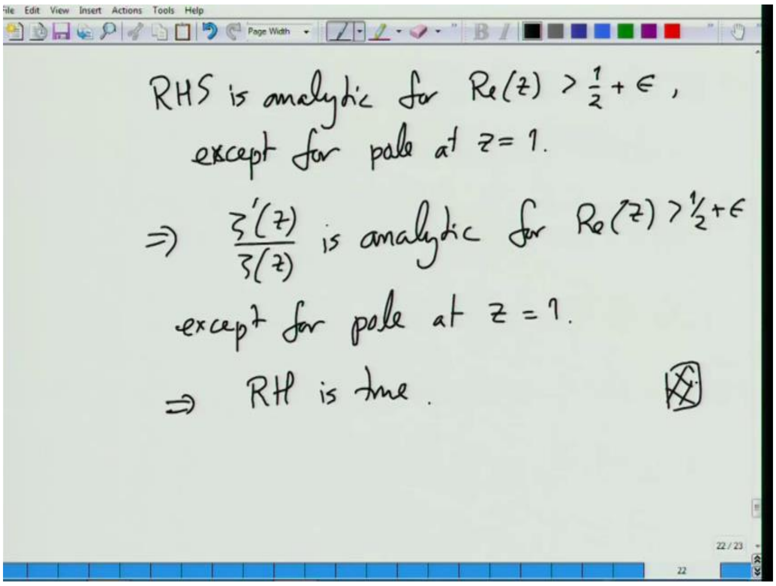

What does it mean, it means that zeta prime over zeta analytic for any z such that real z zeta prime over zeta is analytic for except for pole equals z equal to 1. Well, we already know the pole equals 1, but if the Riemann hypothesis was not true and there was which was there is 0 of zeta at real z more than half then zeta prime over zeta will have a pole there which will contradict this statement. Therefore, it is extremely simple, it is ten minute proof, I got lost somewhere otherwise it finished more quickly the other direction of course to forever.

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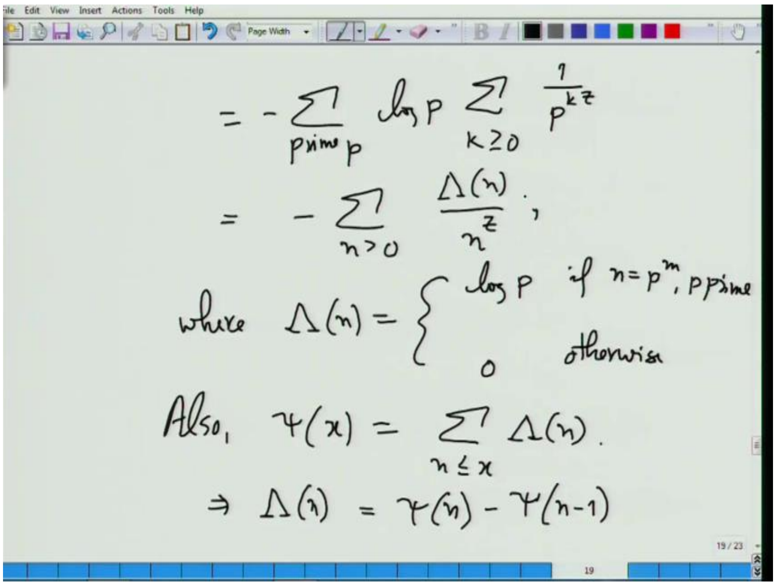

Both sort of arise from the same idea that is this expression for zeta prime over zeta, you have this expression and for psi you have this sum psi of x is partial sum of these and so that is this to have a very nice and tight relation. In fact later on in this course, which is about next step, once I am done with all of this and the remaining lecture what I do is two things.

One is to generalize this, so there is this I mean if you look at it from the higher level psi x is a partial sum of some quantities there is zeta prime over zeta in our case. In general some other function zeta prime over zeta is this infinite sum with same quantities in numerator and n to the z in the denominator. This is zeta prime over zeta lambda n over n to the z for all n psi x is sum and less than equal to x lambda n. There is this relationship one is complex plain object one is a number theoretic object and there is all these were to establish a nice relation between this.

Now, it turns out that this we can do not only for lambda n, but many other numbers and there is an entire theory which has been developed on this. There is really some real remarkable result which I will only be able to give very brief glimpse, I do not either have time or full understanding to explain all of that to you, but this is essentially the starting point for theory of modular forms.

