Riemann Hypothesis and its Applications
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Lecture – 2

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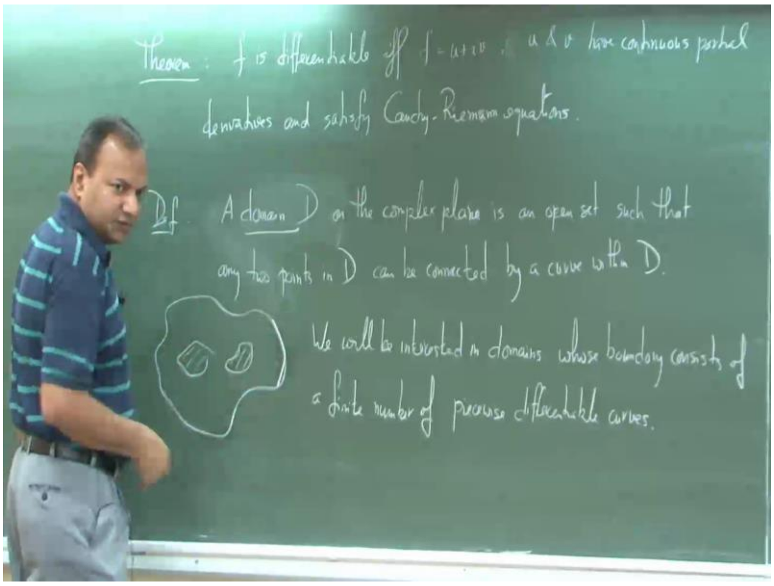

So, where were we were so that is the thermo we did last time and I had problem with this which is right this should be continuous; partial derivate need to be continuous for this sequence hold. The proof of this thermo I am going to leave up to you, to work out and see and convince yourself. As I said that, we are going to be interested in functions which are at least differentiable on the complex lane.

Therefore, this is a very critical or important thermo for us. Now, let me put define specifically the given name to the functions of the kind we are going to be interested in; before that I need to define domain. So, this the kind type of sets we will be interested in these are the domain on the complex plane. If these are open sets first if there are any 2 points in the can be connected by, the credenda formal way of saying something very simple that D.

A region on the complex plane, which is not which does not consist of disjoints parts it may have some holes on it which is fine. So, this is most general kind of sets we are going to sets of points we going to work on and. In fact, more specifically we can rule out some funny boundaries also 1 can have boundaries which are not continues or happen be even a bizarre way. So, let me write down that as well.

So, we are going to be interested only in the domains of the kind whose boundary of the domain is this outer curve as well as, the things inside which throughout the piece the points inside. So, the boundary of the domain consists of the finitely many piece wise differential curves. So this is 1 curve, this is another curve, this is another curve and they are differentiable they do not be having any funny way. 1 could have think like this which is fine or finitely whenever, I say domain I would mean a set of points of this line.

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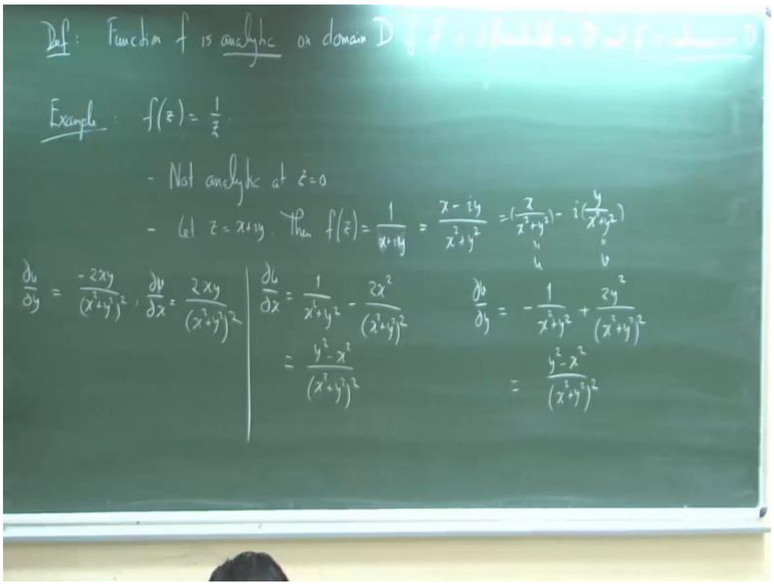

Now let us, come to our main definition we call function f to be an analytic function on the domain D If it is differential everywhere on D and is derivative is continues. This condition is actually very super flues, as long as if f is differentiable everywhere on the domain D it actually follow that f primes continues. But we will not worry too much about let us assume, this is the case that I will not even going to prove why this is super close because, it is not important.

