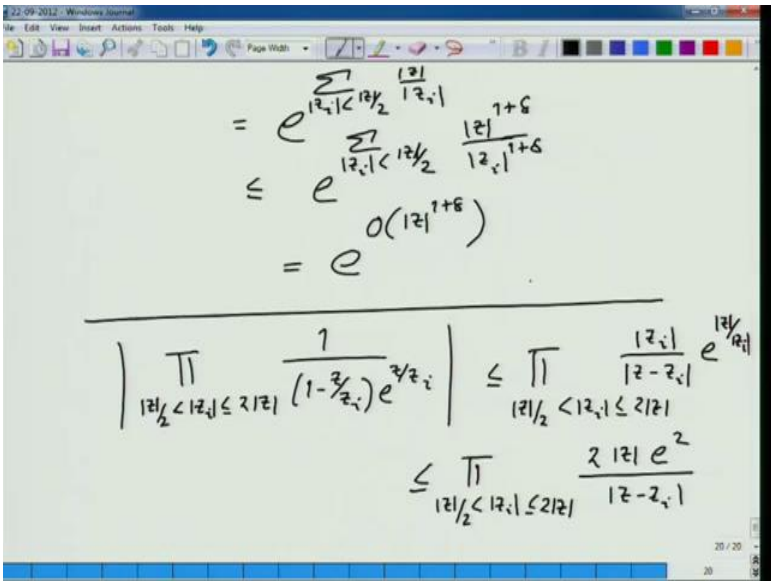
Riemann Hypothesis and its application
Prof. Manindra Agrawal
Department of Computer Science and Engineering
Indian Institute of Technology, Kanpur

Lecture – 18

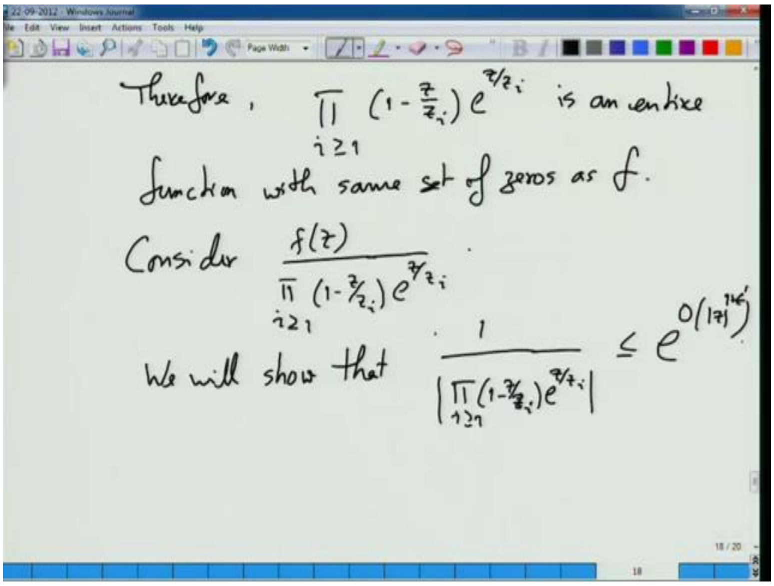
(Refer Slide Time: 00:29)



Let us look at the second one, first and second we sorted out last time. This second one is product of the absolute value is less than equal to product of right and now notice that absolute value z i lies between half and twice of absolute value z. So, z or z i therefore lie the absolute value z or z i is going to be between half and 2, so this quality in the exponent is square and absolute value z i the at most 2 times the absolute values z. We are left with absolute value z minus z i in the denominator, now this is where we run into problem because what if z is equal to z i. Then, this of course is unbounded and this which may happen right z zi lies between and absolute value between z by 2 and 2 z. So, it may happen that is if I choose my z in such is in such that it equals zi in a precisely then this is unbounded, then therefore I cannot bound it.

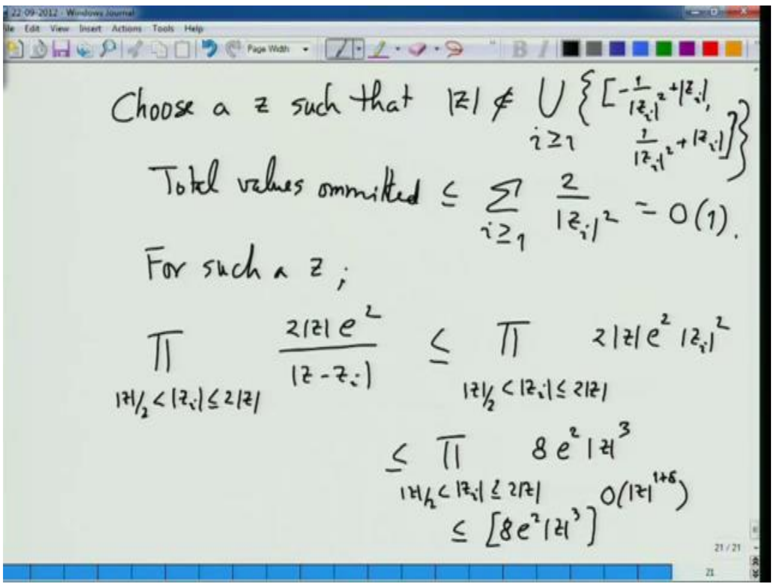
If you are call the roof of the bound or the nature of the entire function of order one without any 0 s in that proof that there function must be of the kind e to the a z plus b, what we used was that for infinitely many r's mod f z adds mod z equals r. It is bounded by e to the order r to the 1 plus epsilon, we did not know this property to hold, we did not require this property to hold for all r's you only required it to intently many r's and that is what we are attempting to do here.