So, there are objects called modular forms which I will define at some point which are functions of the kind of properties like the zeta function and there is a whole beautiful theory around this. It is not just a theory, one can use this its different forms to derive different number theoretic results this is as we did for the zeta functions. The second thing I would like to show is hopefully I will have time to do that is at least one domain in which we can prove Riemann hypothesis.

You cannot prove Riemann hypothesis over this complex plane of this kind, but as I said we can do if we are abstracting out the basic ideas. We can take this relationship of number theoretic functions and complex analytic functions form of different kind and form relationships with them. Then, we post a same similar hypothesis make a similar hypothesis about the those complex analytic functions about where the 0s lie and relate them to the property of number theoretic functions.

So, we can come up with various versions of Riemann hypothesis, now most of those versions remain unproven conjecture were unproven, but some versions have been proven. So, I will give you one example which is also is very interesting on its own which is of elliptic curves, so again in the number theoretic objects elliptic curves are thought of as number theoretic object. Then, we can associate a corresponding Riemann hypothesis with these objects or rather some certain numbers associated with these objects and then prove that Riemann hypothesis.

Before we do all of this I still have to tie a few more things over here one is the psi x versus pi x that business. The second thing I want to do before we move forward is to prove a basic version of not quite Riemann hypothesis, but something. Let us say the starting point towards proof which is what I will prove is that on the line real z equals 1. There are no 0s of zeta function and that is enough to prove prime number theorem because that means the error is in psi x is psi of x is x plus order x to the 1 minus delta for some tiny delta.

Essentially, the error term cannot cancel out x and that is that proves the prime number theorem good and as it turns out all of these is very easy. So, I can do that quickly, so let us investigate this psi and pi psi, we know what it is pi we know what it is so how do write one in terms of that see how does psi x get calculated.

You go through all the numbers in sequence whenever you detect a prime power you add log p pi x get constructed whenever it is your prime you add 1. So, the calculation x I can divide it as follows I can split it into stages stage one only consider primes whenever you see a prime add log p stage two consider prime powers. Whenever we see a prime power p square add a log p and that log p, therefore, is I can instead of talking of log p I talk in terms of the number that you see.

Whenever you see n to be prime add log n wherever you see n to be prime power add half of log n log of root n, which is half log n, wherever you see n to be a prime cube add one third of log n. Now, how many such things you add, how many primes you see you see exactly pi x primes for each one of those primes you will add log n how many prime squares you will see p of root x exactly you will see exactly pi of root x prime squares. Similarly, you will see exactly pi of third root of x prime cubes right and for each prime square you have to add half of log n.

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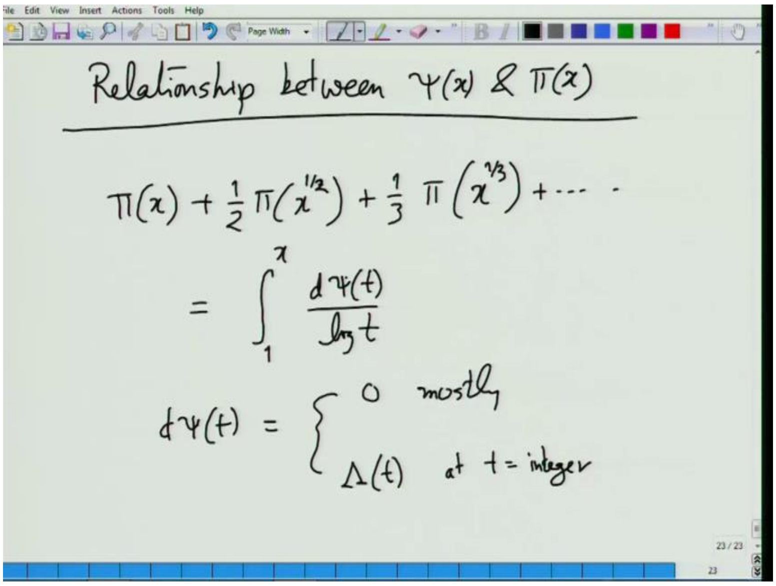

So, this thing I can now let us just do it this way, so you add for each prime in psi you add log n, but if I derive divide psi. So, I am barring psi and whenever I see a change in psi it is, because either I have seen a prime there or I have seen a prime square or a prime cube there or prime 4 power.

