**Riemann Hypothesis and its Application**
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**Lecture – 19**

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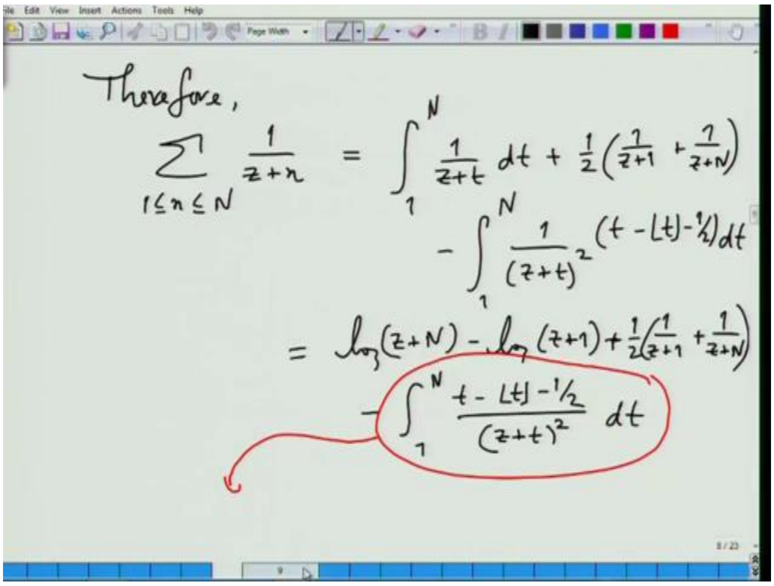


So, we were here last time we were trying to estimate this sum which we used the Cauchy McLaren formula to estimate to be this. So, this we treat as error term error term and we estimated the error to be this bounded by this quantity, so what we can do is, therefore we can write this.

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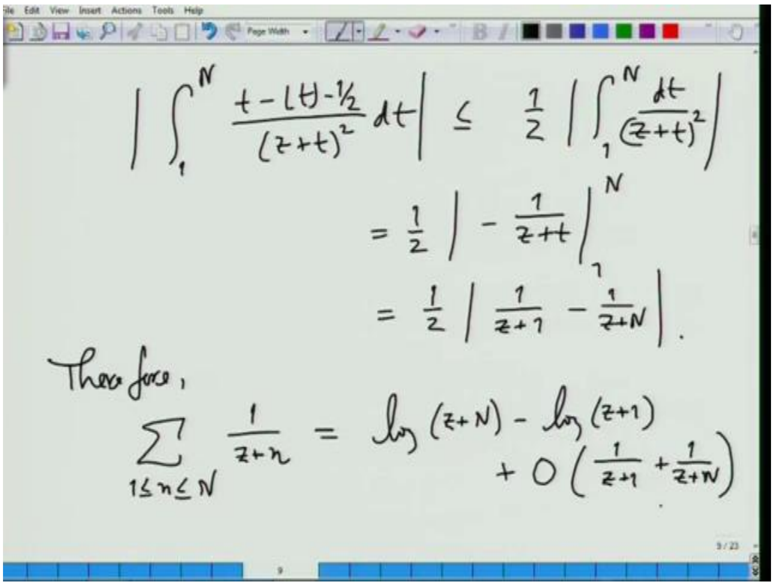


This sum to be equal to this minus this plus this and this is also sort of error term 1 over z plus 1 1 over z plus n this also one over z plus 1 1 over z plus n, so that is like where are we lock plus ordered and that is what we error term.

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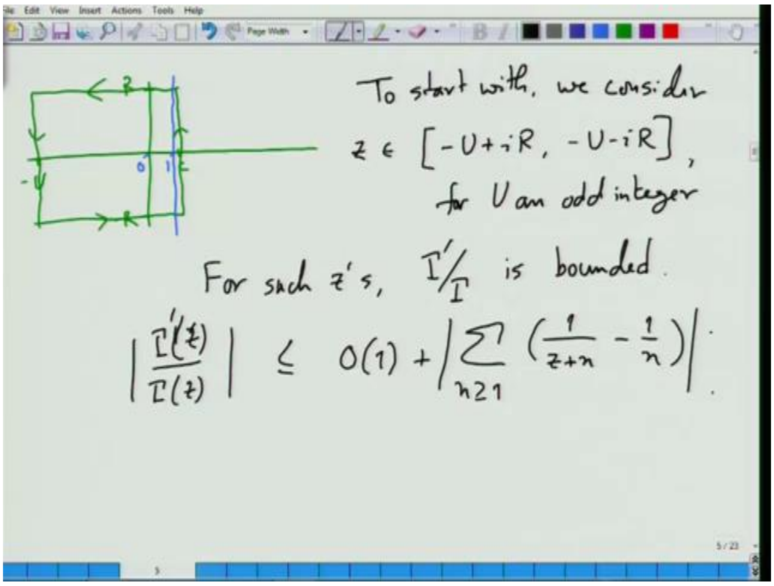


Now, going back to this original estimate, do you remember what we were trying to actually do is to estimate gamma prime over gamma where z is in this interval. So, we derived that in this interval gamma prime over gamma and absolute value is bounded by absolute value of this quantity and this we have just now estimated to almost.

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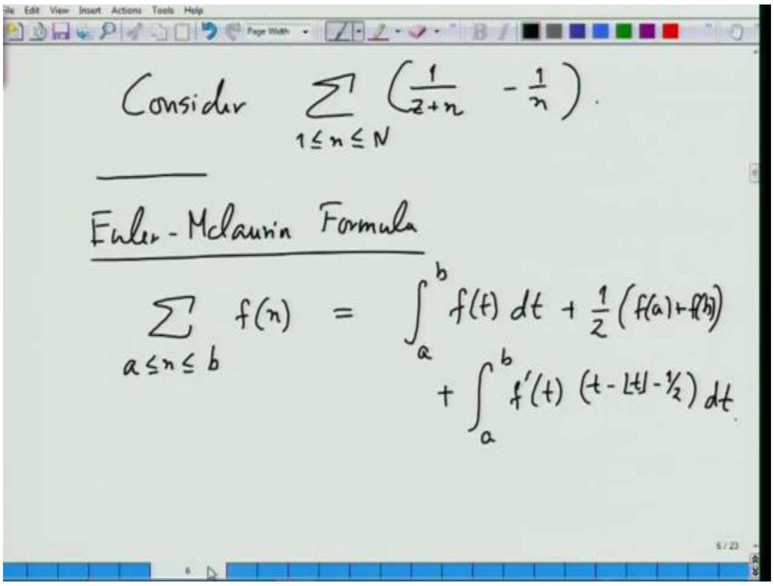


So, this we have studied this by that quantity.