(Refer Slide Time: 04:01)



If we go back, so we want show that this is an entire function of order 1 with no 0 s, which we know of course and the order. Before that, all we need is to show that this absolute value of this whole function is bounded by e to the order mod Z to the 1 plus epsilon for infinitely many mod z's not for all mod z's, so that gives us some flexibility in choosing the z here.

(Refer Slide Time: 04:43)



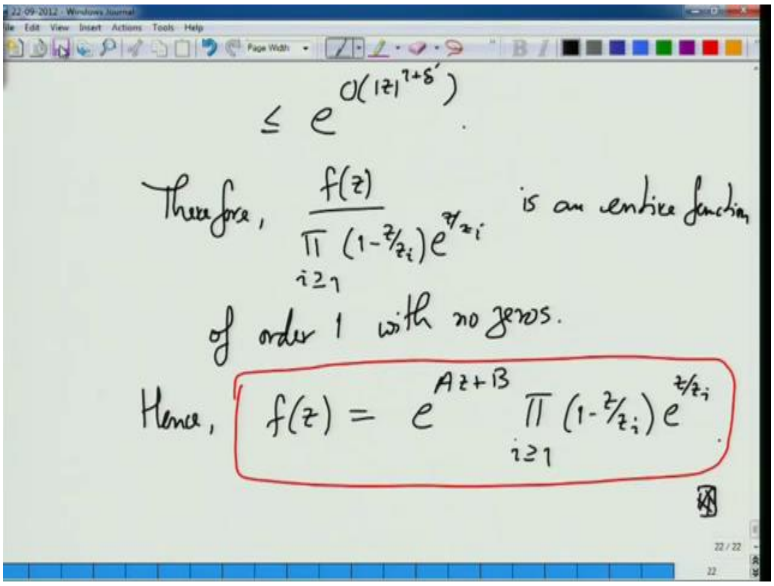
What we will do is we will choose a z such that z the absolute value z is not in this set, consider the interval of numbers which is between mod zi minus one over mod zi squared mod zi plus 1 over mod zi squared. For each zi, take the union of these intervals and we want to take a z, so is at mod z is not in the any of the intervals is that possible can we choose infinitely we can. What is the range of this one single interval, but what is the exact value that we can complete the exact value one interval corresponding to i what is the range of values. So, what is the difference between the ranges, so that is that is number of a value, which are vomited right for your zi z to be.

So, the number of values vomited or the range of values vomited as 1 by zi square right total values vomited is 2 by zi square. Actually, it is less than equal to because for two different I is this intervals may over lap and this is bounded, therefore only finitely when the value are ruled out. So, you still have whole range infinite many values for you will set to choose from, so you can always choose a larger z satisfying this property. Now, for such a z, let us go back this is the quality we want to estimate not this, sorry what is the difference between z and zi at least 1 over z I square right because z is not in this interval for any zi.

So, zi that z is therefore, with respective any zi it is at least 1 over zi square away from that. So, we can replace the denominator here by 1 over zi squared and upper bound and zi is up upper bounded by 2 times set. So, you get 8 e square mod z q and how many times are you multiplying this or the number of times you are multiplying. This is number of z I in this range how many zi are there in this range, we know the count e to the mod z to the one plus delta at most.

So, this is less than equal to 8 e square mod z cube whole to the power order not e to the power sorry the number of zi's number of roots are bounded by order mod z to the 1 plus delta. That can be shown and this is of course the thing inside here does not move the scale here at all I take everything in the exponent, you just were we had multiplied this with 3 log n and so which does not move this delta to anything just sort of stays the same.

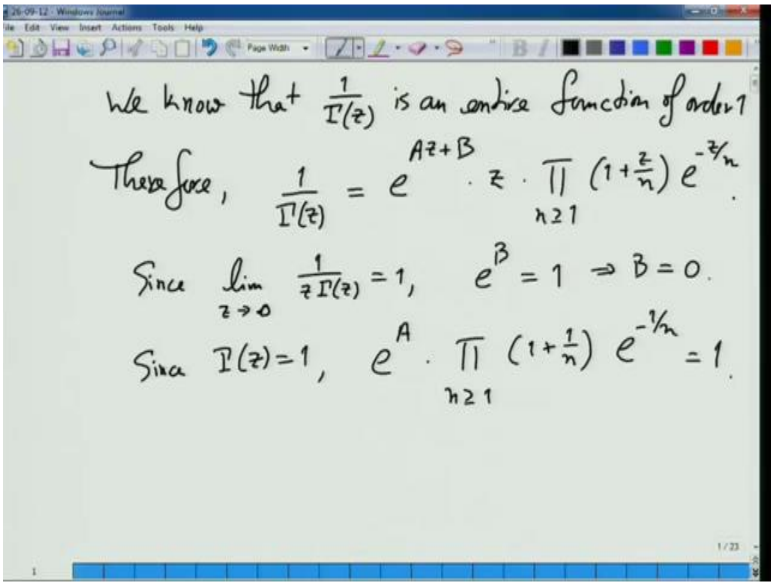
(Refer Slide Time: 11:43)



