Riemann Hypothesis and its application
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Lecture – 20

So, let us continue with our investigation into zi and before I dive into this little more, let us first clearly understand what we out want out of zi.

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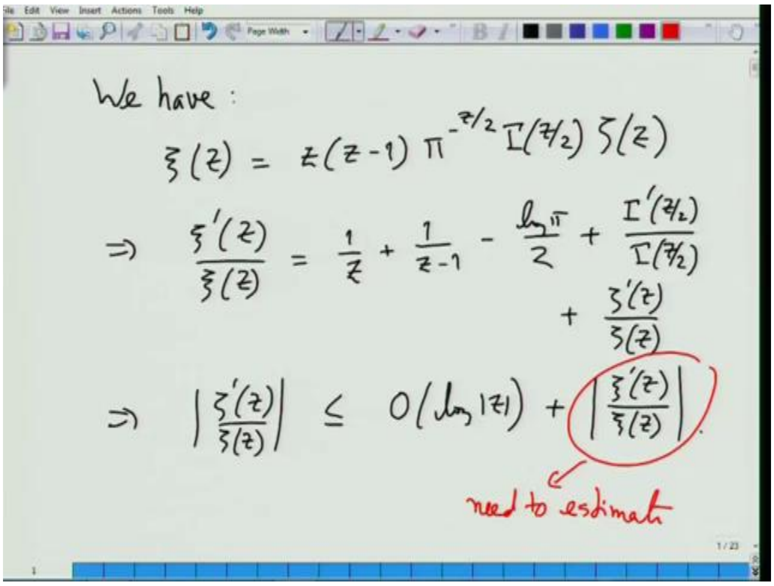

As we know that zi z is z minus 1 pi to the power minus z by 2 gamma z by 2 eta z our eventual target is to estimate log not zeta prime or zeta. So, we do this take the log and differentiate we get zi prime over zi equals 1 by z plus 1 by z minus 1 minus log o pi by 2 plus. Therefore, if you look over zeta prime z over zeta z in absolute value that is formed it by gamma prime z over 2 or gamma z over 2, which is which we already know like order log of z. Again, in the range we are interested in which is z, there what is fine and the first three quantities there are tiny, so they all got into order log z plus zi prime by zi z. So, that is that is a quantity we need to estimate in the range, where z varies from a little more than 1 plus I R to minus 1 plus I r.

So, we already have done this job for gamma function estimating gamma prime or gamma through this first we showed an entire function without any 0 s. Then, we looked at 1 over gamma, which is an entire function without with 0 s, of course then we expressed. Now, entire function of order 1 and then we expressed 1 over gamma as product which gave us an expression for gamma prime or gamma. So, that is a same approach we use for zi, also we know that zi is an entire function we know that its 0 s are precisely the non prevail 0 of zeta function. The only thing we need to establish at this point is the order of this, zi is an entire function, but what is the order of zi.

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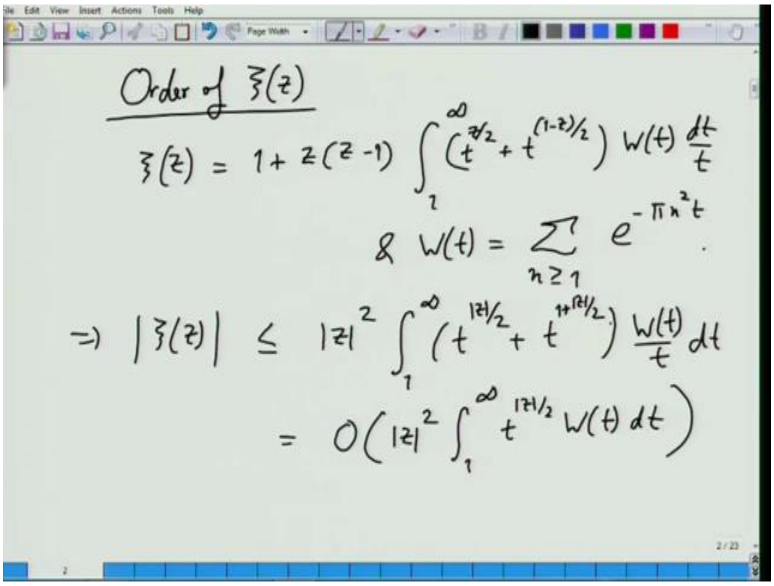

Now, let us just recall the definition of zi, there are two definition, one is through this product, which we already have written here, but this does not give too much clue. Of course, gamma we know exactly the order of zeta, we do not know the order, so instead of this, we look at the other definition of zi, which is zi z is 1 plus z z minus 1 integral from 1 to infinity t to the z by 2 plus t to the 1 minus z by 2 w t d t by t.

Here, w t is the sum n greater than equal to n e to the minus pi n square t, so knowing this as a definition of zi, can we estimate the order of zi, what does this gives the order of zi. That means absolute of zi z we want to estimate, this is let us say less than equal to mod z square for this part.

Then, integral going on to infinity t to the z by 2, what is that bounded with that is bounded by clearly t to the mod z by 0 and t to the 1 plus mod z by 2, then w t over t d t. Now, of course it is a sum, so we can equal to this, therefore write this as order in the order you can always equal to z square 1 to infinity t to the mod z by 2 w t d t because the first integral. That is corresponding this a sum the first integral is going to be less than equal to second integral, because it is a it has a higher power of t. We then cancel our this, so we want to estimate this quantity right and again it is not too difficult to estimate it actually.

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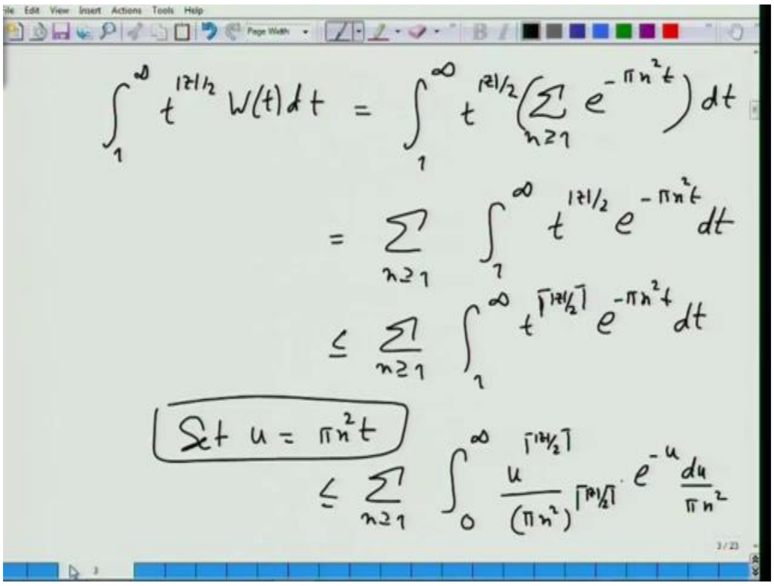

If you look at this 1 to infinity t to the mod z by 2 w t d t just plug in the definition of w t sum and greater than equal to 2 e to the minus pi n square t d t. Now, here we can exchange the sum with the integral although both are infinite, but again using it the fact that this quantity is uniformly convergent, we can exchange the two. So, we will write it as n greater than equal to 1 integral 1 to infinity d t, now what do we do with this, then there is a simple way to handle this actually this is mod z is some real number. So, let us this is the most 1 to infinity t to whatever is the ceiling of mod z by 2 in integral the integral. The integer just higher than mod z by 2 e to the minus pi n square t d t, now this look familiar t to the integer e to the minus pi n square t d t.

This is almost like the gamma function except that pi n square sitting in up there that we can get rid of very quickly even does a variable substitution, so set u equal pi n square t. So, we get this is equal to n greater than equal to 1 when t goes from 1 to infinity, this will go from of course you can say 0 to infinity for u all set gamma function is integral from 0 to infinity, I think right gamma function integrates from 0 to infinity. I think that is right, so 1 to infinity I can always replace by zero to infinity and put it less than equal to here and here you get u to the mod z by 2 divide by pi n square.

