Riemann Hypothesis and its Applications
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Lecture – 17

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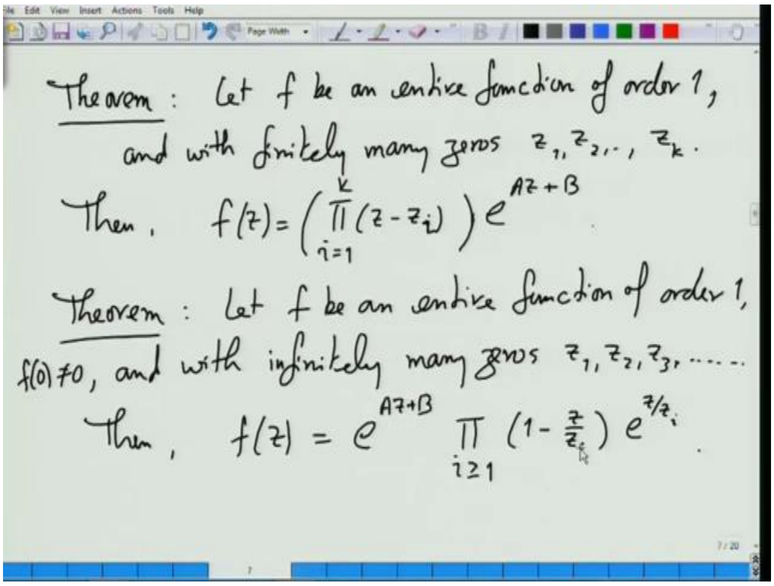

So, this is a theorem which we want to prove today. So, one the first things that you asked time was why does this product converge at all and once we insure that these converges. Then there is still one more thing that one needs to show which is let us say f, f we know is of order onethis function should also be order onethan that division will be an entire function of order 1 without zeroes.

Therefore, it is equal to e to the A z plus B, so these are two things we need to show that this product converges, not only it converges it is actually an entire function of order 1. Now, to show this we will necessarily have to use some properties of this roots z i this will not be a convergent function for all possible z i actually. So, the property that we use of z i is actually the fact that f the function we started with whose roots are the z i is an function of order one so this already says something about the roots.

See roots are very closely related or say the number of roots are very closely related to the order of the function think of polynomials a polynomial of degree k has k roots and that is what the number of roots is what determines a degree. Now, degree is what determines the asymptotic growth of the function, the more the number of roots is the higher the growth is.

So, this seems somewhat at a first class that you would expect that more roots are there the less will be the growth of the function. But, actually eventually more roots allow a higher order of growth, and since we know that the order of the growth of the function we can conclude something about the, about the roots. But, of course the number of roots is infinite, but we can say that within a certain region how many roots are there and that count is what we will derive now.

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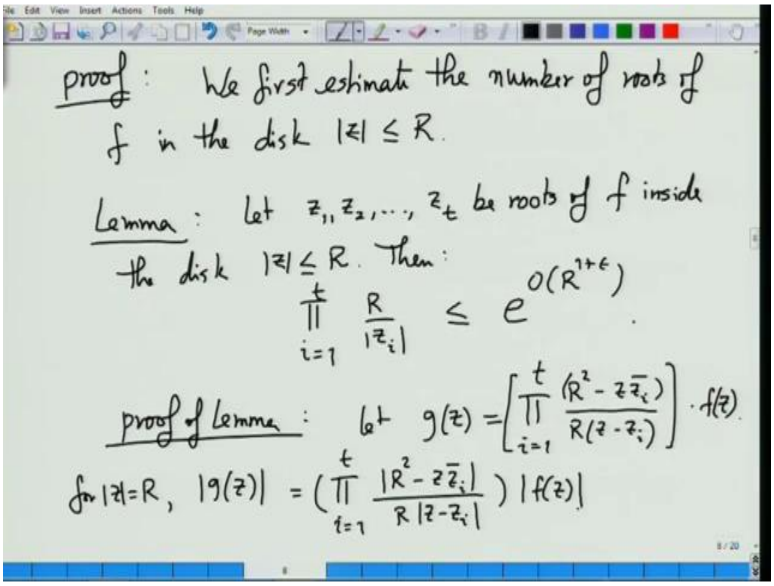

So, let us say we estimate the number of roots in some disk of radius R and for this we the following let me first that. So, let us say it has t roots z 1 to z t inside the inside the disk so I am ruling out the boundaries in fact I will take boundary R, so that there are no roots there. So, that I can always do than this product of I going from one to t R divide by the absolute values of z i that is bounded by the order the right hand side is essentially the order of the function at distance R.