Then, delta prime with slightly bigger delta and that is it, so this bounce this entire this product for infinitely many z's and therefore, and for all z's we know that f z is bounded by the same before together. This whole absolute value for infinitely many z's is bounded by e to the mod z to the 1 plus epsilon prime and therefore f z over product, i equal to 1 is an entire function of order 1 with no 0 s. Hence, f z equals this, so this is the so at this point I am going to end my diversion into the theory of entire functions.

I will step back a level up and what was that level gamma function, you are trying to get a bound on gamma prime over gamma that was the starting point. Then, I said that will completely analyse the gamma function and then will derive from that analysis bound on gamma prime, so let us step back into gamma function and we know that.

(Refer Slide Time: 14:22)



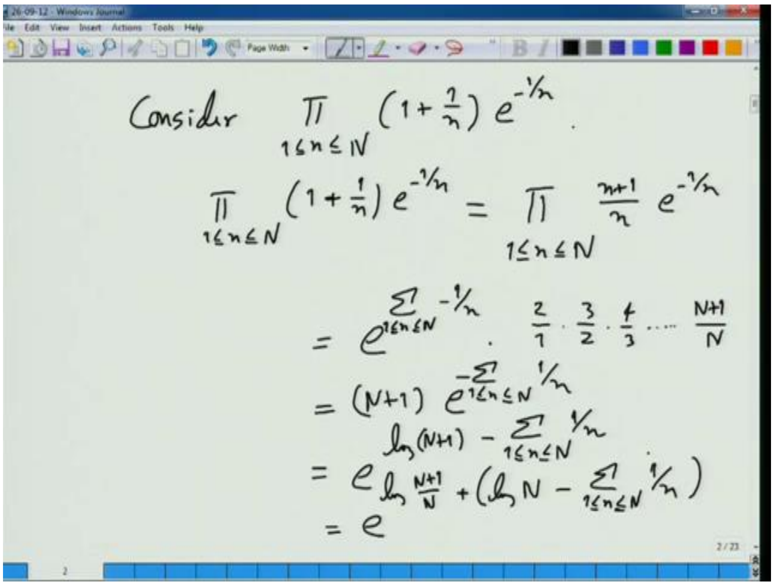
I think we already shown right that 1 by gamma z is an entire function of order 1 is it have any 0 s no of course it has plenty of 0 s because it has it has pole set all the negative interiors. So, it have plenty of 0 s, so we stick that in, but does it have 0 at z equal 0, it does gamma 0 is what is gamma 0. It is undue unbounded right because gamma 0 is gamma 1 over 0 which is unbounded, so it does have a 0 at 0 gamma said. Therefore, by this whole previous analysis, how can we write gamma function or inverse gamma function e to the a plus b times z, this z is for the 0 at z equals 0.

It is a 0 order 1, so we just multiple with z and leave it at that times product and greater than equal to 1, 1 plus z over n e to the minus z over n. Now, here we still haven't fixed the value of this constant a and b, that is not difficult to do, what is z comma z at z equals 0 is gamma 1 is gamma 1 because gamma z is gamma z plus 1 divide by z z comma z is gamma z plus 1 as it goes to 0 gamma z is gamma 1, which is 1. So, 1 over z comma z is 1 at z equals 0 and we plug this into this, what expression do you get when z is 0 this all product is 1.

Everything here is 1, z equals 0, this is e to the a z vanishes, so you get e to the b equals 1 this b is 0, so that takes 1 constant whatever the other constant 1 over z gamma z is 1. So, you take this in, so you take this z down here and then take the limit, now whatever gamma z at z equals 1. Of course, that is 1, so let us plug that in also for z equals 1 what do we get e to the a times product ad greater than equal to 1, 1 plus 1 by n e to the minus

1 by n. This is equal to 1, but this is not so straight forward to work out, but we can do a little tricky here to work this product. What is the value calculate the value of this product or does anyone know expression for this point, that is a good idea take the log and see what is work out, but before taking the log.