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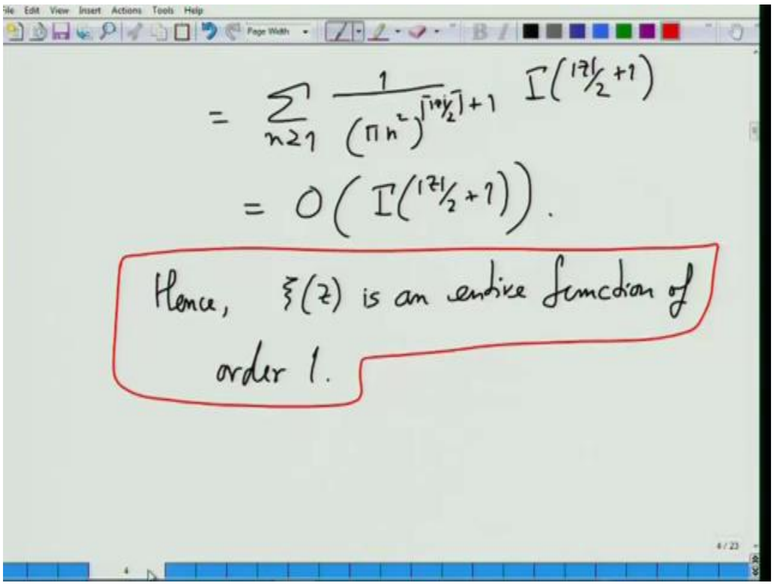

Now, just take everything out that you cannot integrate and this is now gamma function which becomes 1 over pi n square to the plus 1. What is inside is simply the gamma of mod z by 2 plus 1, which is of course has nothing to do with n. So, this is a common actor outside and we get the sum, now what happens to the sum this converges right it is like your summing of 1 over n with a certain power that power is more than 1, at least 2 more than that. So, that sum is bounded quantity, so this is whole thing is ordered gamma z by 2 plus 1, which of course is order 1. Of course, I have looked at this multiplication factor mod z square, which does not change the order at all good. So, the moment we establish it, we can bring in the whole machinery we developed for analyzing the zi.

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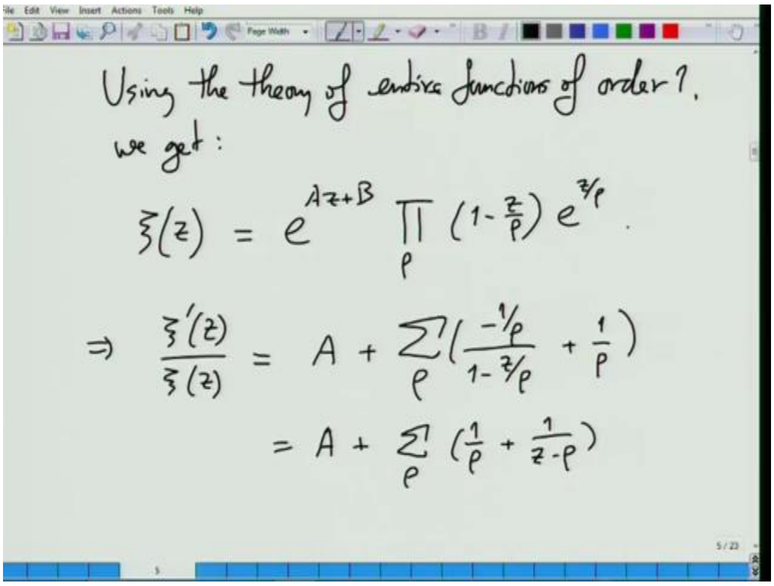

What can we write zi as is zi 0 0 or it is not, I think zi 0 is not 0, we already argued about that. So, this no multiplier of z, so this is like e to the a z plus b for some constants a and b the product which runs over all the 0 s of zi, which are precisely all the 0 s of not revealed zeros of zeta function. So, we will use the symbol rho to run over all not revel 0 of zeta function, so whenever I write product over rho implicitly, it means it is running over all non revel 0 of zeta function. So, that is a good short hand, otherwise I need to write that every time and we will not use the symbol rho anywhere else.

This is the only place we are going to use it, for the 1 minus z over rho e to the z by rho, now that we have this product expression what is zi prime by zi, that is easy that is a plus sum over rho into the minus 1 over rho here, divide by 1 minus z over rho. Here, plus 1 over rho, so this is what we get, so again this is familiar expression we saw already gamma prime or gamma, but of course their sum is over non revel 0 instead of integers. So, this is little more difficult to analyze, but we already get some very interesting things, for example if you recall we said that entire function of order 1 in a radius of R is going to have at most how many 0 s R to the 1 plus epsilon.

We can throw the same on this also, now of course we know that in this case the 0 s only lie on the strip between 0 and 1, so at a height of R and that is the height, we are interested in, we are interested in integrating.

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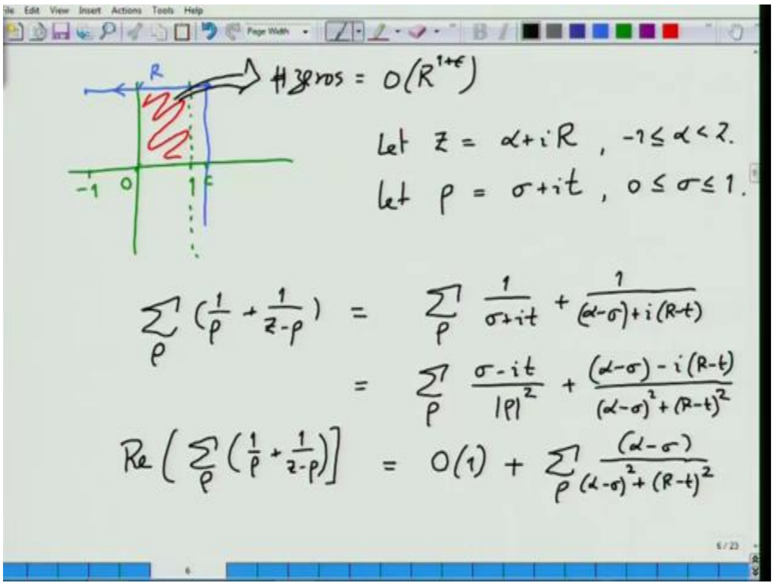

We are interested in this integral, this is at r, we are just interested in integral between from here to here where at height R and what we know as a thing this region got some maximum number of 0 s of order R to the 1 plus epsilon that is directly from this result. Of course, it is not useful, but still interesting to know that there is some immediate quantity; we can put to the number of 0 s, so it is not something very unready. Of course, we still do not know whether exists infinitely many 0 s here although I have shown here the product, now this product is valid whether there are infinitely many 0 s or finitely many 0 s. So, that is we do not know if there are infinitely many, in fact it turns out that number of 0 s at up to height R is I think R log R plus some error term.

So, you can actually very precisely found the number of 0 s up to height r, but I will not prove that not yet because it is not really useful for us. So, coming back to this that is the sum we want to estimate and we want to estimate this sum when z has a specific range of values. So, let us set things up lets write z as for the range of values we are interested in as alpha plus i R because that is we set the height R we are interested in.

We know that alpha is between minus 1 minus 1 and 2, it is actually between minus 1 and c when c is less than 2. So, instead of writing c every time, let us just write this and we also let rho which is 0 is sigma plus I t and sigma of course we know is between 0 and 1, so with this notation fixed let us analyze this sum, A is constant. So, we can just hide it away in order 1, so what we really want is to understand this sum, so let us just consider this sum over rho. So, with the notation we just introduced, I can write this as sigma and this again it can release easier if we get rid of the complex numbers from the denominators.

So, let us otherwise summing this inverse of complex numbers is very little intuition you can stick to, so just take this up using standard methods. Now, let us look at this expression for a moment, the first part can you bound it not clear the t here can become ever bigger as rho increases the mod absolute value increases the value of t also increases. So, it is not clear whether this is going to get bounded even, but you can just look at the real part of this the first question is that bounded that is because sigma is always between 0 and one and sum over rho or one over absolute value rho squared that is bounded.