Now, they bound on the magnitude of the function and is bounded by that magnitude proof of lemma is fairly straight forward, let us define function based on f which is the followingneed to multiply this with R. So, what do I want here, so this g is simply f multiplied with a certain product see the idea is to take away all the zeroes of f inside the disk. So, what we are doing here is we are dividing f by the product z minus z i which take away all the zeroes.

But, the rest of the multiplier is to ensure that the absolute value of g does not blow up or it stays bounded by absolute value of f around the boundary. Now, that is clear by the fact that if you look at firstly note that what should I say let us look at the absolute of g z for when absolute value of z is R what is absolute value of g z.

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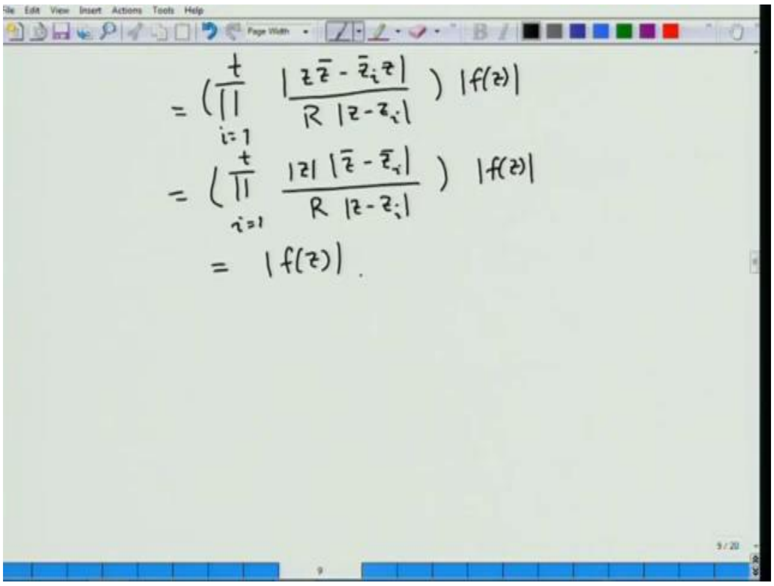

So, this is equal to there is only trick I am going to play here what is R square when absolute value of z is R than z z bar is R square. So, I will replace this by z z where are we, now we take absolute value of z out in common, now we look at the product absolute value of z is R which cancels with the R here. So, absolute value of z bar minus z i bar is same as absolute value of z minus z i because one is conjugate of the other. So, the absolute value does not change, so the product is actually 1 and so what we get is absolute. Now, for g what we know is g is, g is an analytic function on the disk there is no pulse, yes in the disk inside the disk it has to have finite number of zeroes.

So, because every 0 is isolated for any analytic function, every 0 is isolated that see something we showed long time ago. Therefore, inside a finite disk or any finite region there can only be finitely many zeroes and if there are infinitely many zeroes than there will be a yes. But, then they will converge see isolated means there is a delta, so that delta disk around that point has no zeroes, for every 0 there is a delta disk around it which has no zeroes.

But, in this case if these zeroes converge to one point than that point or around that point you can never find a delta with only with no zeroes else. Well, if that very does not matter whether that point is a 0 or not just look at that point for any for any delta disk around that point we actually have infinitely many zeroes for any delta.

Now, 0 is isolated this cannot happen because if a 0 is isolated than take any point on the plane there is small enough delta you can find around which there is at most one 0 or inside which there is at most one 0, either that point is a 0. So, there is small delta, so that no there is no other 0 if that point is not 0, then there is a small delta which, so that in that disk there is no 0, not convinced, let us look at this fine, let us look at this, let us go here.