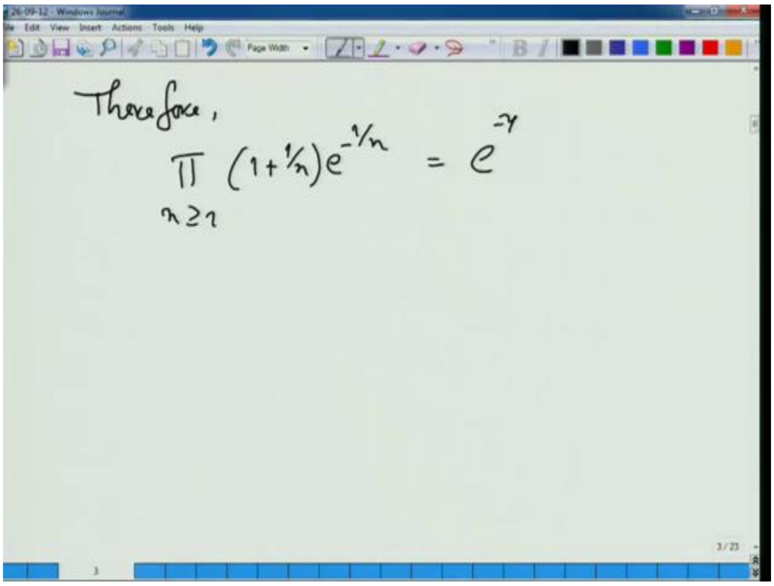
(Refer Slide Time: 20:15)



What I am going to do is bound this up to some capital n and then eventually we will take some capital n to infinity to get the value. Now, take the log, so what is the log value here log of this product, this equals lets simplify this, let us not take the log, let us just try to simplify this this is equal to product 1 less than equal to n less than equal to capital n, n plus 1 divide by 1 to the minus 1 by n. So, this is e to the summation 1 less than equal to n less than equal to n minus 1 by n times product of 1 going to n going for 1 capital n n plus one over n. So, what is that product so it is like 2 by 1 times 3 by 2 times 4 by 3 times dot up to capital n plus 1 divided by capital n, this just leaves out capital n plus 1 do not even need to take log n.

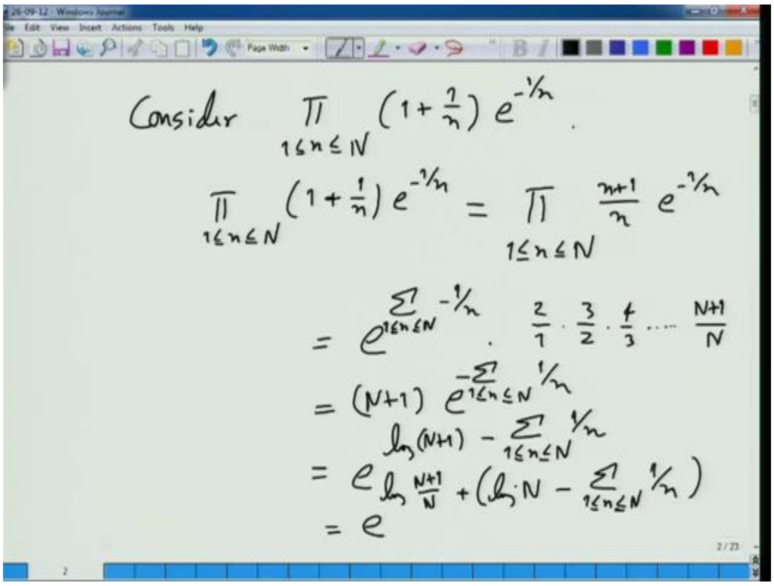
Now, this is the expression in the exponent, where is this one the sum this is harmonic series capital n and this is log of n plus 1. In fact, let me right this as it is. So, look at this expression log n minus the harmonic series at capital n as n tends to infinity what happens to this that is a very familiar quantity. You remember log n is very well approximated by this summation up to n 1 plus one half plus one third plus one quarter up to 1 by n and the difference between log n and this sum as n tends to infinity tends towards affix constant gamma which is like 0.40. Something that is that constant was first described by Euler it was called Euler constant as n tends to infinity, what happens to log n plus n over nit goes away 0.

(Refer Slide Time: 24:55)



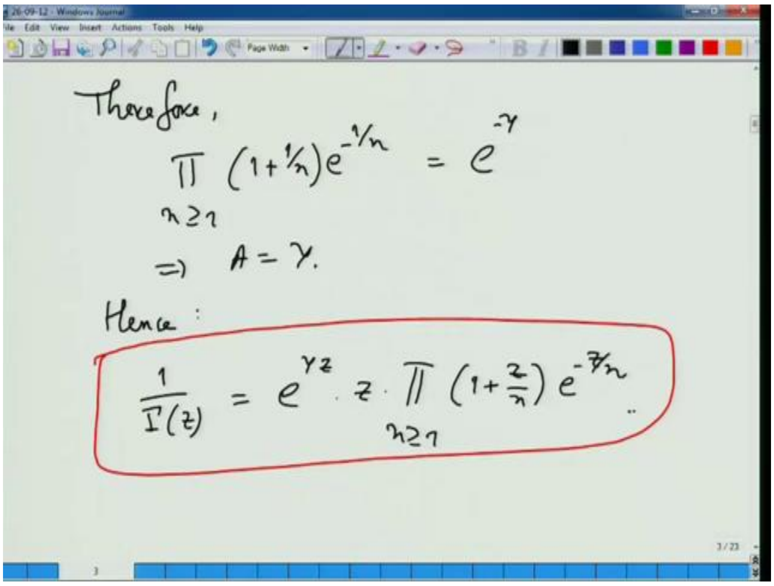
Therefore, we get that this product e to the gamma now I Think i should put a minus sign here because this sum is always bigger than the long say when capital n. Then, log 3 is very close to 1 because to the base e that is very close to 1, where this is 1 plus half plus one third it has bigger and it stays bigger.