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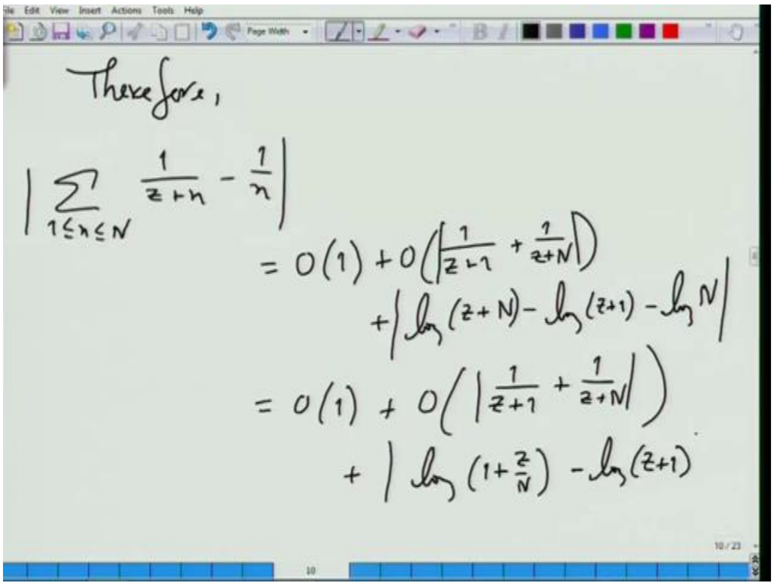


So, therefore, so this is equal to this is bounded by order 1 plus order 1 over z plus 1 plus 1 over z plus n plus log of z plus n. But, in fact I can put this, an absolute value, do you agree with this is the first sum inside the absolute sign is bounded by this plus a little bit more. So, that is sort of absolute value that is bounded by this and the last one sum over n of 1 over n is about log of N plus. So, this high loads constant which is order 1 to that goes out it is important remove to keep the signs here because as see this is order 1 plus order we just take this 2, we get log 1 plus z over n minus log of z plus 1.

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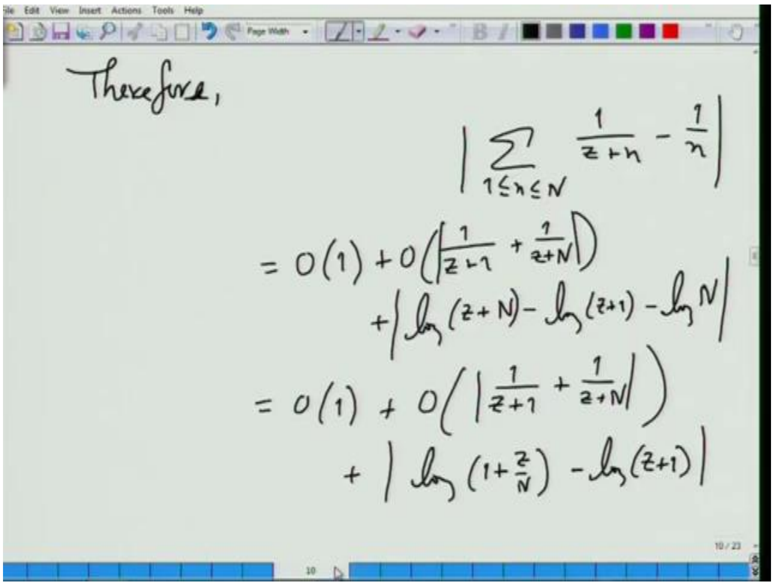


Therefore, absolute value gamma prime z over gamma z is what, now we can come to this that is limit N going to infinity of log of right and what is this as n goes infinity this vanishes. So, this also vanishes and what we are left with is an absolute value plus order, now z plus 1 in absolute value is totally bigger than 1 for the z that we are interested in.

Now, so this all order 1 let me just mention this over z in the interval minus U plus i R, so there we have it a very nice pound gamma prime over gamma. Now, if you go all the way backs, what we trying to do we were trying to calculate the integral along this line. So, what was inside the integral zeta prime over zeta, so let us just go back and recall whatever we have left behind?

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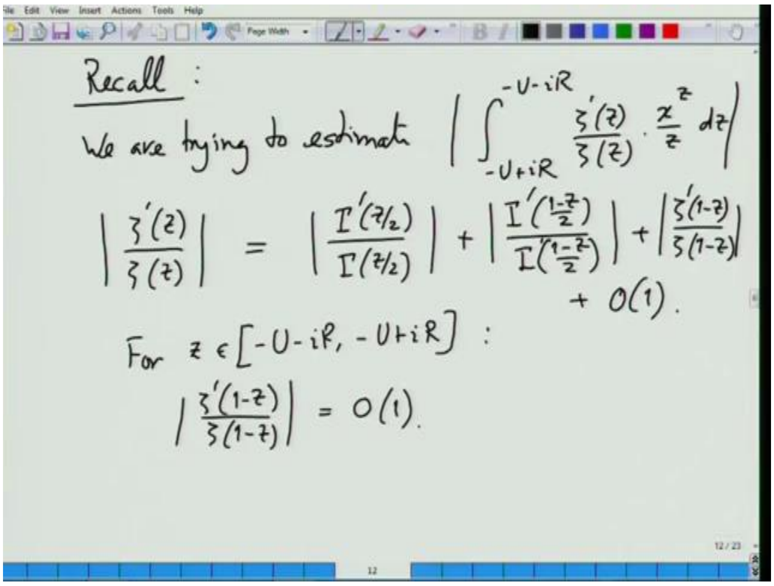


We have already seen that zeta prime over zeta in absolute value is bounded by what quantity we use that functional equation for zeta. So, do you remember I will have to pull that out then it is gamma prime z prime upon gamma plus gamma prime 1 minus z by 2 upon gamma 1 minus z by 2 plus order 1. So, this we had using the functional equation taking the log and differentiating and taking modulus everywhere we get this.

Now, for again z in this interval what we observed was that zeta prime 1 minus z over zeta 1 minus z is order 1 why because 1 minus z is a very large really part and zeta prime as well as zeta on very z. So, this has a large real part converges to some value which is bounded by a constant, so that is taken care of what about gamma prime of 1 minus z by 2 of gamma 1 by 1 minus z by 2. Now, again this will have a large real part, but this is gamma prime over gamma we do not really have an estimate here except that.