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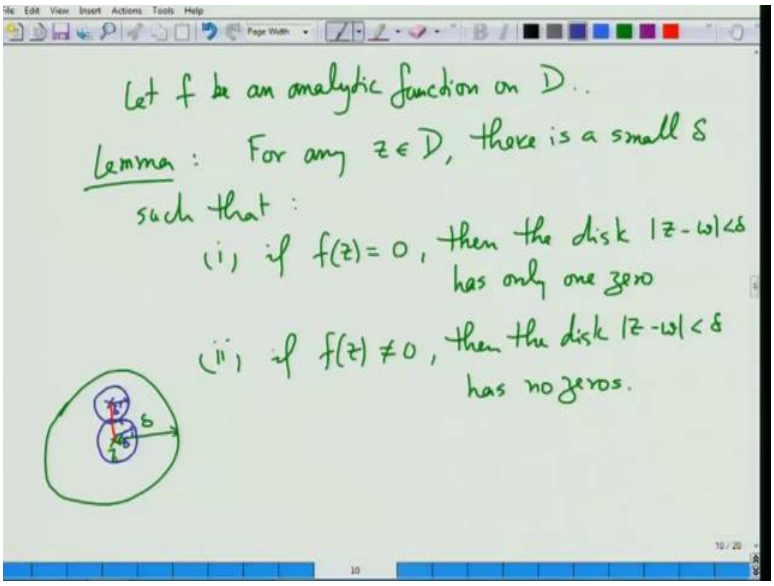

So, let us establish the following property on some domain D, now let us do the lemma for any z in D there is a small delta do you believe this lemma f is on domain D. So, take any point in the domain if f z is 0, f z is a 0 of f then by definition of the fact that the zeroes are isolated there is a small delta. So, that delta disk around z has no 0, no other 0, there is only one 1 on the other hand if f z is not 0 then it is some finite distance away from any 0 of f inside the disk.

So, take the, no that is that is not a precise argument, so if f z is not 0 then you take a small enough disk around f z and suppose it does contain a 0 fine. Now, around that point which is a 0 there is a delta prime disk which is has no zeroes, no other zeroes, so let us, so let us see this. So, this is a domain this is let us say this is z this a disk around delta disk around it suppose it has a 0 here, now this is a 0 then there is a small disk around this which has no other 0. So, let us say if we look join this line from this point to this point then, so I am trying to say that argue that, take the closest 0 to z.

So, the question is can one precisely define the closest 0 to z, then it has to be 0, that is, that is a, so that is simpler good. So, then there has to be you cannot than there has to be a closest 0 to z and if this was a closest 0 than you take a disk centred here of radius. So, if this is delta prime than you take a disk of radius whatever delta prime here than it will notdelta prime or whatever some small number than this distance. So, it will not contain any 0 so with this lemma in place, now you can argue that in a finite domain there can be only finitely many zeroes.

So, because if not than there will be a infinitely if there are infinitely many zeroes than if we keep dividing these domain into smaller pieces every piece there will always be one piece which has infinitely many zeroes. But, keep shrinking it to stay at infinity infinite and that is not that is going to violate this lemma at some point I am saying compact. So, as I am saying to take athis is a compact set of domain, so a finite disk is a compact set, so then we go back.

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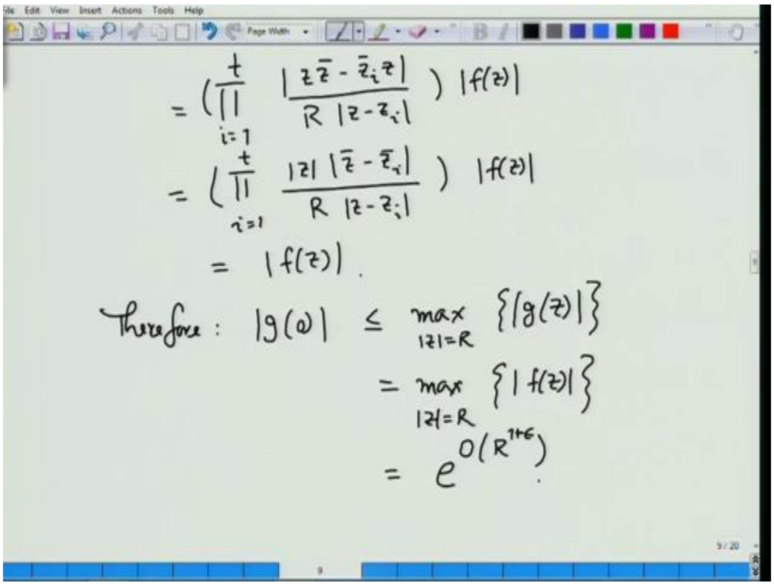

So, the finitely many zeroes this sorts off fine and so that is, this was point of this trick that the absolute value of g is same as absolute value of f on the edge of this desk. Further, g has no zeroes actually that is not very important, but g is analytic on the disk the reason g is analytic is that all the zeroes of this or the poles of g. So, let us say possible poles of g are also zeroes of f, so they get cancelled out, so g is analytic and since g is analytic we can invokes the Gaussian integral formula to conclude that g 0.