That is likely which proved this using this fact that the number of 0 s is bounded if we recalled we proved that for entire function of order 1 if z 1 z 2 z zi are 0 s. Then, sum over i 1 over mod z i to the 1 plus delta is bounded, so certainly 1 sum over 1 over sum of 1 over mod z i squared is bounded and this is what exactly what is happening this summing over all 0 s of this entire function of order 1 and 1. So, if we just look at the real part of this that is equal to this is order 1 plus here, what can we say again, we can forget about this complex part alpha minus sigma.

This is at most 2 because alpha is between minus 1 and 2, when I have already specified the range we are interested in and sigma is between 0 and 1. So, this is at most 2 alpha minus sigma for the moment plus R minus t squared, actually I will be wanting in the upper bond here. So, I want to replace this yeah that is fine I want to replace this with a quantity, which is now greater than this, so let us keep it that way so what we get out of this.

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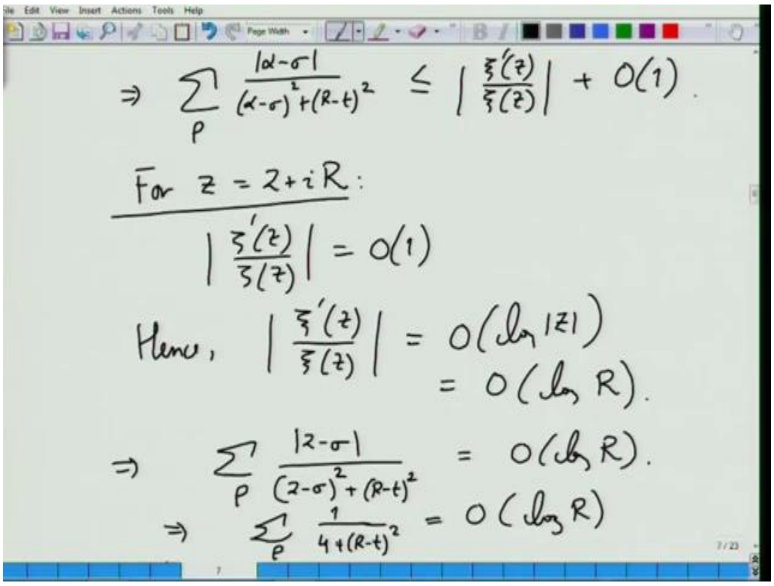

Therefore, if we put everything together all of this all way, then what we get is sum over rho alpha minus sigma and let us take the absolute value here by less than equal. Now, of course this part is small this part is small, so what we eventually we are getting here is sum over all 0 s and then inverse of R minus t whole square that sum, where t is the majority part of all the 0 s, what is gone? I have only looked in the real part, real of this real part of this is equal to this, so this is at most the absolute value of this real part of this is real of a plus real of this. So, I can say that real of this is real of zi prime over zi plus order 1 a is fixed and that is also equal to this.

So, I can say this sum is equal to zi prime over zi plus order 1, now real part of zi prime over zi is less than equal to zi prime over zi. So, I am getting an interesting sum here on the left hand side which tells me about an upper bound on the behavior of inverse square of the complex part only the complex parts of the only the complex part of the 0 s. Of course, I know that this converges that we know anyway inverse of imaginary part inverse square of imaginary parts will converge because the real part will anyway is bounded. So, it is going to converge, but with this analysis I can actually derive a relationship about with this sum and the quantity that it converges to and the quantity will clearly depend on r.

That is the only parameter here everything is being summed over, so that is what the target is so for in order to achieve that we need to get a bound estimate on this, but that was the original problem anyway, so what have we solved? Well, the nice thing is this relationship is available or is true for all values of alpha, remember that is from c to minus 1 and for all values of alpha this are available in fact forget about c when from 2 to minus 1 it varies. So, when I set alpha equals 2, then I know everything because at that z which is equal to 2 plus i r, I can bound zeta prime or zeta I can bound gamma prime or gamma.

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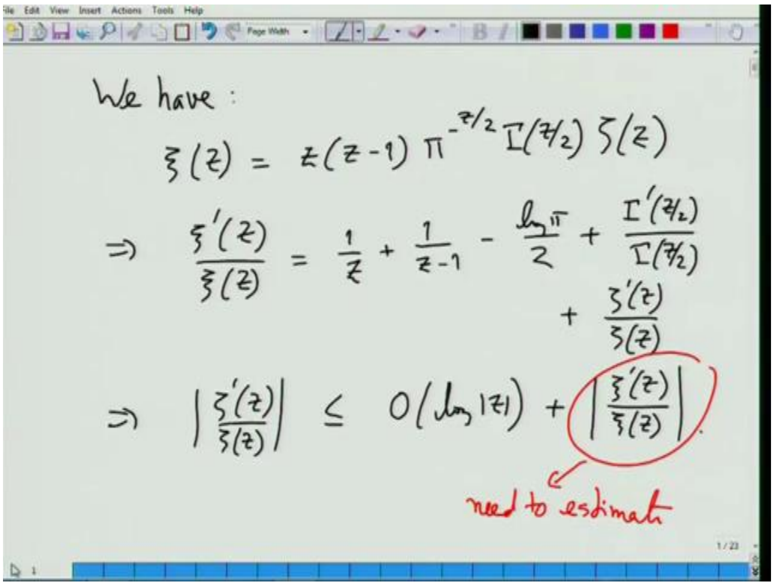

Therefore, I can bound zi prime or zi after all what is zi prime or zi, it is here that is zi prime or zi once I stick in z equals 2 plus i R this is order 1. So, let me just put that in black and white for z equals 2 plus i R zeta prime over zeta is order 1 and hence zi prime over zi is order log of z, what is order log of z its log r, so all together we get 2 minus sigma in absolute value squared with order log r.

Now, I want to just simplify this expression little bit, so all I will do is I will replace this expression with something which is smaller than this. So, that smaller expression will continue to satisfy this equation and to make the quantity smaller what I will do is I will make the numerator smaller and denominator bigger what is the smallest value.

This can take 2 minus sigma 1 because sigma varies from 0 to 1, so I will replace this by 1, what is the largest value that 2 minus sigma can take 2, so I will replace this by 4 and to make it even simpler I mean these are trivial stuffs. I will get rid o this 4 also, how do

I get rid of this 4, I will multiply this by with 4, so that becomes bigger, so this becomes smaller and then take this four this side.

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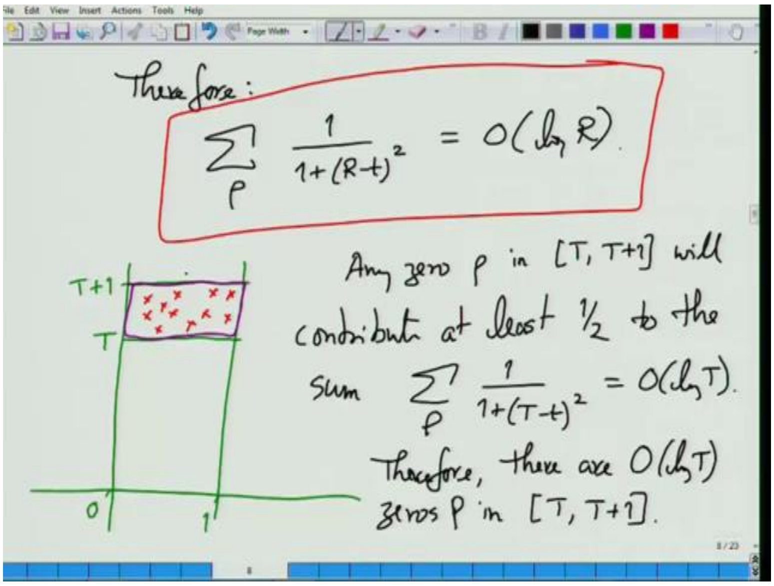

This is going to be a very important relationship for us because this is giving us something very interesting, let us put out again that diagram. Let us look at this region many zeros will lie in this region possibly between t and t plus 1 t is and integer some t and t plus 1. This expression is going to give me a value to how many those 0 s can maximum number of 0 s can be how just substitute for R capital t. Then, every single 0 in here, so what I know this sum order log t, so every single 0 in this region is of course will be counted by this will be contribute something to this sum.