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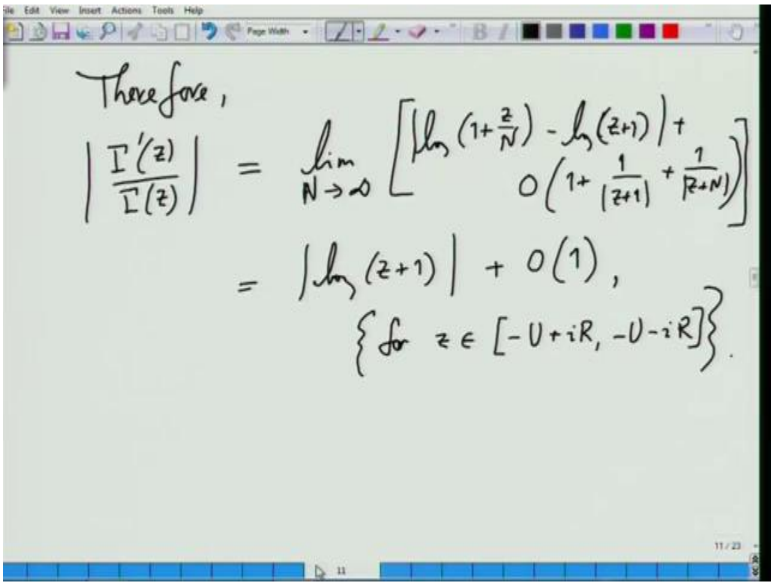


We can just invoke this just obtain this bound right and this part is valid if you recall for all z which are which do not lie in the negative real axis for all other z, this equation is valid. So, in particular for 1 minus z by 2 certainly not be lying into negative real axis when z real part of z is negative. So, it is real actually their on the positive side, so we can just throw this pound at it, and now again real part of z is very large is actually something like U, U plus 1 by 2 actually real part of 1 minus 2. So, again this will go vanish away this will be order 1, so actually a same bound applies here, so we can just go back.

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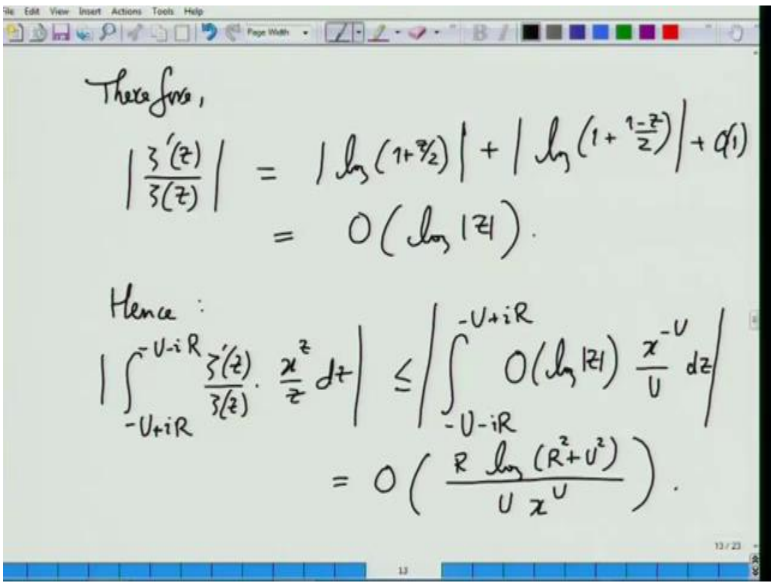

So, is log of 1 plus z by 2 this is what we get from here gamma prime z over gamma z is this and here we have gamma prime z by 2. Then we have gamma prime 1 minus z by 2 which is log of 1 plus 1 minus z by 2 plus order 1, now if we recall this log of absolute value of log of z is what see z is if you write in polar. So, co ordinate is like R times e to the i theta, theta varies between whatever minus phi to plus phi it is never equal to for the z of interest it is never equal to minus phi it is.

So, the absolute value of log z would be log R plus log square R plus theta square and the whole square, now theta is a constant and whenever log of we are taking log of z theta is a constant. So, when if we are taking theta of log h of z for same function h then theta may not be a constant because that really depends on how much rotation that h give. But, if it z or even 1 plus z by 2 they do not give any rotation this are, so this is essentially, therefore either, in either case we can write this as a order log absolute value.

Then when we try to bound this we get less than equal to integral zeta prime over zeta is bounded by order log of z x to the z in absolute value is bounded by is actually exactly x to the minus absolute value x is z. So, absolute value of z is at least u, so it is since in denominator stick that U there and this, and this is equal to again. Now, this everything in fact absolute value of z we can put an upper bound which is square root of U square plus R square and then take everything out the integral is 2 i R.

So, we just get order R log of R square plus U square U and this is perfect for us because remember our target was to send U to infinity. So, when you send U to infinity this vanish remember x is bigger than 1, x was a, was number of x was that number we want to count there is always going to be bigger than 1. So, here x to the U sitting in denominator U sitting in the denominator and the numerator is only log u, so it vanishes, so that is good.

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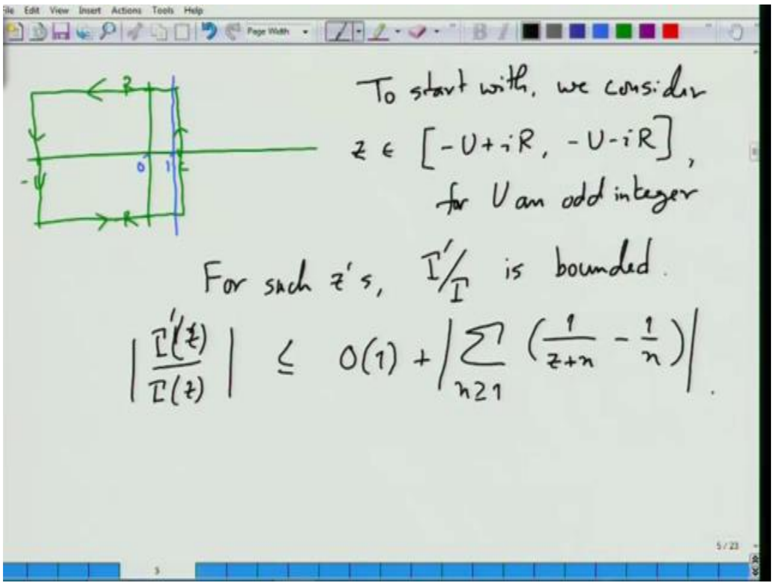


Now, this takes care of this integral, now we look at this two are cousins, so if we will take care one we will take care of other this, because this will symmetric stuff, so we have to take care of this, now little go back and draw this because where the fun begins.