So, the absolute value of g 0 is bounded by the maximum value of f z around the circumference, in fact this happens that oh sorry maximum value of g z around the circumference. So, this is same as we just saw maximum value g z around circumference in absolute value same as f z around circumference this is max f z and what is max f z around circumference because f is an order 1. So, at z equals R f grows like e to the order R to the 1 plus epsilon and what is g 0, g 0 if we look at the definition of g when we plug the 0 in here what do you get.

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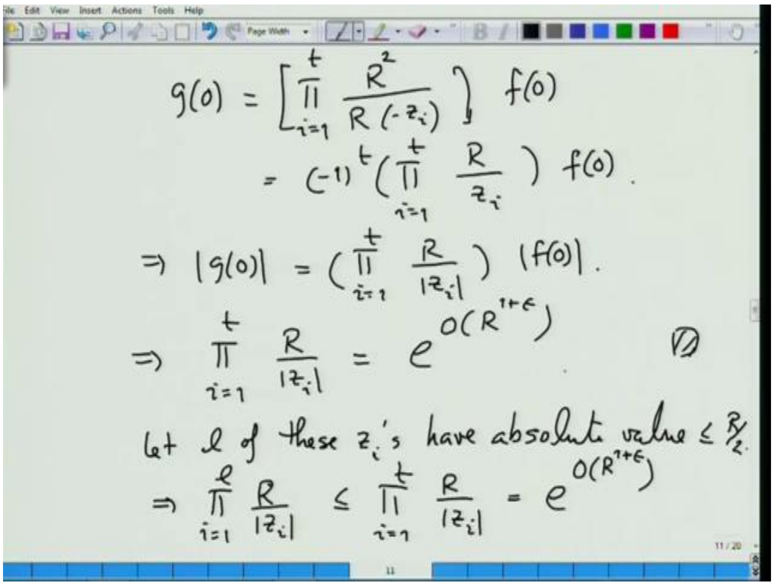

So, product of I going from 1 to t R square up there R down here and minus z i here times absolute. So, absolute value of g 0 is, therefore product I going from one to t R by absolute value of z i times absolute value of 0. Now, f 0 is some finite it is not it is non zero number by assumption some finite value whatever it is does not matter. So, we just put this this together with the bound we just derived on the upper bound on absolute value of g 0. So, what we get is that, that is what lemma and now you can see that this expression already says that you cannot have too many zeroes here because absolute value of z i is always less than R.

So, this ration is always more than 1 and you are taking product of t of that and we know an upper bound on this product. So, you cannot have too many of these, in fact these already you can derive suppose we just take try to count how many z i are there with absolute value less than R by 2 l of these z i. So, then we get i equate to 1 to l R by absolute value of z i, this is surely less than equal to I going from one to t R by absolute value of z i. So, this is e to the order R one plus epsilon and what is this this is greater than equal to 2 to the l.

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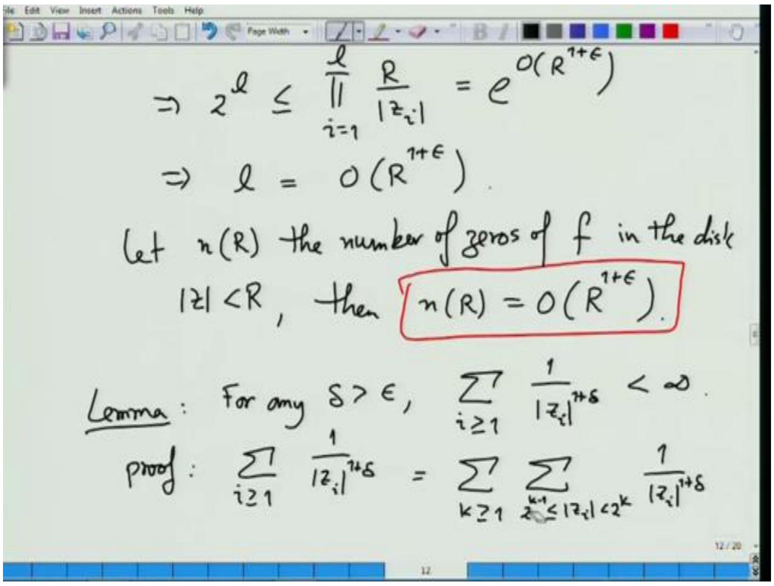