(Refer Slide Time: 25:47)



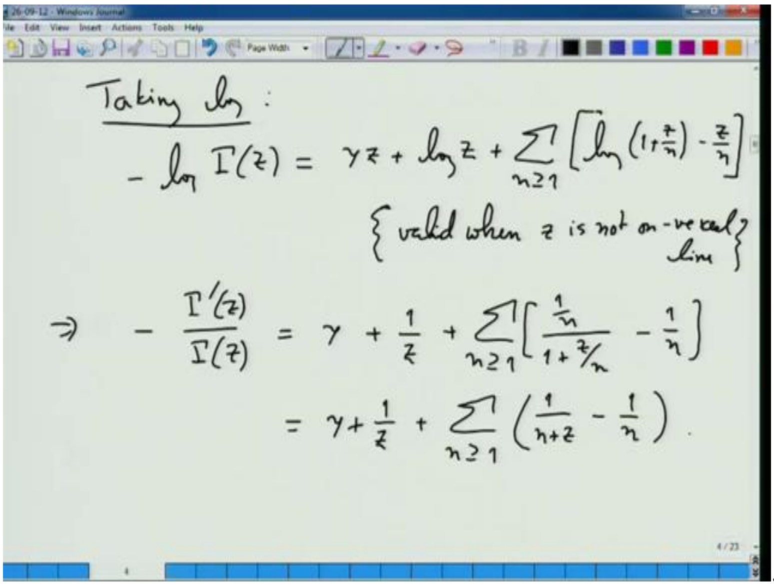
So, the difference between this two are this minus this is gamma this minus this, so the difference is minus gamma, so that is what you get, and therefore what is it e to a.

(Refer Slide Time: 26:07)



We just were described this is e to the gamma so e is gamma, now this completely describes the gamma function. There is nothing left out here you know each and every single constant appearing how does it factor how does it add up everything is there and this is the expression for gamma function. Now, we will use to give it bound on gamma prime over gamma and the ways to do it is very simple take the log of both the sides and differentiate.

(Refer Slide Time: 27:25)



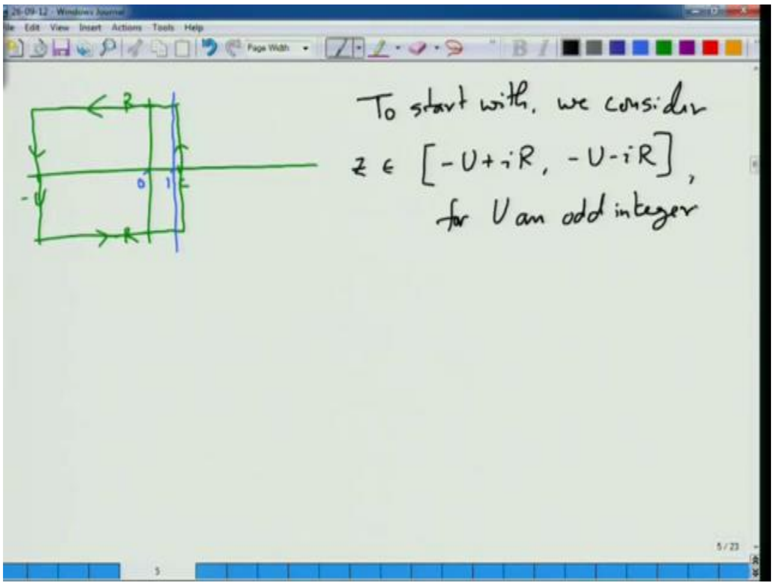
So, we take the log of both the sides what do we get minus log of gamma z equals small gamma z plus log z plus summation n greater than equal to 1 log of 1 plus z over n minus z. Now, because we have taken the log again we have to say something about the validity of the expression. So, this simplest way of describing the validity of this expression is to say that just take cut out one line that for whatever log of whatever thing we are taking that quantity. We for that quantity we cut that line from 0 to negative infinity that we will line and see that rest off the domain it is well defined, but we are taking log of on the right hand side of log of one plus z over n z.

Here also, if you think about what we need to avoid it turns out that we need to avoid essential the real line. Well, actually not the whole real line I think let us see n greater than 1, I think negative real line is what we need to avoid.