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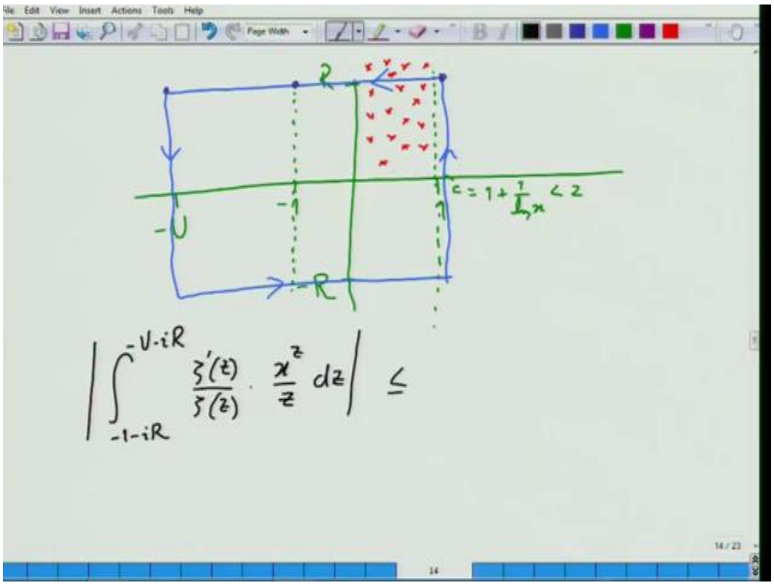


So, this is the domain that we have and let us mark out, now and this is c, c if you recall was one plus one by log x which is less into fine. So, the integral here is starting from here it is from c which is the number slightly bigger than one going all the way to minus U and stays their [FL]. Now, as c travels around here and then keep in mind then what we are integrating we are integrating zeta prime over zeta times x to the z over zeta. So, as we travels on this side x to the z over z is very well behaved, but zeta prime over zeta is not at all well behaved because there may be lots of zeros sitting here.

So, we need to not only avoid those zeros, but, also we need to worry about even if we can avoid the zeros zeta prime over not. So, this are poles we not only need to avoid the poles we also need to make sure that the path we take as the zeta prime over zeta does not become too big because if it becomes too big. Then we cannot make the error terms form on the other hand once we have cross this boundary and we are on this side.

Now, we know there are no poles and actually we know, we far away from any other pole because the remaining poles are here. So, this are good reasons for us to integrate and further can be by may be a little unsure about this part because there may be a pole very close by. So, by the time we have come here we have still not got rid ofthe effects of that pole, so what we will do is we will split this integral into two parts.

So, one is from here to here c 2 minus 1 and second part is minus 1 to minus c this one be this will be easy to handle in fact I will handle it within 2 minutes using the tools you have already done. So, this will be not so easy to handle, so we will then move on to handle this one, so what about this one. So, if you look at integral of minus one minus i R to minus U minus i R zeta prime z over zeta z x to the z by z at the absolute value this is bounded by. Again we take the modulus inside how do you bound zeta prime over zeta prime over zeta, again recall the functional equation just go back.

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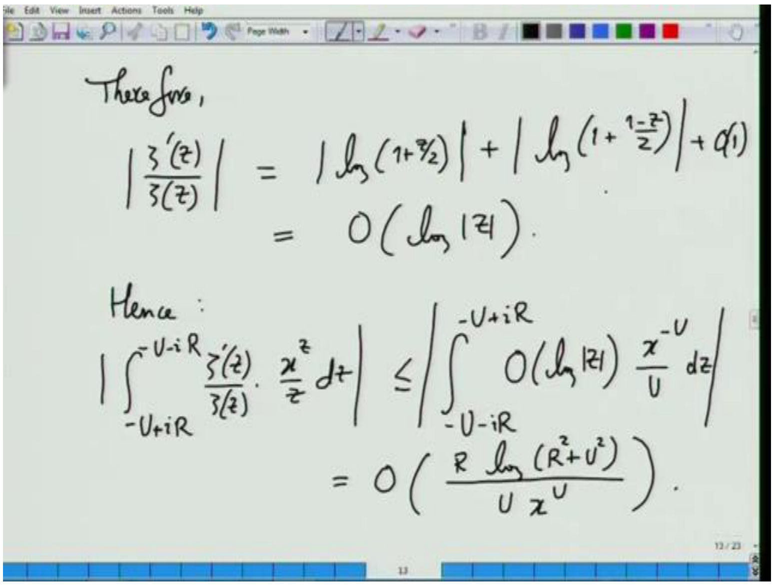


So, this holds, why does this hold we have to be a little careful, now z is less than the real part of z is less than equal to minus 1. So, that is another reason for you know going minus 1 and below if real part of z is less than equal to minus 1 what about the real part of the 1 minus z. Now, we are at this equation real part of 1 minus z is going to be greater than equal to 2 and when it greater than 1 equal to 2 well zeta prime of 1 minus z over zeta of one minus z is bounded that we know. So, that any above one is bounded, so for 2 it is surely bound, so this goes away, so we their move into this part.

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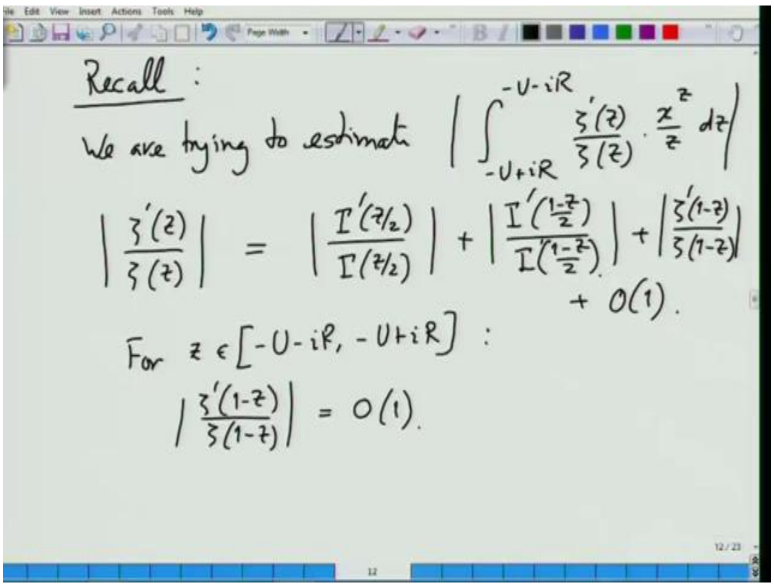

Zeta prime over zeta is bounded by this sum this plus, this plus, now what about gamma prime over gamma well we have already derived an expression for gamma prime over gamma that is valid the only in valid was part was the negative axis that is valid here.