So, at that point I add appropriate logarithmic term, so instead of I can alternately state this way that whenever I see a change in psi and divide at t if I see change in psi and then divide it by log t. Then, either I could in stage one counting all primes in stage two, I will be counting half of pi of square root x one third of pi x to the power one third. So, this is equal to I have to go through numbers and sequence t starting from one and notice change in psi t at whichever t there is a change in psi at whichever t, there is a change in psi t that point is divided by log t.

So, this is captured by this integral of course, I had to now define this a little more formally because psi is not a continues function psi t. Well, it is continuous, but it is not differential it has it is a step like quality, but we can define the this d of psi x as measuring the delta value. So, between psi of this d of psi t is well as t varies. Let me write for d psi t is 0, most of the time and it becomes lambda t at t equals integer.

At integral points, there may be a jump inside, so that point of course, it is a instantaneous jump. So, I will just fudge around this meaning of this integral, this differential d of psi t to mean that at around integer d of this is measuring this jump amount of the jump and the jump is exactly lambda of t and everywhere else it is 0. So, this is not quite the original meaning of differential this is called Stieglitz integral there was a mathematician called Stieglitz and lot of i g s in that spelling who defined this first.

Then, actually formally analyzed that and proved in what situations is this a sensible definition and then because this allows you to work with a larger class of functions even those which are you know step wise functions as long. It is continuous not necessarily differentiable, we can work with such function. So, that is the meaning we will assume here and what are the things he showed was that we can given the reasonably decent condition on psi of x psi of t. We can assume this to be functioning almost exactly as a normal differential and this again psi of t does satisfy the condition fairly mild conditions it should not be all jumping up.

It should not be a function which will jumps up and down it is monotonically going up function. So, some simple properties which as long as they are satisfied you get go on to new page oh it is entered here what do I do about this select all cut paste. Now, we have a few page and before we run into similar problems, let us insert some more pages, now you see this integral and think of this as normal integral all the more.

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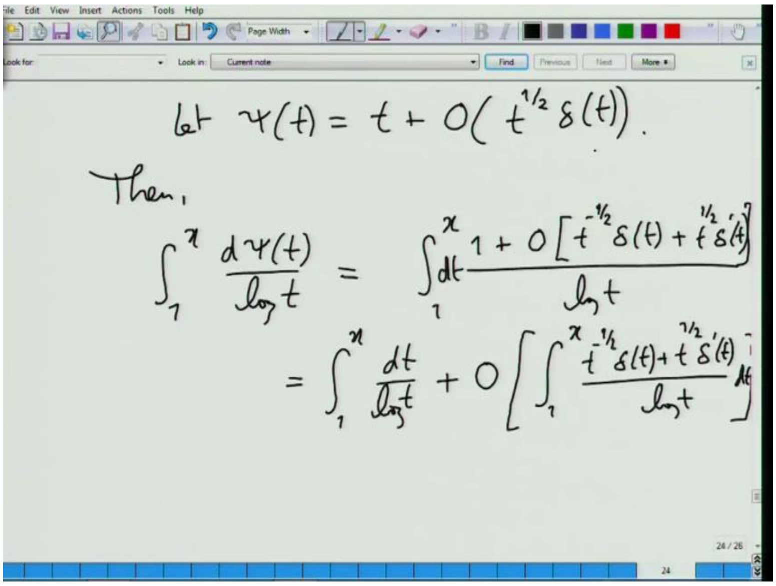

So, because I have a nice expression for psi t psi t is t plus order, so let psi t 80 plus order square root t and put another function of t let us say delta of t, then let us just first look at this integral going from 1 to x d psi t even log t. This is equal to now d psi t i can with this expression for psi t i can write it as psi prime d t just divide and multiply by d t and now you take differential of psi t with respect to t. You get 1 plus order what happens here you get t to the minus half delta t plus t to the half delta prime t and d t.

Now, look at the first one what is that log t in integral from one to x that is x over log x is very close to the x over log x, how do you show this this looks should be simple enough d t over log t integral does not this have a closed form formula integral of 1 over log t. Now, we will tackle that later let us look at that part what happens here see this is maybe just to make life easier, let me just take keep it delta t and let us assume for the moment delta t is log square t which comes out of the Riemann hypothesis.