Everything is positive here, so everything adds up will contribute something to the sum what is that quantity is going to contribute is 0 in this. This is simple at least how much what no R is capital t, so I know that this sum 1 plus inverse of 1 plus t minus small t whole squared is order log of capital t and this sum is over all the 0 s all the way out.

So, in this region when the rows are in this region pick any one of these rows it will have some value small t what is that value small t it is going to be between capital t and capital t plus 1. So, whatever that value is if plug that value in here this is capital t, this is something between capital t and capital plus 1. What is the minimum number that you get the maximum, this difference can take is 1, so this number the contribution of any of these 0 s here will be at least half to the sum and all the numbers here are positive. So, how many 0 s can be here at most log t there cannot be any more than that because otherwise the sum will go beyond this and this is true for any t.

So, this tells us how to choose my R of course I want to choose my R to be a certain around a certain value, but within that, remember we wanted an r, so that to avoid any of these poles here because we cannot afford to integrate over a pole. Now, we know that there are at most log t zeros between t and t plus 1 we can conclude therefore, that there is there are at least two 0 s. So, t is the value around which I want R to be, so what I do is look at this region I know that there must be a strip between this of horizontal strip of width of one over log t and there are no 0 s in that strip. So, will send my line arch R choose R to be the middle of this strip just cut through like this which will guarantee that this line is away from any of the poles by a distance of at least order 1 over log t.

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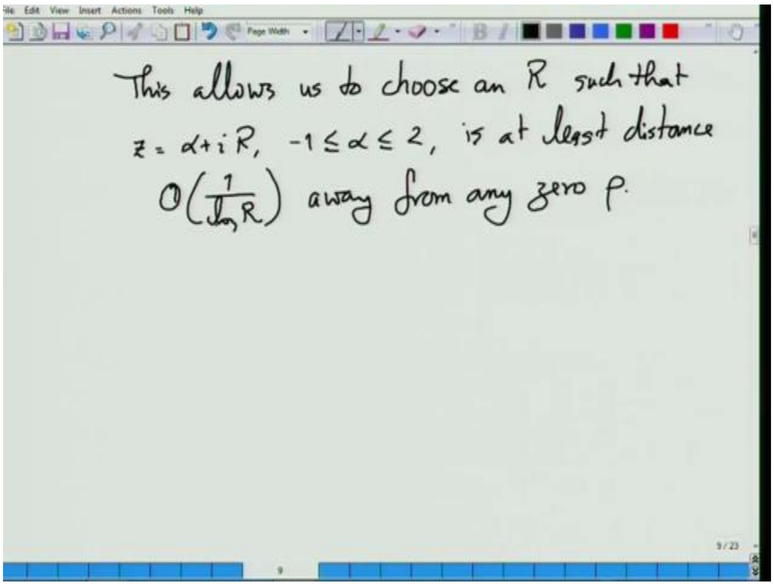

So, that is what I am going to do, so instead of writing distance of order 1 over log t, I am just writing log 1 over log R because t is same t is R plus minus 1 so that it gets solved. So, that is the first thing we learnt out of this, now this was obtained by looking only at the real part of the this and we ran away from the complex part. That was diverging and this was simpler to manage, we still are not done because we are still not able to bound zeta prime over zeta.

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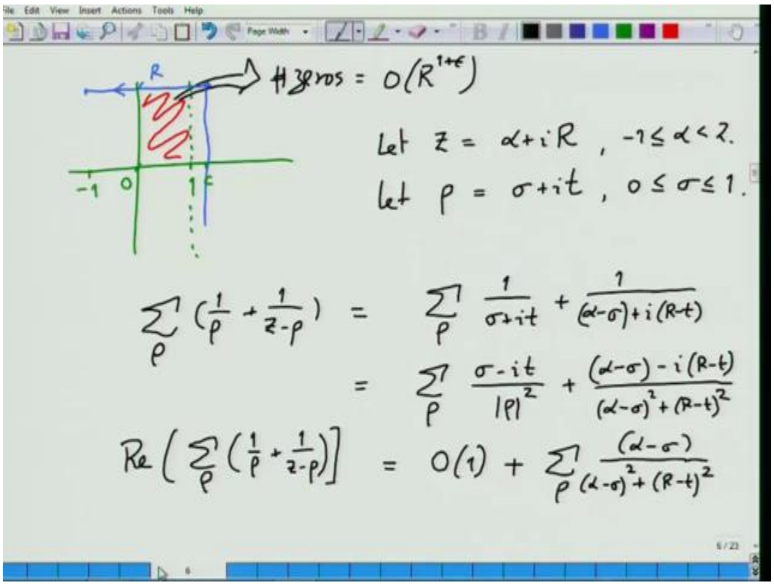

We have done everything except bounding zeta prime over zeta in that region, so that has to be done and for that we have to get an upper bound on this quantity an upper bound on this quantity for z varying between in this region, we sort of in this analysis. We just say this because we cannot vary z in this region this fixed z has 2 plus i R and we know that zeta prime and zeta is bounded there.

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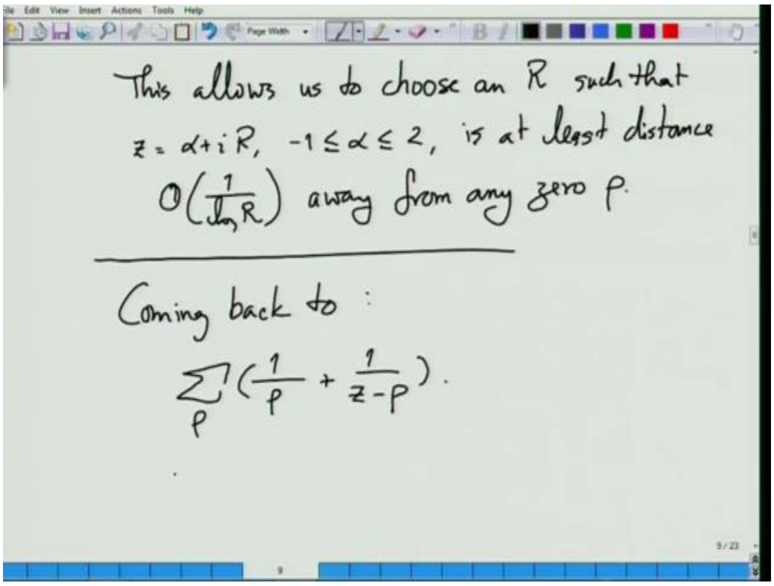

So, we do not have to work hard, but now I have to work hard, so let us get back to this fortunately the hard work is not too much, so you can manage it hopefully within this class let us see. So, now after this lets come back to this if am not allowed to look at the real part of this then this quantity 1 over rho itself becomes troublesome because it is not clear whether its sum over rho or 1 over rho is not given at least not going upon it.

This may not mean much because this quantity which in negatives, it may it may cancel out things, then eventually they get some sensible value and we know that it gives some sensible value, it is bounded. We just need to know what upper bound is, but for the sake of analysis we cannot easily have this, so we employ another simple trick which is that we consider this quantity.