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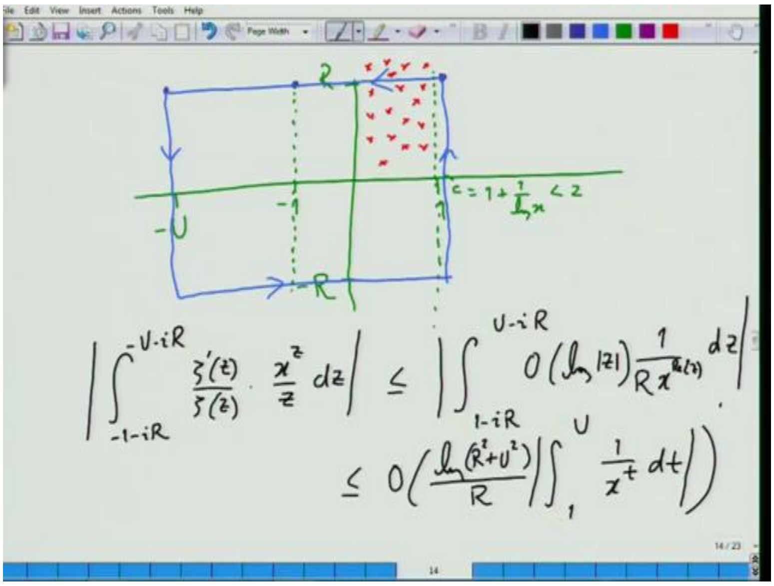

So, we can throw this in still get a log of absolute value of z, so doing all this we get this is order log of absolute value of z what is absolute value of x to the z. So, this we cannot say because it is now the real the part absolute value of x real part that is varying from minus 1 to minus c. So, x is in the denominator, therefore z is the denominator and we can always substitute the smallest possible value which is x. So, let us just do that and again absolute of z the smallest is R and again the absolute value of n is the largest value of this is square root of R square.

So, you just pick that in and this going from minus 1 to minus U, I cannot afford to do that, sorry I have to 1 over R is fine. But, I cannot afford to have replace x to the z by x because then this going to diverge, so what I am going to do is keep that x to the real part of z. So, this less than equal to log of absolute value as I already said this we can bound it by order log of R square plus U square divide by R.

Then we have a integral, now in the integral, now the only things are wise is x to the real part of z. So, we are essentially integrating over a real quantity like because we are moving from here to here, so only really part changes, so this are also only move the real part, so basically this is into U 1 over x to the t d t this integral.

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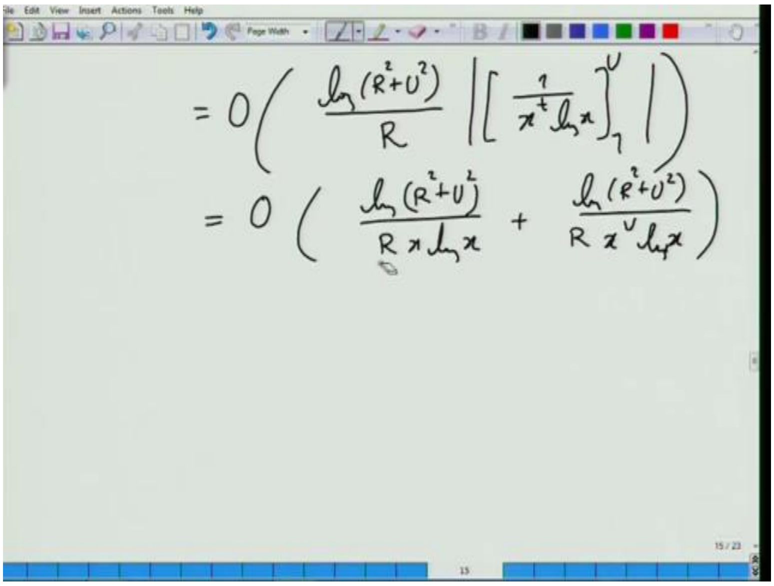

This is simple to estimate.

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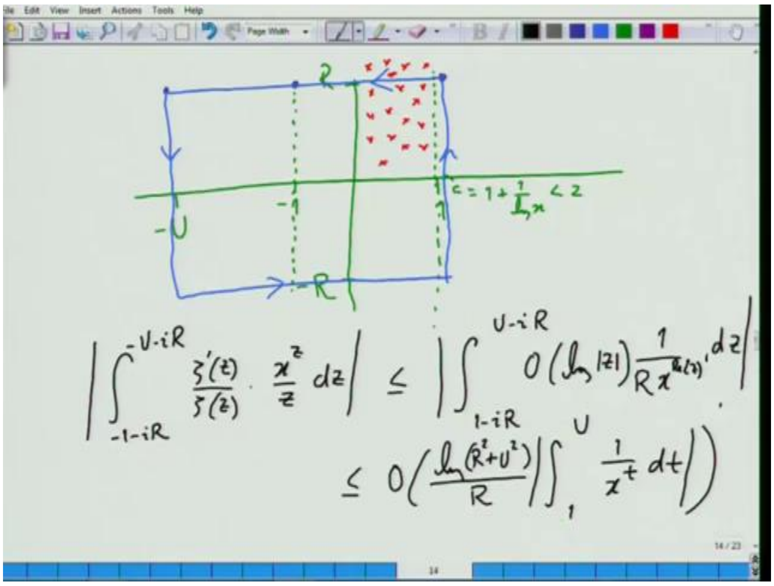

Integral of this is going to be, no it is not good enough because this integral is going to be probably is, now it is not x e to the minus t log x and integral of that is going to be 1 over t log x into U. So, absolute value of this for t equals minus 1 t equals 1 over x log x you know that is what it mean as you go this vanishes, this diverge. So, I have made a mistake somewhere, this is fine maybe I should not have replaced this absolute value of z by R will never send R to infinity we will at some point.

But, only for a short while we will actually be setting R to be square root of x, no we do not want to find the integral on the whole line. So, see if you recall for any value of R we had keep this integral related to psi of x with some error term and our target is always going to be to minimise the error term. So, we will choose that value of R which minimises the error as we will see later on that value of R will come to about square root of x. But, that is a story for future how do we end this story, zeta prime over zeta is bounded by order seems like a very reasonable conclusion.