So, it is the analytical functions that will be our object of interest and study. The reason I have defined an analytic function on a domain instead on the whole complex plane is that, the many of the function. So, that we will study are not analytic over the entire domain or over the all the or over the entire complex plane. But they are sort of piece wise analytic on the different regions of the complex plane.

A simple example is let us, take f z to be a 1 world let see if it is analytic, 1 thing is clear in any domain which contains the point 0 this is not analytic because, diverges is at point 0. So, what about other points? At least it does not diverge. So, it may be analytic how do we show that this is going to be analytic? We have to verify the f is differentiable and verify that f prime is continues. And to verify differentiable we have to use this theorem.

So let us say, so we can write as this form this is the reason for writing it in this form this is going I can take this to be function u and minus of this to be function v. And I need to be verify the Cauchy Riemann for the questions of these, so this is u and this is v. So, what is del u by del x? And del v by del y. So, are these 2 are equal they are equal. And similarly, let us look at del u by del y that is going to be and del v y del x that is going to be and this is negative of this that is fine.

So, what is f prime it is differentiable that is good to know what is derivate of f? What usual derivate yes, but 1 has to actually justify that over real's yes we know who derivate who to calculate derivates; whether the same law apply complex number also is something that needs to be actually verified. And that process I will leave it to you it actually turns out to all the laws of differentiation which hold good over real's just carry over to the complex numbers. So, at some point latter we will see a general principle why that happens. But for now let us just continue using usual laws of derivative differentiation.

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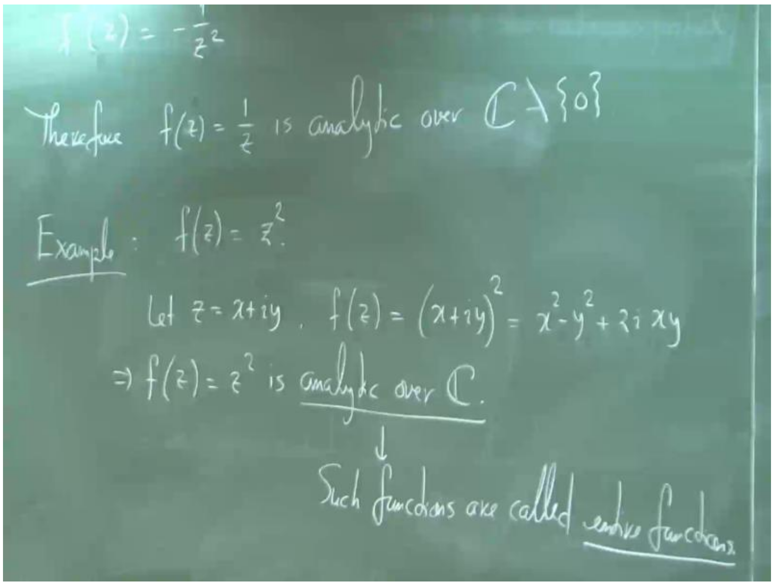

So therefore, f prime z. The, f prime z is minus 1 by z square and we need to just verify 1 more thing, that this is continues which is of course, the case for all points except z equals 0 this is continues. And second order of the partial derivative these are also continuous, as long as x and y both are not 0 which is something we are ruling out. So, this together implies, so take the complex plane and take out the point 0 this is the domain it is an open set with 1 boundary inside and this function is analytic holds good.

Let us, equate another example is this analytic? An f z over what domain do the same thing. This is u this v and with this we can verify almost directly del u by del x is 2x; del v by del y is 2x. So, they are equal del u by del y is minus 2y; del v by del x is 2y. So, it is negative of this both are continues functions over entire complex plane. Therefore, f is differentiable over entire complex plane, its derivative is again following the same which is 2z which is continues function over the entire complex plane.

Therefore, in fact there is a name for these property functions which are analytic over entire complex plane are called entire functions. So, you would naturally expect entire functions to be a simpler kind of functions, do not behave strangely at any point and that is indeed the case. And there is very nice theory of entire functions, which tells us lot of properties of these we will not now, but when we go into zeta functions at some point we will have to touch upon entire functions also.

And at point we will see some of the interesting properties also some should we look at more examples that is going to be so what, time wasting 1 should expect. There are some general theorems 1 can prove, about what kind of functions are analytic which is true and 1 of the real basic 1 which also us to construct more and more analytic functions is this theorem.