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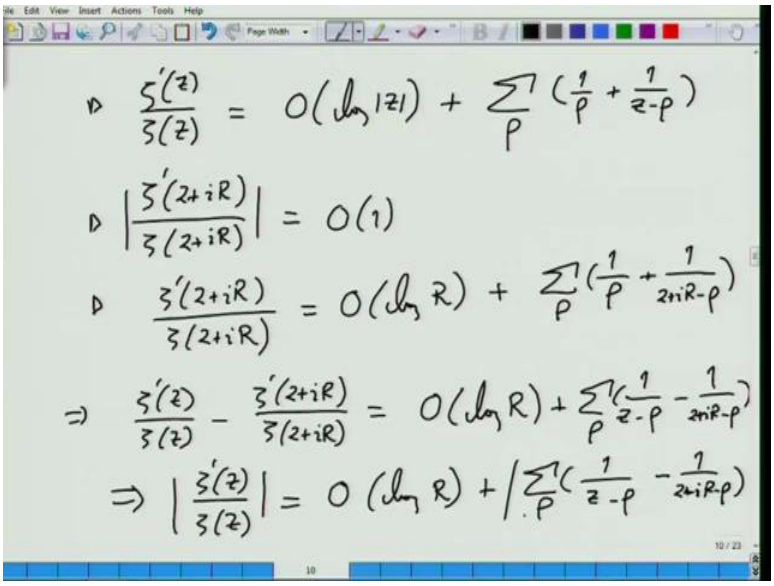

So, we had that zeta prime or zeta that is equal to order log z plus the sum, so what we do is let us say simply, we already know we have used this is order 1 and absolute value absolute value and this of course satisfy the equation as well. So, we put all this together, our target is to bound this zeta prime over zeta, we will derive an upper bound zeta prime over zeta. We are going to use this, but this is hard to bound, so all we do is we subtract zeta prime over zeta from zeta prime over zeta and 2 plus i r, what is this equal to its order log R actually order log z. Here, it is also order log R because z is varying from minus 1 to c is always the real part is always bounded.

So, this always going to be order log R is order log R is order 1, so plus the nice thing is that sum over 1 over rho and sum over 1 over rho cancels each other out. What we are left with is this and since we know that this is bounded this constant, so absolute value, this is order log R plus absolute value, so the problem now is to bound this quantity, so let us focus on this.

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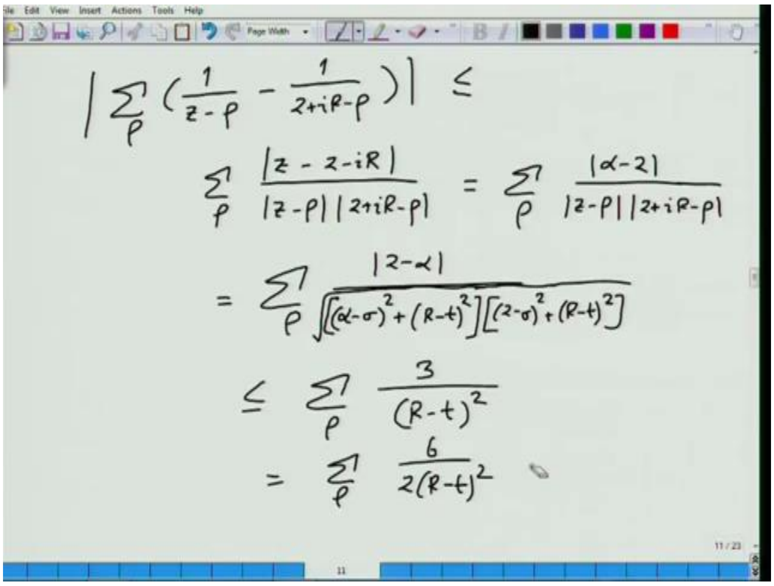

We are looking at the absolute value this is equal to and if we say less than equal to this is equal to z is alpha plus i R where alpha is varying quantity. So, if you just take that in then alpha minus 2 alpha plus i R minus 2 minus i r, so that cancels i R and now unfortunately I will have to expand this also, what is this z minus p, we know that this is alpha minus sigma square plus R minus p square root. Let us take square root outside what is this part that is simply the only change is 2 minus rho square plus R minus t square. Now, I need to get an upper bound, so I will replace this with smallest possible value of alpha minus sigma what is that 0, we will throw it off what about this 2 minus sigma it is not 0, but assume it 0, throw it away.

This 2 minus alpha I need to replace with the largest value 2 minus alpha largest value is three that is much simpler expression. Now, we are very close that means if you see this is this familiar we just derived something similar to this where was it? It is here that sum over all rho 1 over 1 plus R minus t whole square order log r, so that is giving us a good upper bound. What we have here is three forget 3, 1 over R minus t whole square, the only thing missing is one plus here, but for this we would have had our upper bound where we need order log r.

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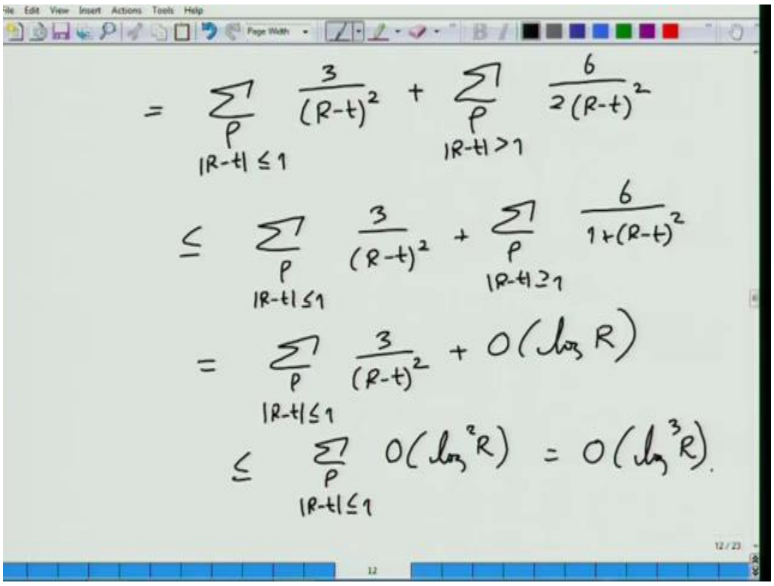

We are not too far off because I can write as again 6 by 2 R minus t whole square which is less than equal to which I will break as sum over rho s such that R minus t less than equal to 1 plus sum over rho s equal to R minus 2. This we can bound because absolute value of R minus p in this sum is bigger than 1, 2 twice R minus t I can surely replace by 1 plus R minus t whole square and lesser value. So, this is number 1 and this is a sum over all rho s actually except for a few 0 s, we know that even if you sum over all 0 s this is bounded by order log r.

Now, we saw this, but this is sum only over finite limits like R is there is just R plus 1 and R minus 1 in their strip whatever the 0 s are we are summing over this and this is the quantity. We want to sum and this is where the choice of R plays the role I just fix that R was to be chosen, so that it is away from every 0 by about order 1 over log R what that translates to in terms of this t s represent the 0 t s represent the height of 0 s and R is the height.

Here, we cut that region and that height is away from all this t s of 0 s by order 1 over log r, so R minus t, therefore for all t s in this is at least one by log R some 10 by log r. We see this is in the denominator I can replace this quantity by R minus t by 1 by log r, so what we get here is order log square r.

Of course, this quantity has nothing to do with the sum we are just summing over all the 0 s in that strip how many 0 s are there in that strip order log r. So, this is order log cube

R that is it we have bound zeta prime over zeta in that region while cutting across it, now I am done.

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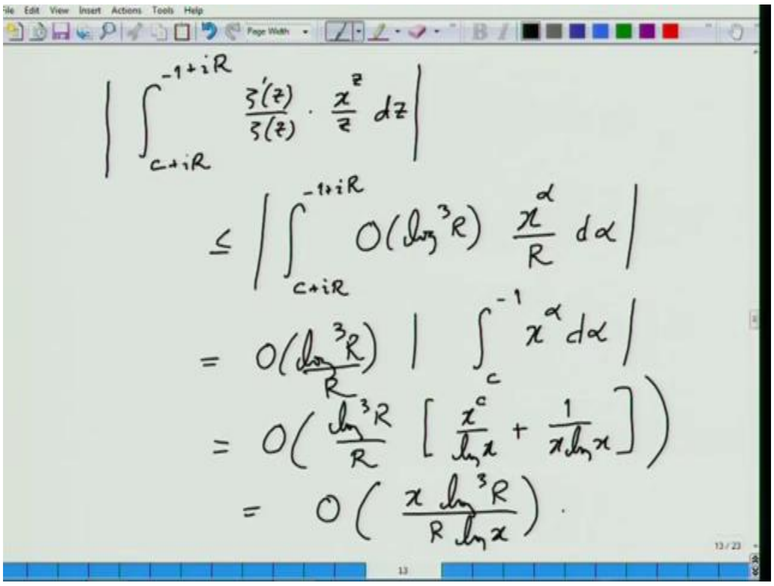

Therefore, the integral as we went along from c to minus 1 plus i R minus one plus i R of zeta prime over zeta z by z d z. This is bounded by zeta prime over zeta, I can replace by log q to the power x to the z is replaced by absolute value of x to the z, which is real part of z. So, I do not want to mess with this, so I just replace with x to the alpha mod z is at least r, so I can divide this by R and here d i that is simply d alpha. This is order log cube R times the divide by R of course here times the integral going from c to minus 1 x to the alpha.