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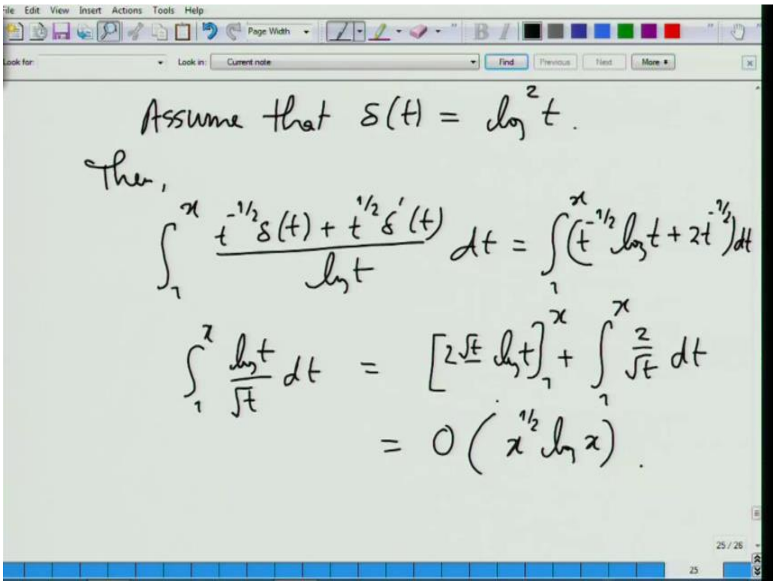

So, then your 1 to x t to the minus half delta t this equals delta t islog square t so to the minus half is log t delta prime t is two log t by t so this becomes previous integral. There is a one sitting here there is where log t becomes 0 diverges with it becomes funny integral to handle, but there is somewhat very simple trick which a log is what happens here. So, that delta prime as I said was 2 log t over t, so this becomes log t, so that is 2 to the minus d t clearly the first is bigger than the second one. We are bonding the error we can ignore the second one and then what is 1 to x log t over t square root t d t what is this?

We are again in this situation or we may be not maybe not this is we do integration by parts because here log t is integrates 1 over square root t, what you get is root t by half. So, that gets two root t log t one to x, same thing log t differentiate when you get one over t, so you get 2 by root t d t. Now, you see that this is of course and if you integrate this you get root t, which is again dominated by this and this is all of it is order x to the half log x. So, that is what the error is and the main part it should turn out this is basically where is it?

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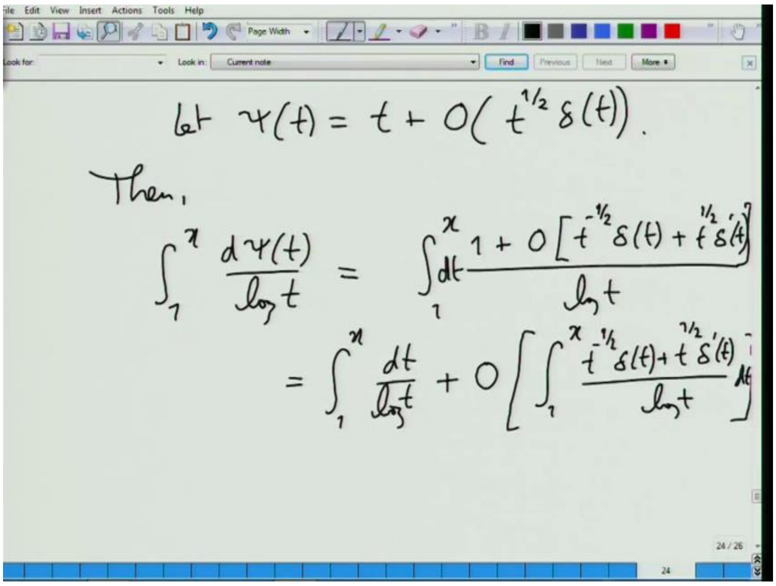

This is x by log x plus some small order term which gets absorbed into the error, so it should actually what I want to see here is basically x by log x plus order square root x log x, that is what should come out model.