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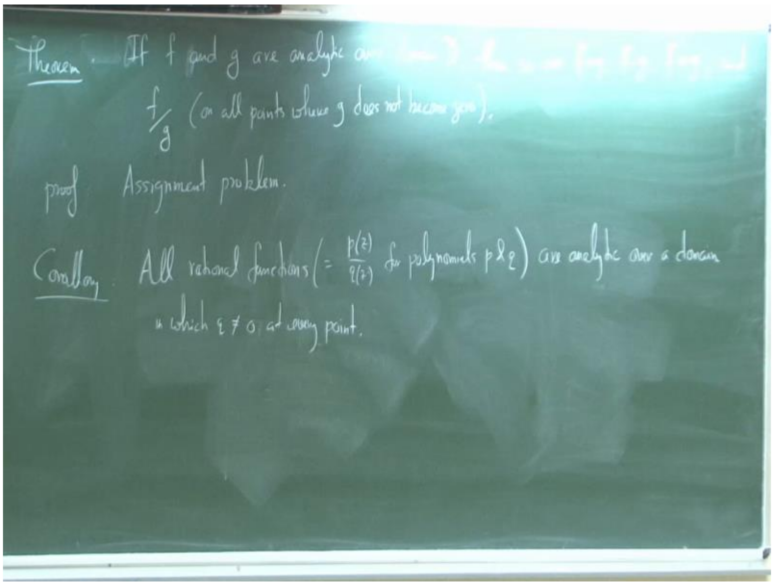

So, if 2 functions f and g are analytic over some domain D, then all there arithmetic combinations are also analytic; except there 1 has to be careful looking at the f by g that g should not become 0 so that it diverges. So, as long as that does not happen f by g is also analytic; the proof of this I will leave as assignment problem it is very simple you just have to blindly follow through this the same technique used f to be u1 plus v iv1 g 1 to be iv 2 just do the operation and verify whether Cauchy Riemann equation holds good.

So, I keep give this assignment problem and at some point I will ask for you to submit those assignment, so submission to those assignments. So, do not postpone solving these problems because, if do that you will probably forget what were the assignment problems. And if you come and ask me, I will also would have forgotten, so do not take that risk. Because, I will remember by looking at least 1 of you would have remembered the problem right.

So, that I will grade according to that those who have forgotten about the problem will then loose so now, with these we are in good shape what immediately follow from this is all polynomials are analytic. In fact, all rational functions are analytic over the domain, where the denominator does not vanish. Pick a domain in which q does not vanish at any point, and then p by q is analytic over that domain this is good to know.

But if you keep our goal in mind the zeta function, that is not a rational function it is a infinite power series actually. So, we will need ability to talk about power infinite power series in order to analyze the zeta function. So, that so we will very soon come to analyzing whether a function give as a in power series in analytic or not. Well it is not right actually the way is given it is not even a power series; it is an exponential series; which we can write in terms of power series yes.

So, there is still lot of work to be done before we start discussing properties of zeta function. Now, very brief detour here look at an analytic function is differentiable everywhere and not only that its derivate is continuous which means, that if you think of the take the analytic function and look at its derivate at every point on the complex plane.

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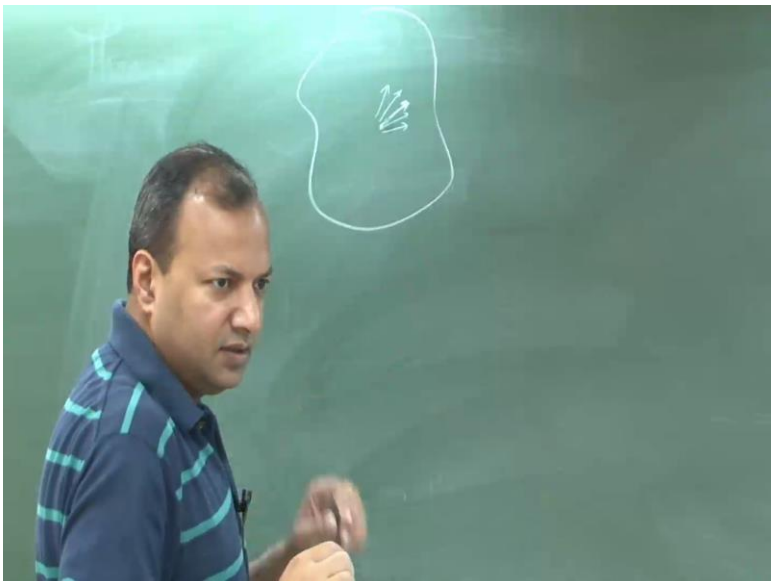

So, let us say this is some domain and you look at a derivate take any point look at derivate of that point; at that derivate of the function, at that point think of that derivate as giving you a vector which is what a derivate does, it is a tangent vector right. So, and look at the derivates in the neighboring points again as a tangent vector. So, what the analyticity of f implies is that as you move from any neighboring points towards this particular point.