This is of course familiar this is equal to order log cube R by R x to the alpha will integrate to x to the alpha by log x and then we have going from c to minus 1, we get x to the c by log x plus 1 by x log x. Now, use the value of c was 1 plus 1 over log x, so this is actually x, x in c is essentially x and this is clearly bigger than this, so I can throw this away also, this becomes order x log cube R divide by R log x. This quantity is going to be bigger, let us now I will have to really go back where you will end that one, now one of you will tick that out.

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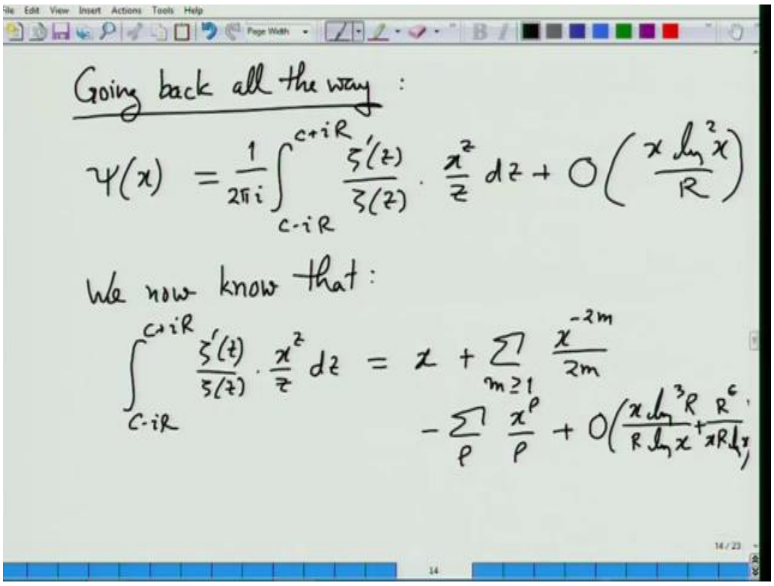

What was the original equation derived of psi x psi x is, yes that is right, this we estimate it to be is integral c minus i R to c plus I r, I think there is 1 over 2 i also somewhere zeta prime z over zeta z x to the z by z d z plus order the question is what is the error? We simplified that x log square x by R that is what we had and now we know that this integral this is equal to x. Now, pull out this x minus what was it this is the residues are all 0 s all residues and all poles, so there was a residue at pole at x equals that equals one which was x one pole at z equals 0 that was zeta prime over zeta 0, that is order 1.

Forget that then there were residues at all negative even integers what were the residues there was a sine also. I think it was like x to the minus 2 m by 2 m, now I do not remember the sign whether plus or minus this plus and then there is a negative right sum over all rho s x to the rho by rho. The error terms now what are the error terms, so we had this three error terms the integral from minus u plus i R to minus u minus i R that vanished went to 0 because we have sent u to infinity.

So, that just leaves two integrals minus 1 to minus u what was that integral we estimated this last time of course something like R to the 1 minus epsilon minus something it means let me pull this out. So, as long as R log x is common, so as long as x is bigger than R to the epsilon over x which is x square is bigger than x is bigger than R to the epsilon by 2 this is going to be, but anyway let us keep this.

Now, we put this here what so we get psi X equals this is 1 by 2 pi i, let us take this out and push shut that in here we get x plus by the way this is familiar what is this sum over m greater than equal to 1 x to the minus 2 m divided by 2. Just differentiate this one this sum what do you get you get sum over m greater than equal to 1 x to the minus 2 m minus 1 and what is that sum.

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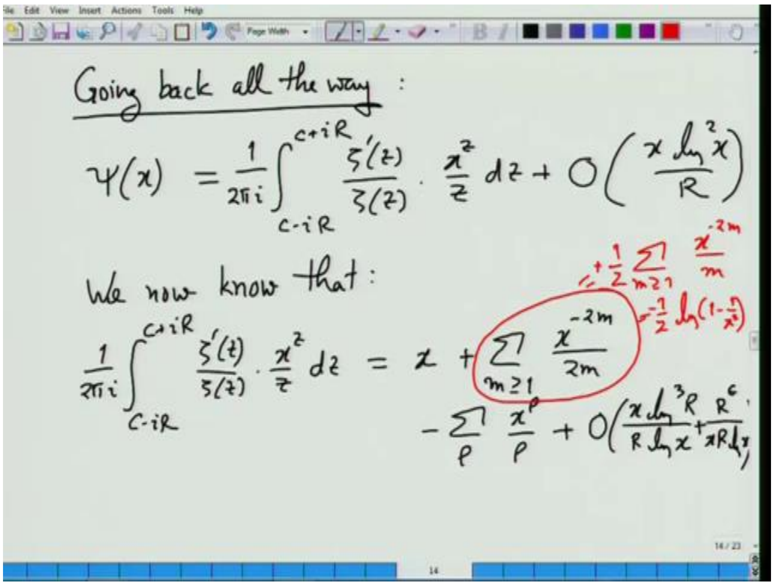

Let us take this away, this may be you do not need to differentiate this, let us just write this as half of its familiar to you anyone what is log of 1 minus z log of 1 minus y let us not talk about that log of 1 minus y the sum of m greater than equal to 1 y 2 d n by m. That is precisely what this is equal to half of log 1 minus 1 over x square and there is a minus also here there is a plus here. So, there should be a minus there is plus there minus log of 1 minus y is minus of sum greater than equal to 1 y to power m by n.

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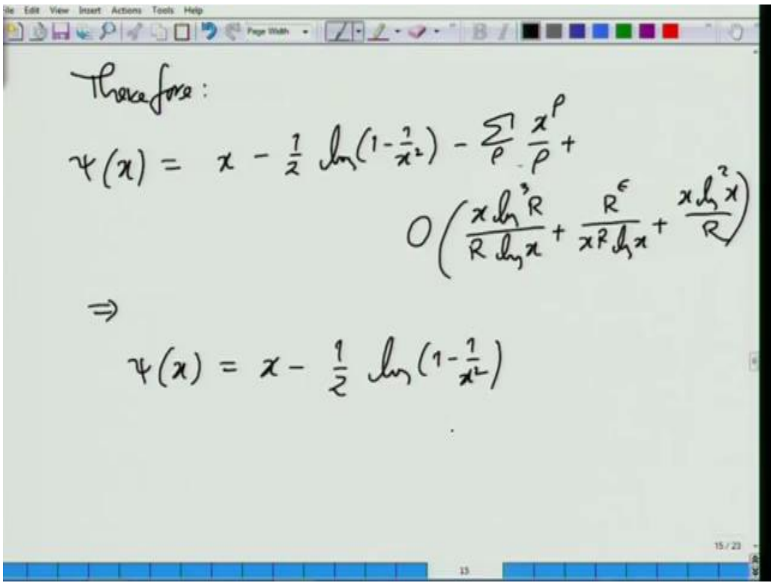

So, this actually is a well known quantity, so I can replace this with minus half log minus x to the rho by rho plus all the errors what are the errors what does the errors add up to this plus this. Thus the original one x log square x by R and I need to minimize this error, so what is the quantity at which this error is minimized? My only parameter of control is R I can choose my R to be just about anything of course, whatever value of R derive from here eventually I will have to portray slightly in that band of t n t plus 1 to reach the right avoid the all the 0 s.