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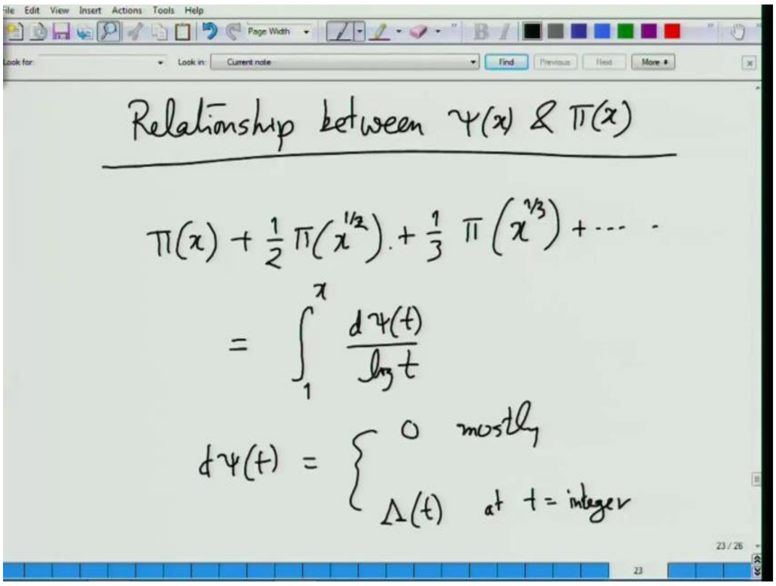

That integral maybe I can leave that as an assignment to you and this, what is it, so it is saying that less than equal to this.

Student: Sir actually the integral of x by log x because the second the second part after the minus sign we will get back exactly the same.

You get the same thing, why should you get half, x by log x your point is good yes why have I replaced psi by this order. This actually not true that we cannot use yes psi is this t plus order this, but I cannot therefore, write psi prime is one plus order this derivative because all this is saying that psi is in this band grows like t. Then, there are fluctuations within this given by this error would those fluctuations make the derivative of psi very large depending on how those fluctuations happens good part.

So, I think this whole derivation is wrong, I will need to think about it then I thought I can do it in this simple way, but whatever eventual derivation is it is going to throw up x by log x plus order square root x log x. That is equal to this left hand side, which is pi x plus half of pi square root x plus one third of pi. Now, what we certainly know is pi what is pi of square root x it is square root x pi of x to the one third is right. So, whatever the pi x is the rest of the sum just gets absorbed into the error order square root x times something 3 times square root x times something just get sucked into that. So, left hand side is pi x plus order square root x right hand side is x by log x plus order square root x log x.

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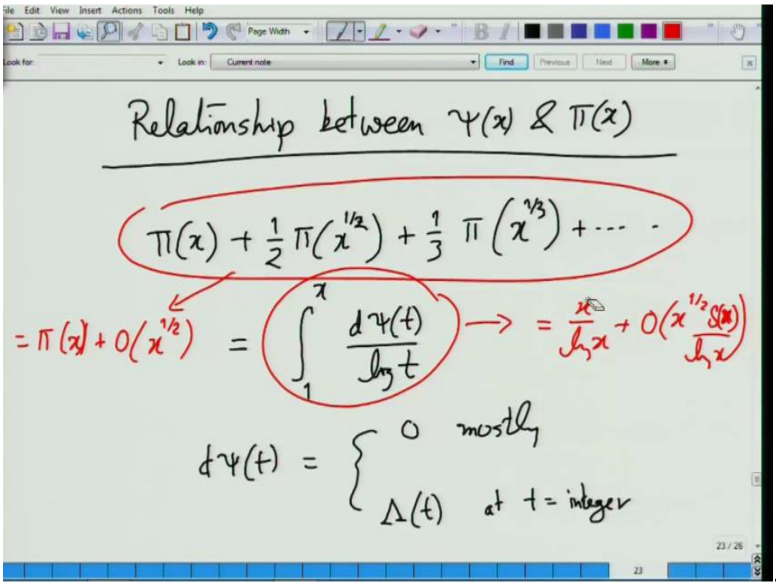

So, this is equal to pi x plus order square root x the right hand side this will be equal to x by log x log x this is Riemann hypothesis and general if you do not assume the Riemann hypothesis you will get delta x by log x. So, whatever is the assumption about psi x for you that is order x to the half times delta x, you just divide by log x this is what the error term you will get and just put these two together.

You will see that pi x equals x by log x plus order square root x delta x by log x and now delta x is log square x it has given Riemann hypothesis and you get square root x log x. So, that is the relation between these two, so the only thing I now have to sort out is how do you show this on the right hand side.