Then, look at the ways the derivates of the tangent vectors evolve they change very smoothly and converts to the derivate value at this point. So, this has a very direct connection with flows on 2 dimensional plane which I mention earlier also, that when a fluid flows on a plane. Then, the with every point you can associate a derivative which is the velocity of the fluid at that point it has a direction as well.

Now, if it is a flow without any turbulence or source or snick, then these flow derivatives will be as this form very smoothly going from 1 point to another. So, that is that is the correspondence between flows in 2 dimensions with analytic functions. In fact, analytic functions are heavily used to study these functions.

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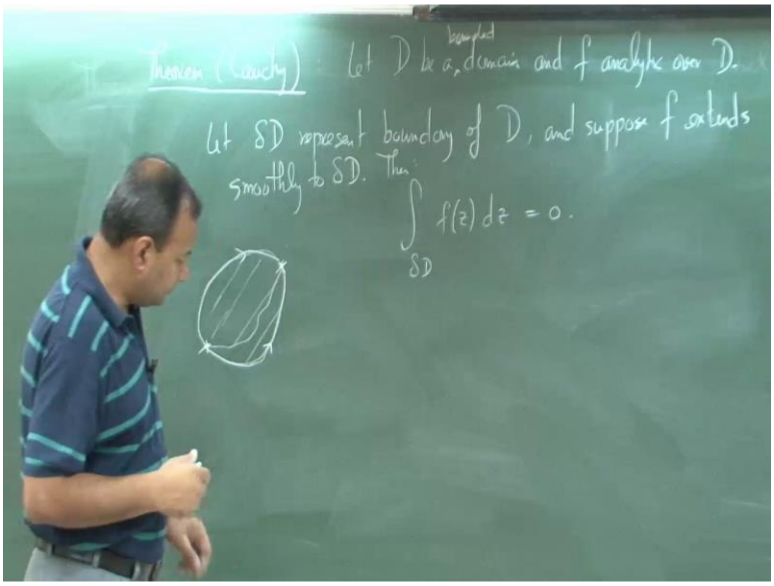

Now, we come to most important theorem on analytic functions almost everything subsequently follows this theorem. Let D is a domain and function f analytic over D, then if you look at integral of f over along the boundary delta D represents the boundary of the domain. So, here should say bounded otherwise this does not make sense, it is a bounded domain it is going to infinite or anywhere. It has a boundary well defined to integrate f over this boundary; we get 0 function f needs to be defined that is true let us look at it.

So, I should write and suppose f extends smoothly not only we want f to be defined on the boundary it is extension be smooth because, D is an open set. So, by staying within the closer and closer to the boundary and in the limit you will hit the boundary and there the function of fl value of f should be in the limit. So, then this integral over the boundary is 0. This is on 1 hand very remarkable theorem, that it does not as long as f is analytic does not matter what other properties of f has its integral is going to be 0.

On the other hand, if you think change your perspective it is not so surprising. Because, what it is saying essentially let us take a simple example, suppose D is an domain like this then what this theorem this is the boundary of D what theorem is saying is, if you integrate f along this lets pick 2 points on this. And integrate f along this path or integrate along f along this path the resulting value will be same. Because, this is equal to this statement because it is if 2 integrals are same, then integral of f along this boundary is if this is first negative of this and that cancels.

So, essentially this theorem is saying is that along the boundary it does not matter which path I can take any 2 points and follow any path along the boundary to reach 1 point to another to get the integral value and. In fact, f is analytic over this entire domain we can say take fix these 2 points and take any path from this point to this within the domain D. This integral value is going to be the same with respective of this path and this is not surprising.

Because, we have when we define the fact that f is differentiable we said that does not matter which direction you approach f you are still going to end up with the same value. So, there is a direction variant inbuilt in the definition and. So, this theorem extends that invariants to longer path instead of localized invariants which is what the differentiability implies this is saying longer scale there is an invariance.