That does not affect the error because it is very small in calculation, so what is the minimum value of this expression for what R does it achieve R equals infinity. Of course, at R equals to infinity this is 0, so what happens, let us see what happens, so this certainly implies that psi x equals x minus half.

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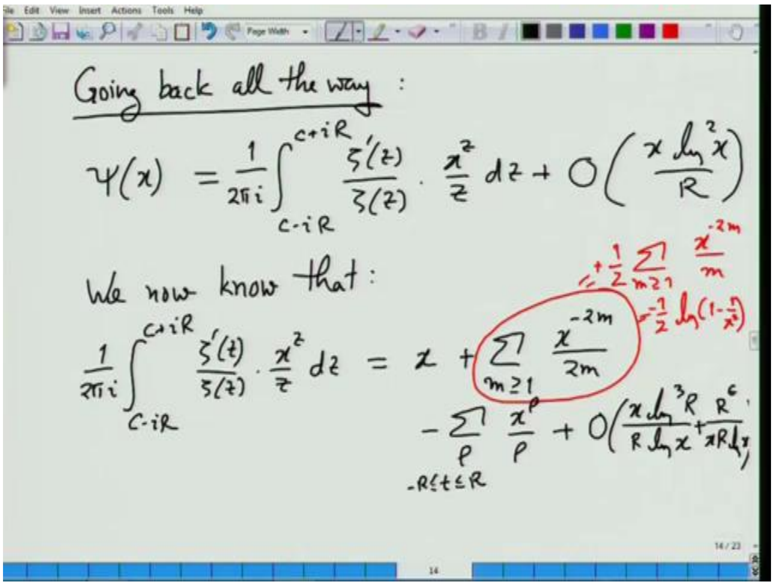

I have been making a mistake here at least here, I made a mistake this some of rho s is only for t less than equal to r. Actually, minus R less than equal to t less than equal to R because that is the reason we are integrating and of course we can send R to infinity and derive this an exact formula for psi x precisely.

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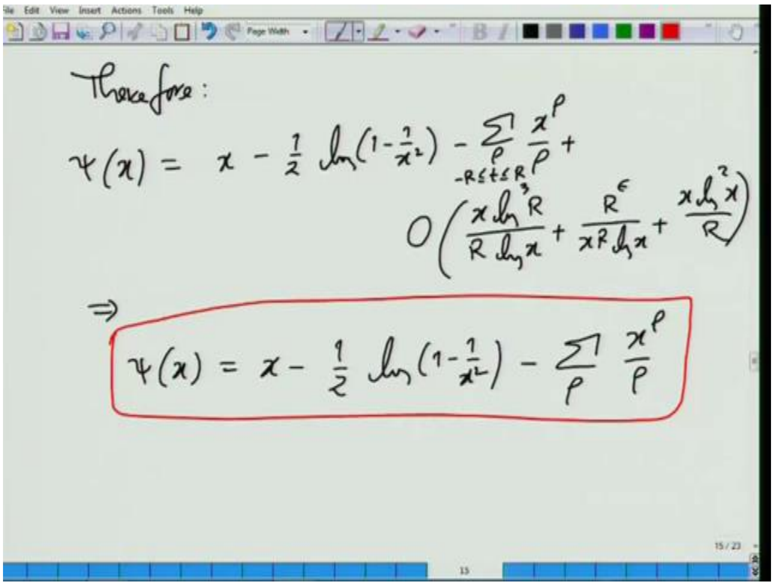

Now, the first two expressions are well understood the last one is we are to understand, but this also shows that this very starkly that the distribution of primes is dependent on 0 s of zeta function non trivial 0 s of zeta function, how they are distributed? What determines what this is unfortunately not very good in terms of estimating size, this is nice formula, but what about this quantity what can we say about this quantity it is an infinite sum first of all. Secondly there is a rho sitting here, which is in 1 by rho which I cannot even say that this is bounded, if there was a power of rho, which is and worse than that there is x to the power rho as well in the numerator.

So, that is all of this put together make this equation useful this is very nice and interesting expression, which is not useful expression in order to get a useful expression we need to limit R to certain limited or bounded value. Then, we can get a error of all this, so this is of course a trade of between what value of R we choose, suppose we just add up all the all of these quantities in the worse case, they will all actually not get added up. Let us say we assume in worst case they all get added up what is the maximum contribution from 1 x to the power rho by rho, well this would be x of the rho will contribute square root x maximum.

What am I saying is X to the rho will contribute the real part of rho is between 0 and one so if the real part is one than in the real bad case this can contribute something odd like omega x. If it does contribute omega x, then the first two expressions are sort of meaningless actually second one is anyway meaningless, second one is order one x is always going to be bigger than 1, then log of 1 minus 1 over x square is very close to 0. As x increases, this actually goes to 0, so this actually redundant only the first one is important if the real part of rho is 1, why it actually sum of such rho s can get together and cancel out this x here. We have no clue what psi x is going to be, on the other hand if these are going to be smaller then why?

Then, we can say that maximum contribution say by real part of rho is sigma is what we had assumed then maximum contribution this can make is to the sigma divided by actually some absolute value of rho. That is smaller than actual sigma and maximum contribution is going, but the smaller the sigma is the smaller the contribution from this quantity is what is the smallest value sigma can take equal to 0. So, if the sigma takes value 0, then contribution here is 0 that is the best case, but that is also the worst case because if there is a sigma equals 0 by symmetry, there is a 0 at sigma equals 1. So, this sigma is going to be 0, but that sigma is going to be maximum, so you gain nothing.

So, taking sigma below half does not gain you anything because symmetry gives you bigger sigma that is it so that sigma equals half is when there is this contribution will get minimized. If all sigma were at sigma equal to half all rho s were at sigma equals half then this contribution will globally will be minimum to this quantity and that will be around square root x. If that is a very rough analysis to say that this error introduced by this in the best case will be about square root x, but then we have to look at this error how much is this going to be? So, let us try to estimate all of this, so what I am going to do is before estimating all of this thus now we can already fix a value of R because anyway the error going to be introduced by this quantity is about square root x.

Also, notice that the bigger R is more there is error because more of these terms are there so we got R as small as possible to reduce the contribution from this, but if we make R too small, then this entire error blows up. Now, we know that the R is the error is going to be square root x, so looking at this x by x by R x by R it makes sense to fix R to be square root x there about. So, if you fix R to be square root x this contribution is also about square root x and that is what we are going to do.

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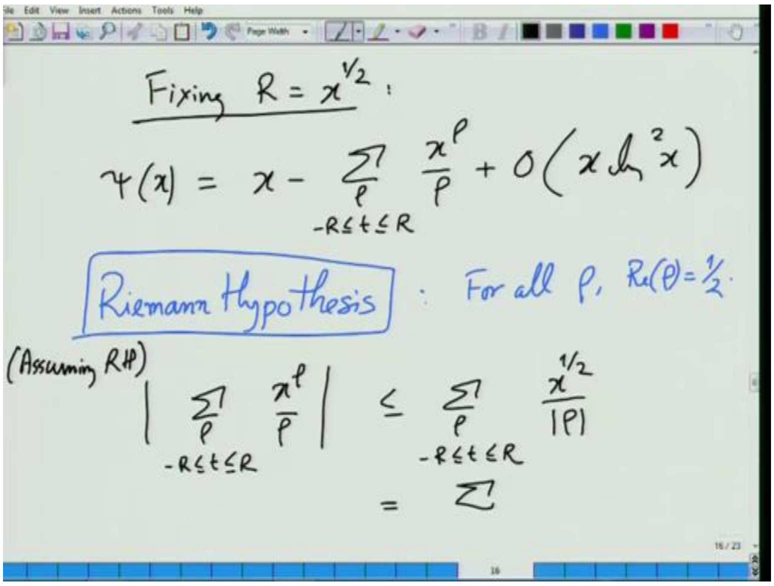

We get psi x equals x minus and we are now going to throw this away the second term because that is pointless and what happens to this term, this becomes square root x log square x this can be safely ignored and this becomes square root x log square x. So, both of these actual terms match, so that also shows that this is the best you can achieve in terms of choice of r. When you want to choose it around square root x, now we come back to the Riemann analysis this is all this analysis was done by Riemann and everything that I have shown you today was more or less done by Riemann. It has taken me what twenty odd lectures to do that which you did it in eleven pages very densely packed.