So, how do we take care of this I think clearly my bounds have been over simplification, so one needs to find a slightly better way of handling maybe yes. Now, maybe I should do that instead of in fact log mod z I should just replace by log of root of R square plus t square then do the integral. So, let us just go ahead do that, so I do not want to end up here, so I will erase all of this lets go back.

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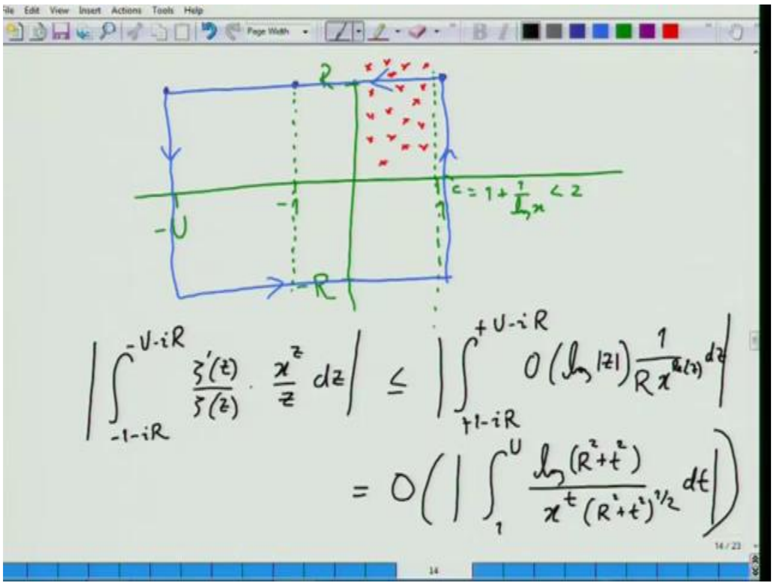

Here, we replace this by take order outside absolute value and, now t we are going to send 1 to U what is order log z it is log of R square plus t square. So, in the denominator we get x to the t and for mod z we get R square plus t square, square root d t, now how do you integrate this can one find some easy estimates here think, so see what we can say is.

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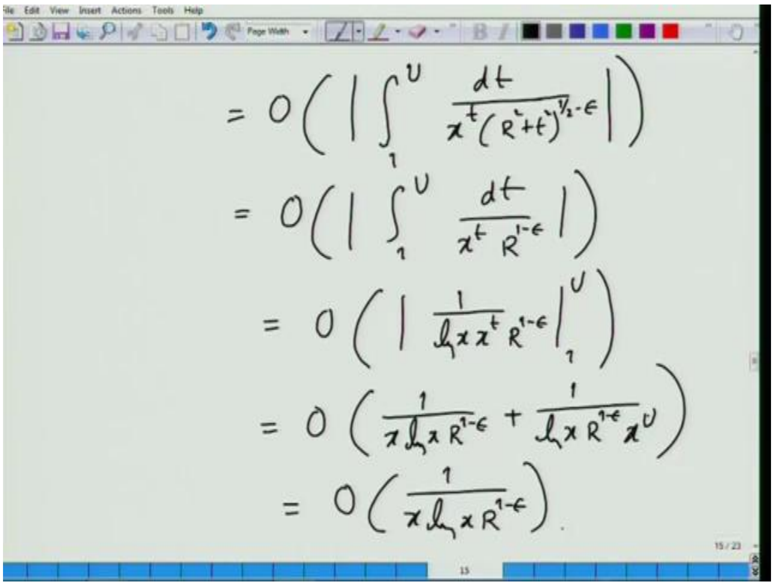


We can say surely this is order one to U and notice that log of this quantity is upper bounded by this quantity to the epsilon for any small epsilon. So, there is some constant multiplier that gets absorbed in this order, so we place this by this upper bound which then gets absorbed in here. So, we get d t over x to the t R square plus t square half minus epsilon, now integrate this how do we integrate, we can try many, we just have to worry about upper bounded do not have to worry about this exactly evaluating this integral.

So, what would be a good way, suppose we throw away R square there this is still remaining an upper bound. Then t square becomes t to the t square to the half minus epsilon becomes t to the 1 minus epsilon if t to be 1 minus epsilon divide by x to t. Now, if we integrate what do you get or I can say or I can keep R square that might be useful later on. So, if we integrate this by parts let us say figure outside t square equals U that messes up things t square equals U I can do that t square plus U then this becomes.

So, there is square root U also which locks somewhere and this becomes x to the square root t that is becomes messy. Now, of course throw away t square t is always greater than 1, so if we throw away this t square from the denominator. Here, you are always going to be in a problem and, now this is a integral that is like one by log x, x to the t R 2, the 1 minus epsilon 1 t when t is 1 you get 1 by x log x R to the 1 minus epsilon.

So, when x is, you get one by log x R to the x minus epsilon times epsilon and, so this is always 1 plus x log x R to the anyway this will go to the 0. So, that gives us a good estimate of this integral, this part of integral which is also same to this part of integral.

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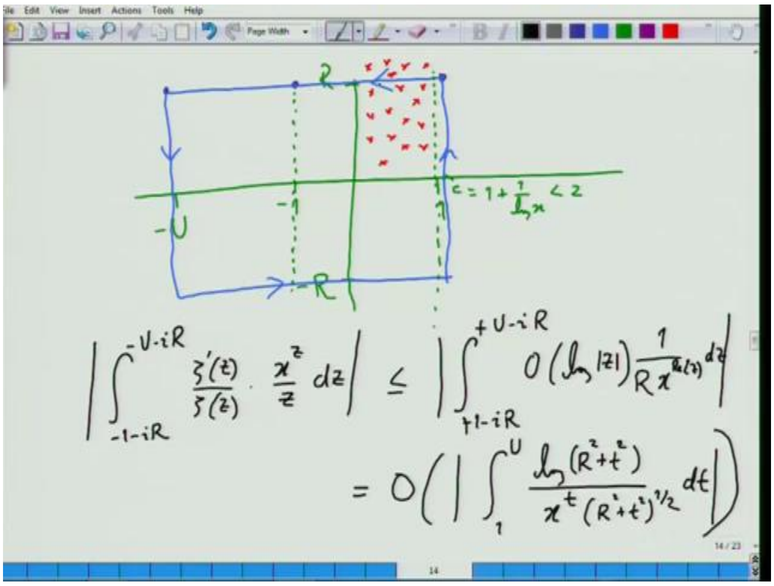


So, it is same thing no difference, so what is left is the messy one.