From 0 to all the way up to minus infinity that negative real line if we can avoid this is well defined because if it is a that is right if z is minus n then this becomes 0 and the log becomes acetify. So, that is the problem, we do not want to run into so we just avoid the negative real line for z values that is it, so let me note it here this is valid and that is what we need to keep it in mind and that also takes care of all this duplicity. Also, once you cut out like this cut, then the log function analytic on the rest of the domain, so everything works out fine log differentiate, now you can differentiate because analytic now log of this you can differentiate and what do we get on the left hand side.

We get minus gamma prime over gamma equals small gamma plus 1 by z plus n greater than equal to 1, 1 by n divided by 1 plus z by n minus 1 by n. So, what come from I do the very simple expression and this is well defined whenever z does not take value is in the negative real line. In fact one can go one step further and say that is this part at least is well defined whenever n does not take negative integer values. Let us just stay with the negative real not z not taking negative real values, now in order to estimate this quantity we need to estimate the right hand side. This seems like not too difficult task integers not too difficult task, but now you have to really dig back and go level up and recall where did we require bound gamma prime over gamma or for what sets do you remember.

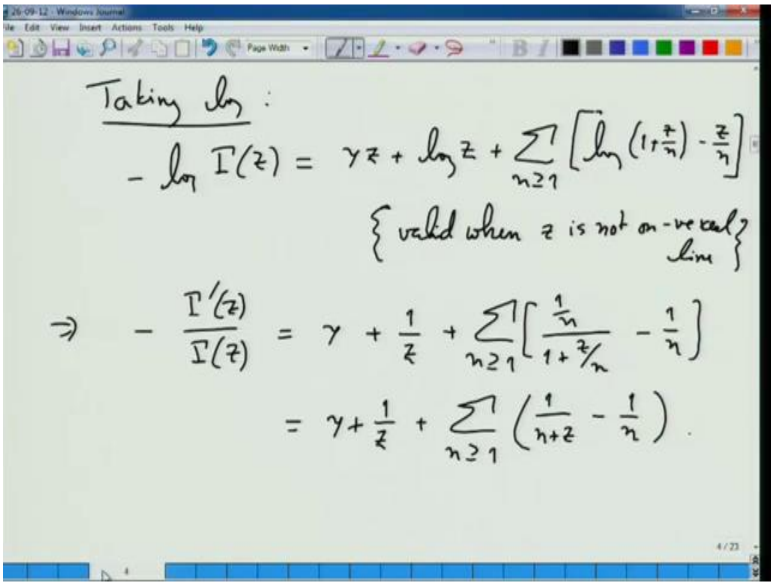
(Refer Slide Time: 33:18)



We were looking at we were looking at this rectangle right and we wanted to show that this is this integral is what we are interested in so you want to get rid of all this three integrals. I said that will first start with this integral with which is the simplest to handle and so we were at this point when we diverged into gamma function. So, let us get back to this, so to start with we consider z in minus u plus I r minus i r and since we want to avoid hitting a pole here pole sole gamma functions here. So, you want to avoid hitting pole we take you to be odd integer, because remember we were actually looking at gamma z by 2 and analyse gamma prime z by 2 divided by gamma z by 2.

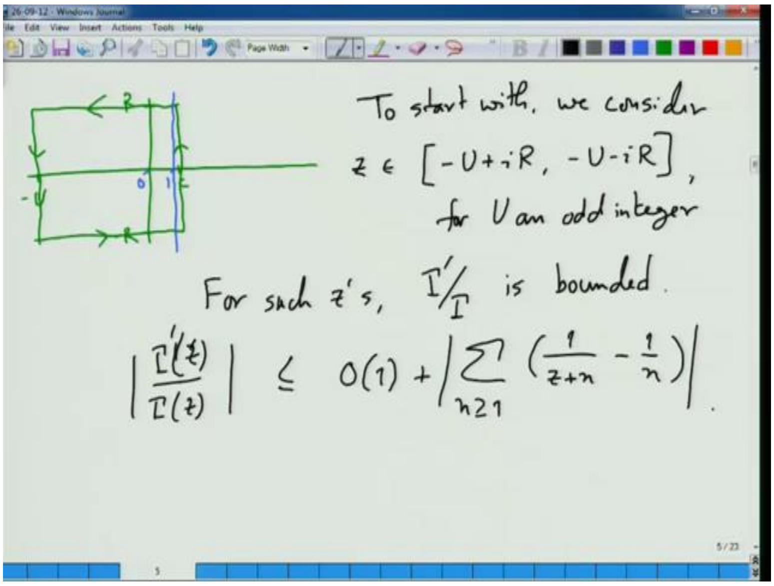
So, when we use a odd integer we are essentially looking at 1, you look at gamma prime z over gamma, it is a the midpoint between two integers is what we are doing. So, we do not, so this line never hits a pole as far away from a flow less pastel, so that is the z, we are looking at. Now, for such a z are in general that is which are reasonably far away from poles of gamma what is the bound on gamma prime over gamma and what is this bound equal.