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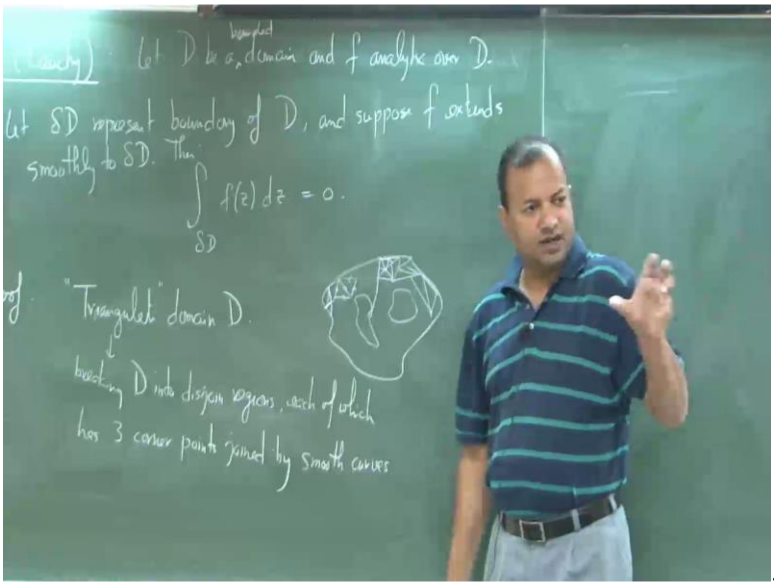

So, this such an important theorem and we will prove it. This also gives a sense of type of proofs which are used in complex analysis. So, we start with domain D and triangulate it. So which means that, we have a domain like this basically you make lots of triangles covering the entire domain of course, you cannot cover entire domain with this. Because, boundaries particularly they will be curved; they are not straight lines.

So, in the boundaries you might have a situation where what you get is not a triangle, but something which has 1 of the sides being a curve. Fine, that is something will accept what basically I want is, division of this entire domain into small pieces till finitely many which have only 3 sides as far as possible these are straight lines, but if not then they can be curves. So, that is the meaning of triangulate for us here; see that is yes we need to prove, but that is easy.

Because, this is a contented 1 by definition domain is connected. So, it’s bounded that is given by the you do not even need connected this for this. Because, as long as if it is not connected since it is bounded it will only have finitely many this joint regions for each region you can do this triangulation. So, this triangulation basically implies that breaking...

So, what I will going to look at each one of this triangle; triangle is the sense of this and integrate f over the boundary of f. What I will show is, f on each one of this boundary integration is 0. So, that is going to be sufficient enough to prove this because, once you add up all this integrals of course, I have to fix an orientation of traversing these regions triangles as well. And by convention we always traverse in the counter clock wise.

So, that is consider positive traversal of a region of a boundary clockwise; traversal is also used at time, that is considered as a negative traversal. So, we will traverse all of them in counter clockwise and. So, when you have a neighboring triangle here this will also be traverse counter clockwise. And if you look at this particular curve or edge once it traverse in 1 direction; second time it traverse in another direction. Therefore, the integral sum of integrals will be 0.

So, each integral along each of these boundaries is 0 when you add them all up; the resulting integral is of course, 0 and the boundary that you get of resulting integral is precisely the boundary of this region everything else cancels out. So therefore, all that I need to show is for take of any of such region for triangle for this general sense. The integral along the boundary of this triangle of f is 0, for that we will start with something a very simple actual triangle.

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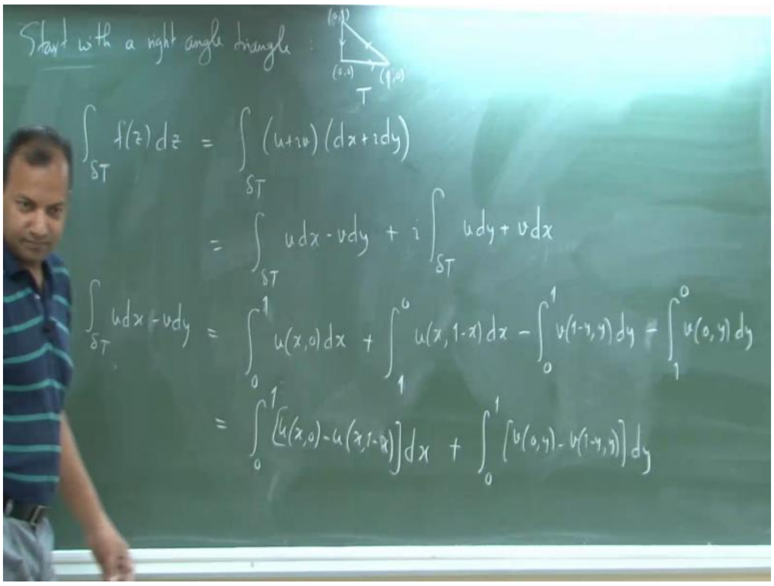