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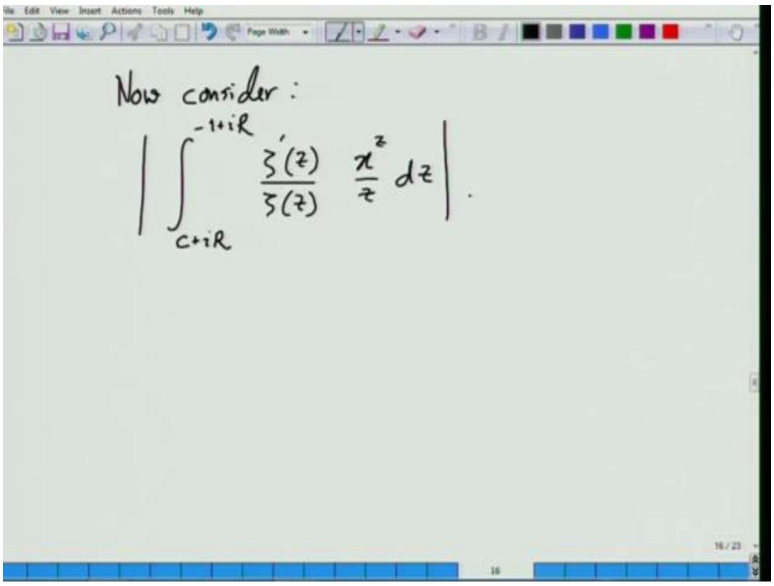


How do we get this estimate, because in here that whatever we are trying to do.

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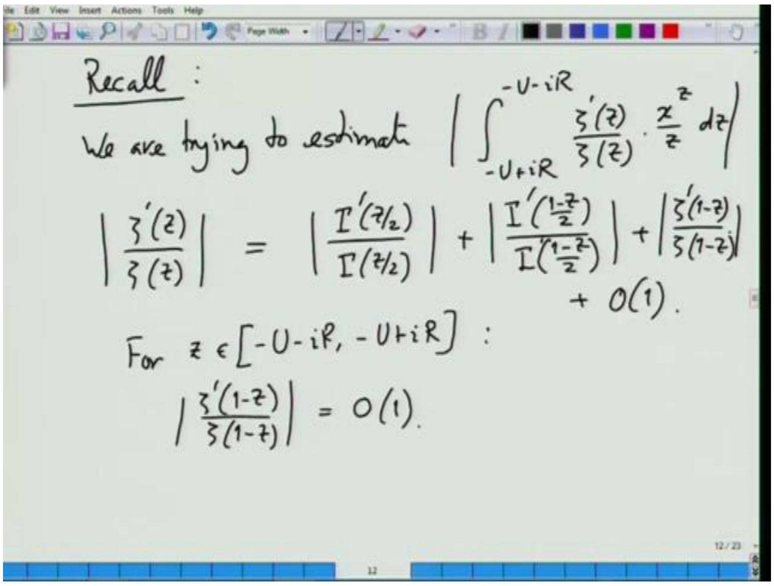


So, using this equation, it does not work anymore because the key thing here is that we cannot get rid of this 2 quantities we know how to handle this 2. So, this are well behaved in that region as well, so we can again plug in log z this 2, this is not well behaved when real z is less. Then negative one, real part of less than 2 well for a while we can still come up to we can keep this bounded. But, as real z becomes greater than zero you do not have any control over this and, therefore we need 2.

Now, we can only express this quantity internal of some order log z times or plus quantity, but that really gives nothing because you want to bound zeta prime over zeta. So, therefore we seems that we cannot use this functional equation then to bound zeta prime over zeta in the region of interest for us which is unfortunately because we spend a lot time to understand this.

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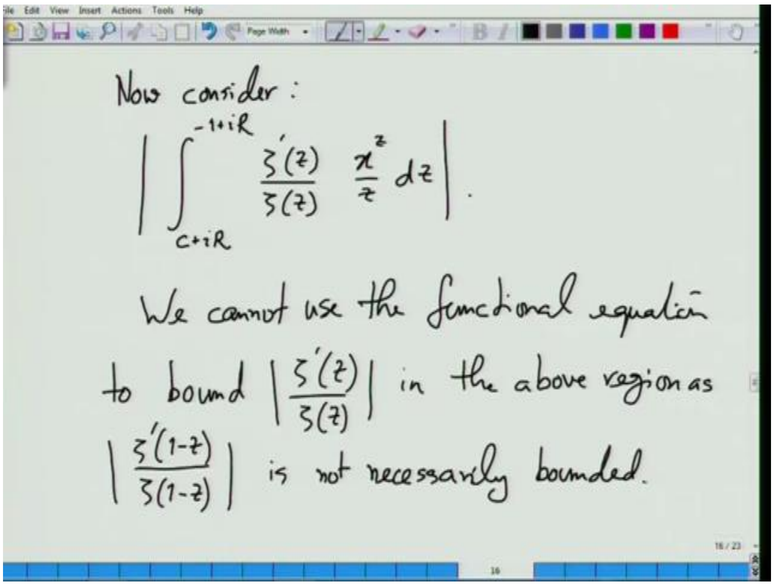


But, you have to work with what you have given you cannot use it so you cannot use it, so you have to use something else so what is that something else. So, let me first write, therefore we cannot use the functional equation, so as I said we use the same thing else as that something else fortunately is not something totally new. So, we can reuse the tools that is already develop we have to approach it slightly differently and the approach we are going to adopt is.