So, what we get is l is order R to the one plus epsilon l is, remember the number of zeroes of f of absolute value at most R by 2. Now, R was the, so this already gives a bound on the number of roots of f up to a certain value whatever certain value. So, you give whatever value you give you get the number, so let us say if we denote by n R the number of zeroes if f in the disk n R is order R to the 1 plus epsilon. So, this is a very interesting conclusion of the fact that f isan entire function of order one now using these.

But, this is still not the full story this is something very useful we will make use of this later on also one of the very interesting things we can use this. So, for this to bound sums of roots some expression of roots, so here is another lemmafor any delta greater than epsilon. So, if we take the sum of 1 over absolute value of z i to the 1 plus delta this sum converges and prove it pretty straight forward. So, I split this sum by or rather group this sum by absolute value of z i and I do group them of absolute value between 0 and 1, 1 and 2, 2 and 4, 4 and 8.

So, successive powers of 2, now this sum if we look at the thing inside absolute this is in the denominator, so I can replace this by always or upper bound this sum by replacing these y 2 to the k minus 1 or 2 to the k. So, which one gives me an upper bound smaller value of denominator will give me an upper bound, so we take 2 to the k minus 1 for absolute value for z i times 1 plus delta.

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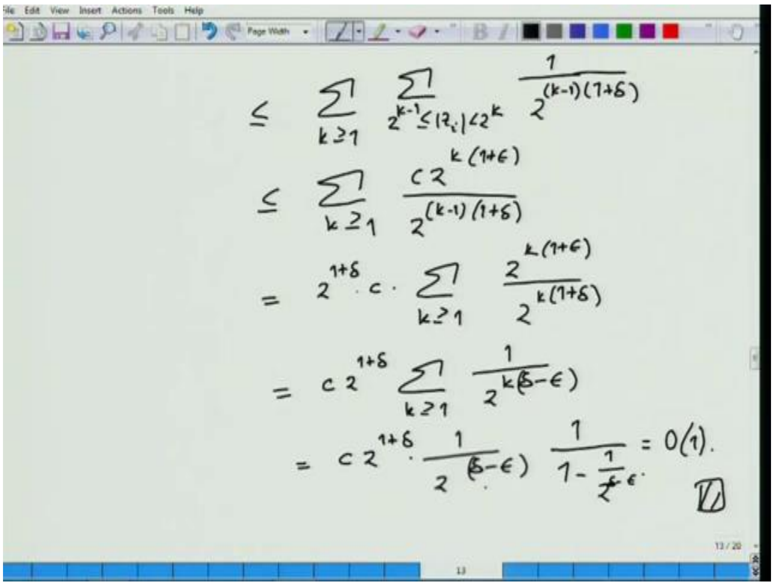

Now, the sum becomes easy, so this is how many roots are there between 2 to the k minus 1 and 2 to the k. Well, that is we just derived that just you forget about the lower bound how many roots are there up to 2 to the k, 2 to the k to the 1 plus epsilon. Now, of course some constant here, so let us just take out the constants parts of these and there is of course convergence this is geometric series converging to some quantity whatever it is I think it converges to. So, that is a very useful property for us and this is what we are going to use to prove the convergence we just saw back to the proof of theorem.

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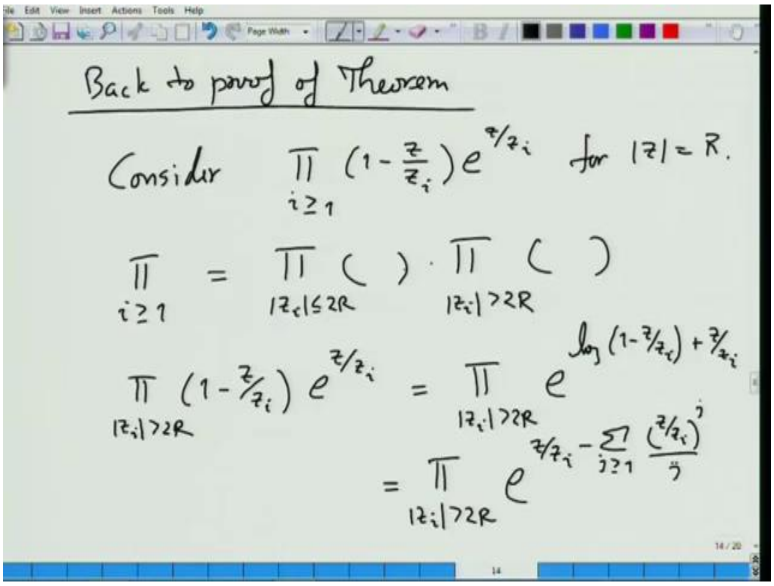