(Refer Slide Time: 36:47)



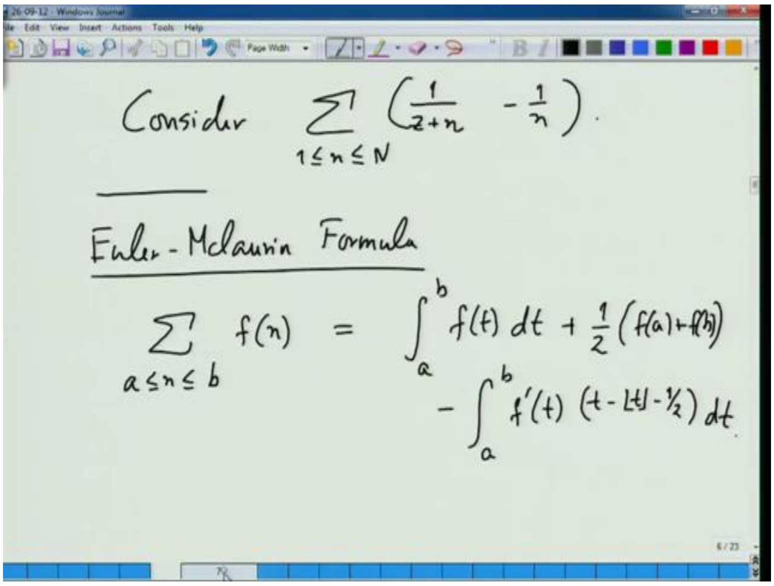
Just look at this this is constant this is one by z 1 by z 1 by mod z actually 1 by mod z mod z is way out there it is at least u which is at most 1, so this gets obsodent to order constant here.

(Refer Slide Time: 37:14)



So, what is left out is plus summation n greater than equal to 1 mod, so if you can estimate this modulus of this whenever z is far away from negative integers, then we get an good estimate. So, how do we estimate this, when you look here this looks again familiar should look familiar. So, to estimate this again, we will use the same trick that consider the sum from 1 to capital n from this quantity derive an expression to it and then send n to infinity.

(Refer Slide Time: 38:48)

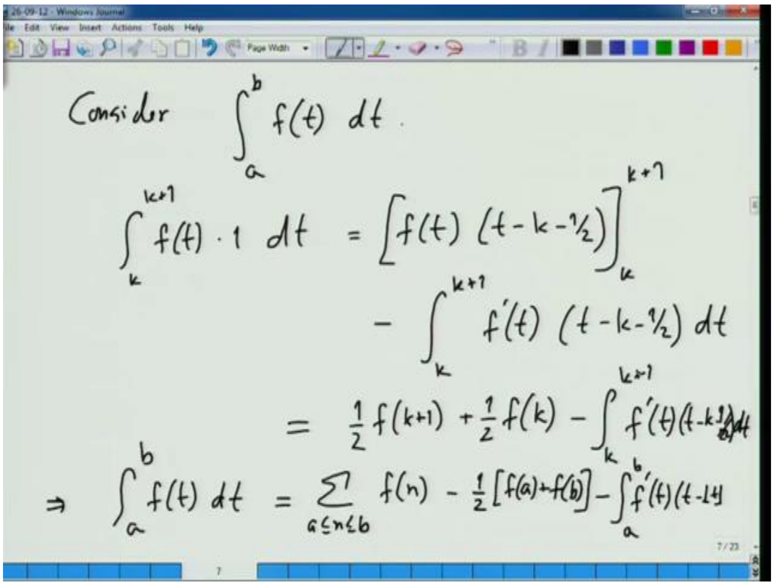


So, if you consider 1 less than equal to small n less than equal to capital n and we want to bound this what is the second bounded with we know that that is like log of capital n plus the Euler's constant. So, we know understand, we understand it very well what about the first gage these are the z taken out in all of the denominators, well it would be convenient if we can replace the sum by an integral we can integrate things far more easily. Now, there is a standard way of replacing sums with integrals that is called think even long time about Euler Aclaurin Euler and Mclaurins is called the Euler Mclaurin formula.

So, we take a bit of diversion from here and take even to Euler Mclaurin formulae, this in general talks about sums like this and what this corresponds to in terms of integrals. In fact, it does not even say that you should be one to n you can replace with a and band the formulae says of course I am sure I will remove it in correctly. So, let me write something and when we derive it we will come back and correct it, so that is the main part is like this sum is same as this integral, but then there is some problems here which are the error terms plus or minus no.

I do not remember let us take a guess it is probably minus prime t and then there is this funny expression this is as I said this is only my guess or what the formulae is, but is close enough, so its derive this does not require anything fancy oh we are out of time. So, let us find out an expression for this how do we do that it is is a simple trick here and there is lot to do with this a quantity.