And will show that this is the point 0, 0 this is a, 0 and this is 0, a and will show that the integral 0. Notice that, I have also give coordinates of corner points that is also we will take care of it later. Let us, call this as T and what is delta T of fz delta T boundary of T, so traverse counter clock wise. There are 3 edges defining this boundary, so we can break this integral down into 3 pieces integral going from...

Now, I will have to first transfer lets us, delta T this is over complex sense lets translate this to the 2 dimensional real structure. So, f is u plus iv, so this write it u plus iv and dz is dx plus idy. So, just collect the real and imaginary parts separately becomes udx minus vdy plus iudy udy plus vdx and since, this is all real we need to show that this is 0 as well as this is 0.

So, let us consider the first 1 now we can talk about this as 2 dimensional plane and break delta t to 3 parts. In the first part, only x varies it goes from 0 to a y does not vary in the first part, so dy is 0. So, for the first part you only get u of x 0 dx in the first part y is 0; what you get in second part, where you make it even simpler. Let us, make it 0, 1 and in the second part we go from 1, 0 to 0, 1 that is the line x plus y equals of y equals 1 minus x that we need to follow.

So, x goes from x goes from 1 to 0 and you get u x and what is y? y is 1 minus x x minus 1 and what about y? y goes from 0 to 1 and you get 1 minus y dy. And what about third region? In third region x is 0. So, dx vanishes minus y goes from 1 to 0 and we get v 0 y dt.

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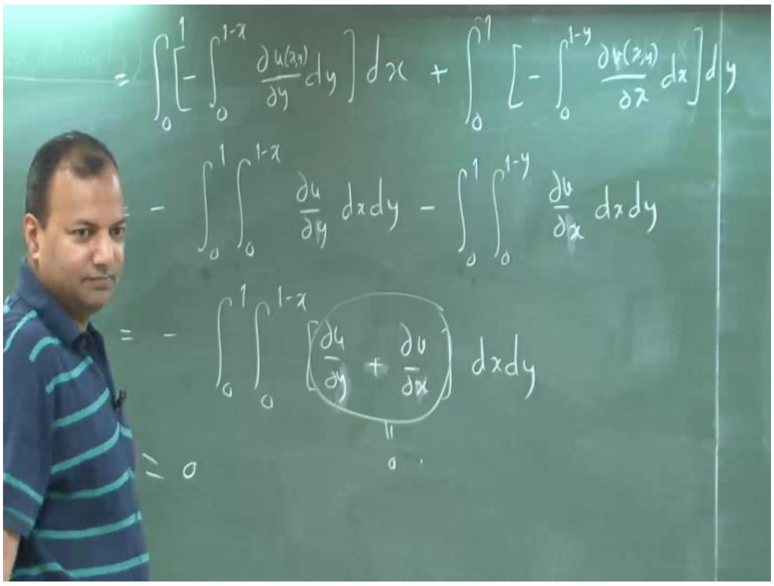

Just rewriting it as now look at this expression inside this is ux 0 minus ux 1 minus x I can write this as, just a trivial rewrite partial derivate with respect to dy integrate with dy this of course, dy going from 0 to 1 minus x. So, that is the difference there is a problem with sign some where no, there is problem with sign somewhere del u by del x. Just inter change the order of integrals here, 0 to 1x 0 to 1 y y 0 to 1 minus y and 0 to 1 minus x. So, this integral is same as this integral this limits del just swap this and add this you get this is 0.

Because f is analytic therefore, it satisfies Cauchy Riemann equations and according to 1 of those this is negative of this is. So, this is 0 and therefore the whole thing is 0 exactly the same proof works. Here except that you will be using other Cauchy Riemann equations. So, that shows that at least along boundaries of this specific triangle there is 1 unique triangle 0 0 1 0 0 1 the integral is 0.

So, we will continue this analysis next time. No, it is a integral this was a this was a area integral; double integral x going from 0 to 1 and y going to 0 to 1 minus x basically here you are integrating inside the region. But you do need the limit in the limit there should not there should not a funny thing happen, that is already guaranteed by the fact that f extends smoothly over the boundary.