He skipped the things which were obvious to him, but not so obvious to the rest of world, so now at this point is where Riemann made this hypothesis for all rho s real rho is half that is the alternate formulation and the reason was to minimize the error. If all the 0 s are at straight line contribution of this sum to the error is minimum, so what is the contribution? Let us just add everything up what is x to the rho x to the rho is absolute value x to the rho is square root x assuming Riemann hypothesis. So, all this calculation is now assuming Riemann hypothesis, then absolute value of rho absolute value and now we take the sum of this from minus R to plus R t going from minus R to plus R 1 over absolute value.

This you can do in many ways, one of the way we can do is to just derive, we derive the one band of it one there are going to be only log t minus 0 s. So, if we just stick that in there, so if I replace this by summation, so by the way there is symmetry around real axis also of the 0 s that follows almost trivially.

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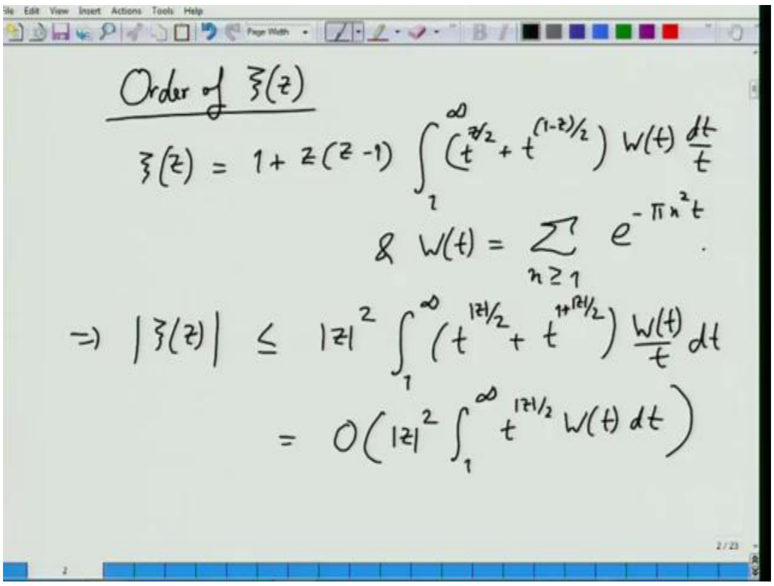

So why is symmetry along, let me write this we had this right psi z 0 s are precisely non trivial 0 s. So, if z is a 0, what about z bar z bar is a symmetric point that right z bar would be z bar here t to the z bar here z bar here right. Now, what do I do, why is that 0, well because that is equal to xi z whole bar xi of z bar when you replace z by z bar when you replace z by z bar everywhere is same as taking xi z taking the complex conjugate of that t to the z.

You take this conjugate what would happen t to the alpha plus i beta, what is the conjugate of that split that t to the alpha conjugate of alpha t to the i beta conjugate is t to the minus i beta. So, t to z conjugate is t to the z bar, so whether you replace z by z bar or take the bar o xi z the answer is the same and that is why 0 less than in fact that for any analytic function forget about this because analytic function you can write as a power series or something again, the same R applies.

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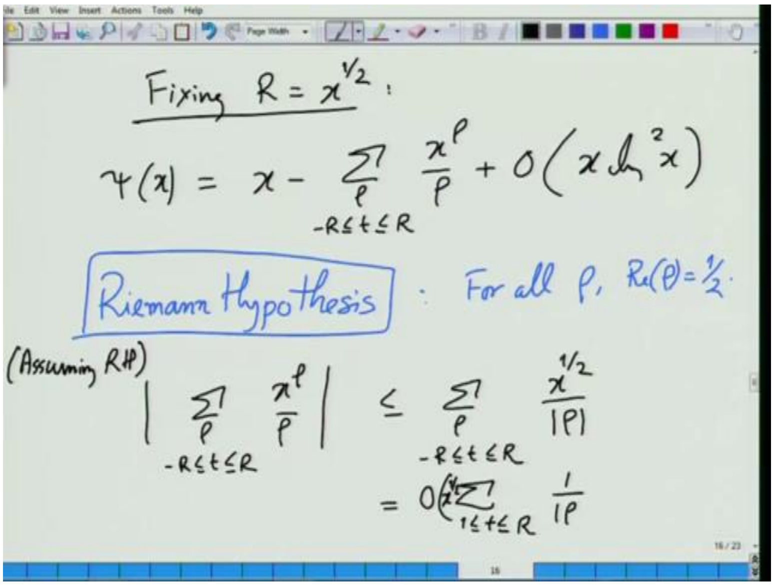

Now, coming back to this, so I can replace this is I can replace this with the order and some t from now 0 to let us say 1, 2 between 0 and 1, there are only finitely many 0 s. So, there contribution is only going to be finite, finitely many in the sense some constant times square root x. So, we can that we can ignore and we can just look at 1 less than equal to t less than equal to R of course square root x comes out i over absolute value of rho.

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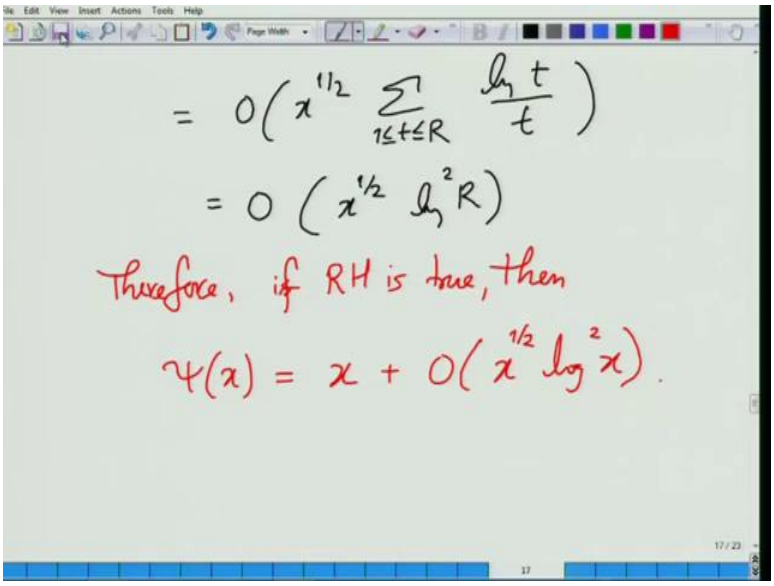

Now, I can replace this by sum over between t and R because how many rows are there for between t and t plus 1 log t by absolute value of rho when rho is about t is how much rho is half plus i t. What is absolute value of this is about square something, which is about t right and how much is this sum. Now, log t over t going from 1 to capital R log t is upper bounded by log R and sum over 1 over t where t goes from 1 to R is first log R square root x log square r. So, therefore if the Riemann hypothesis is true then psi x equals to x plus this is where we close this now we have completely derived the analysis of Riemann and given this is what is psi x if Riemann hypothesis is true.

If Riemann hypothesis is not true, then psi x estimation is not true because this error term then starts becoming bigger and bigger. Now, after doing all this you might believe that we started from estimating psi x gone all done so much and eventually we showed this implication if Riemann hypothesis is true, then psi x is this. You might think that in doing such this implication has such a long prove with, so many steps that we will lose something lose something meaning if psi x equals this then Riemann hypothesis may or may not be true.

We have this is a very long implication, so probably there is a good chance that we lose somewhere the equivalence and only have single direction. Surprisingly, that is not true if psi x equals this then Riemann hypothesis is true and that is a clearly simple proof I will show it next time.