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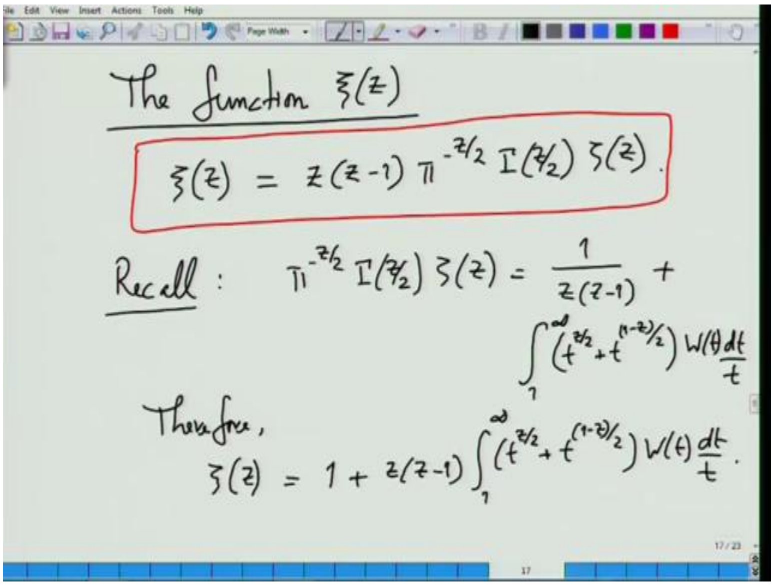

So, we define another function which we call xi the definition of xi is quite simple and it will look familiar to you also if we recall this, the last 3 quantities in this product occur in the functional equation. So, if that xi is motivated from the functional equation, if you further if you recall that we had phi to the minus z by 2 gamma z by 2 zeta z to be 1 over z times z minus 1. Now, I will really have to remember what it was plus minus, here there was a one to the infinity t to the z by 2 plus t to the 1 minus z by 2 W t d t.

So, that is really a long time when we were trying to prove that are trying to analytically extend the function over the entire complex plain. So, this is the equation we use to extend the zeta function is this was this the equation, this d t by t and was the sign plus good. So, this is what we had, therefore the xi of z is 1 plus, is this good, so let us spend a couple of minutes on understanding xi of z.

So, first is it defined over the entire complex plain, yes may be except for some may be some poles it is defined over the entire complex plain. So, because zeta z is defined over the entire complex plain except for one pole gamma of z by 2 is defined over the entire complex plain except for those infinitely many poles. So, this is of course this are defined over entire complex plain that, so xi z is defined over entire complex plain is meromorphic function, let us say is a meromorphic function.

So, if may have something, so are there some poles xi if, yes where are they if is z equal 1 pole which is a pole of zeta z, no why not. Then z minus 1 takes care of it is any of the negative integer a negative even integer is the pole minus 2 is minus 2 a pole for example no true because that is already taken care of by zero of zeta function minus 2 to that only lives out 0. So, z equals 0 is equal 0 is equal is a pole because that where this is bounded, but this is bounded, no because there z is taking care of it and this poles are all 1.

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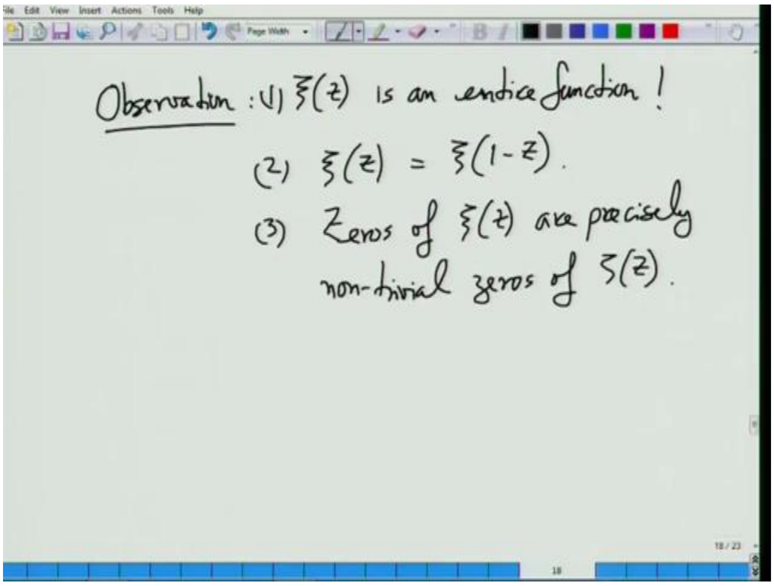


So, actually we can, therefore conclude that xi is an entire function, in fact that was the reason of multiplying out this z minus z minus 1 to take care of the remaining poles. So, there is an entire which is very good and another nice observation which we will not be using anywhere. But, is still good to know that xi is perfectly symmetric along real z xi z equals xi, xi of minus how xi of 1 minus z that follows because it is really is a functional equation.

So, z time z minus 1 is also a similar term, so as I said we will not be bothered about the second observation, but the first one is going to be important because we know about entire function. Now, what we know about entire function is that we can write the entire function n terms of z, so we will have to worry about where zeroes of xi are, so that is the third part of observation where are the zeros of xi.

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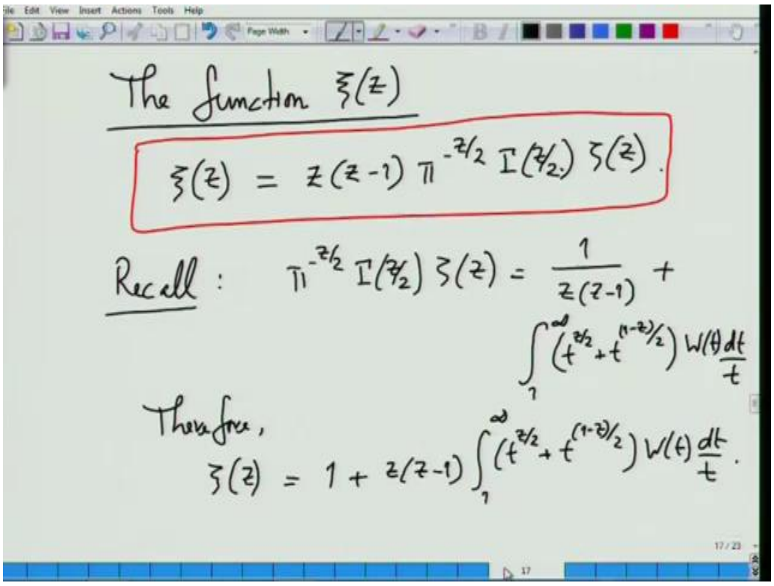


So, gamma has no zeros, we already know, so the potential of zero are z equals 0 that not a 0 because there is a pole which cancels around z equals 1. So, all zeros poles cancels zeros of gamma trivial zeros of gamma sorry trivial zeros of zeta not zero because they cancel out poles of gamma. So, the only zeros of xi are precisely the non trivial of zeta and that is on those any of non trivial zeros this are all quantity which are bounded.

Therefore, now you see the usefulness of xi it is the entire function, so we can use the product formula of xi and in that formula all the non trivial zeros of zeta will occur. So, these are the zeros that we are worried about because zeros gives rise to those poles of zeta prime over zeta that we need to avoid in order to bound the integral. Now, we will study xi to derive some facts about those nontrivial zeros of zeta, so that is going to be our next task, so we are done for the day.