(Refer Slide Time: 44:16)

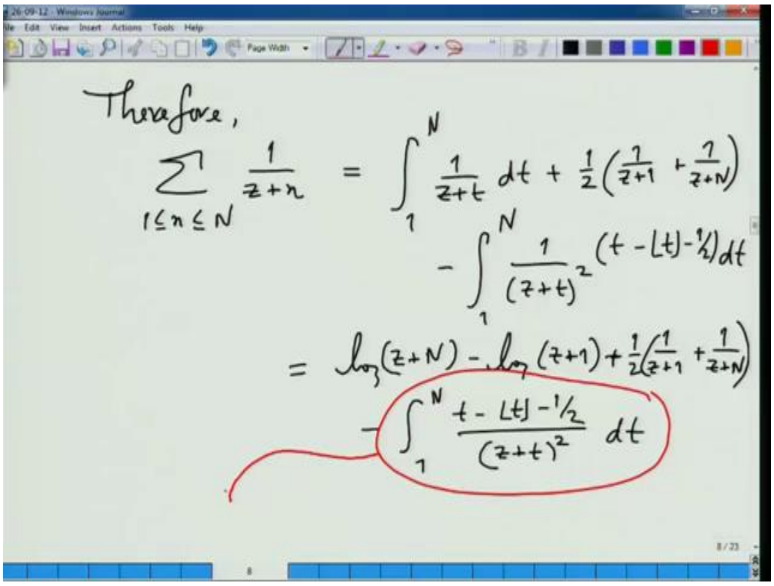


Let us look at this Now that is not right, let us do integration by parts, now what do you mean by integration by the time only one quantity here well. So, what I will do is times 1 times dt, so other part is 1, this is equal to ft and then integrate 1, what do you get when you integrate 1. So, let me do something funny here, I integrate from k to k plus 1 just over 1 integer and then integrating this I get t, but instead of actually there will be t plus I, can bring in any constant. So, I will integrate this tot minus l minus r we are perfectly within my rights to do this, then there is another part is negative f prime t and then I integrate 1 in the same fashion.

These are important, so this becomes t minus k minus half dt, now what is the first one of these if f k plus 1 and what you get there is one half 1 minus 1 minus half, so half of f q plus 1 and minus of minus becomes plus half off k minus this integral. Now, sum we do here form for k going from a to b minus 1, so I am assuming that a and b are integral points is what sum over this for k going from a to b minus 1, so what do you get half of f of a plus half of f of a plus 1. Then, next time I do not want to get half of f of a plus 1 plus half of f of a plus two and then again and again and again.

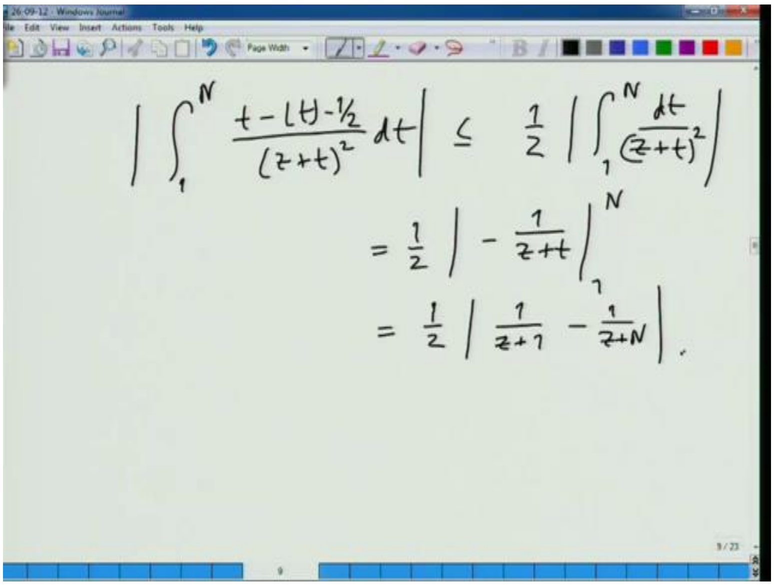
So, you basically get half of f a plus full f a plus one plus f a plus two plus f a plus three all the way upto f of b minus 1 and then integrate half of f b. So, this is therefore, fn minus half of f a plus f bright minus a to b f prime t t minus, now k is same as flowed of t because now I am replacing the limits from k to k plus 1 to a to b. So, in within this range floor of t is exactly k when you go to the next integral range floor of t becomes a plus one and so on. So, that perfectly makes this good I was actually right my first easting were right this should have been a plus, so we just take it on the right side to get this that is it now, let is just apply this on to this summation.

(Refer Slide Time: 50:28)



This is equal to what integral from 1 to capital n 1 over z plus t dt plus half of f of a is 1 over 1 plus no pps 1 plus over not 1 plus n, sorry z plus 1 plus 1 over z plus n plus derivative of f prime that is equal to z plus 2, whole square then minus sign t minus floor of t minus half d t and this is first integral is what log z plus t plus half. Now, in absolute value I will not get into absolute value yet this term just look at this this is sort if the error term for me if you look at the numerator here it is always utmost half numerator value of t. So, numerator is utmost half the denominator for 1 z plus 2 whole squared, so if I try to get a bound on error term the absolute value of this.

(Refer Slide Time: 55:34)



So, let us just look at the error term, absolute value of this is equal to half this integral is 1 by z plus t minus 1 by z plus t actually m going from 1 to n. So, we have an estimate of on the value of the error term, now we have everything what we want in place, now just we will put it together next time and derive the estimate.