So, consider this product I would like to prove that this converges this is an entire function which means for any z this has to take a finite value and let us say for some z which is let us say R fix a z whose absolute value is R. Now, I want to show that this is afinite value at that z, so I split this product as in two parts z i less than equal to 2 R and 1 is z i greater than 2 R this part this is a finite product. So, this surely will converge to some finite value, this all very nicely behaved there are no poles here, so it will converge at some point.

So, if show that this converges to from finite value I am done, so let us just focus on this partso absolute value of z i, i order this in product of z i than only consider. Now, why is this finite we just decided right we just argued that in a disk of radius 2 R there are only 5, so this is finite, so that that argument still holds. So, this is bounded and I want to show that this is bounded, now because absolute of z i is more than 2 R, here if we look at z over z i.

So, the absolute value of this is less than half always for all i, therefore I can write this as e to the this part as e to the log 1 minus z over z i plus z over z i. Now, all I have done is taken written this as e to the log, now again log has come here so you have to be careful. But, here again because of this property that z over z i only moves in a radius of half and it moves around 1.

So 1 minus z over z i its minimum value is half maximum value is in the real line 3 by 2 and similarly in the complex plane. So, it is a disk centred around one of radius half that is a that is a range here, so there log is completely analytic on that disk because 0 is far away. So, I can take any, I can take any log of the when we just choose the principle log which has no addition to this term.

So, I write this not only that because again this is at most absolute value in half, so I can replace this because it is analytic. So, I replace this by power series in that small disk what is a power series like, so let us just bring z over z i first log of 1 minus x. So, what is the power series, there is a negative for with all the terms than j greater than equal to 1 then we get z by z i to the power j upon j.

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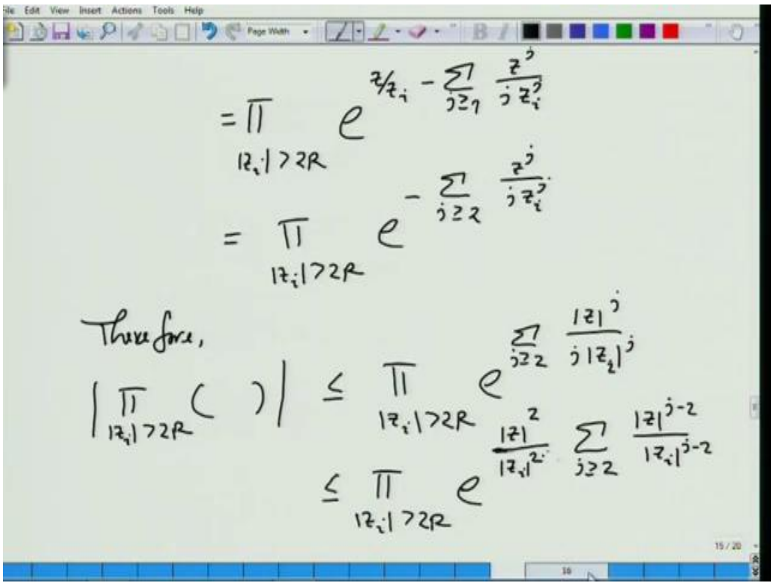

So, z by z i minus sigma j greater than equal to 1 z to the j upon j z i to the j, now the first term of this is z over z i in the sum which cancels with this and that was the whole reason of sticking e to the z over z i there. So, what is left out after the cancellation is e to the minus j greater than equal to 2 z to the j over z i to the j and this is the product, now let us take absolute value of this. So, this product is absolute value or let us say less than equal to e to the, well the real part of this real part I can now let substitute by the absolute value of the upper exponent. So, I get j greater than equal to 2 z to the j divided by j absolute value of z i to the j, no absolute values, so this all gone minus is gone.

So, I am saying that absolute value of this is e to the real of this, now real of this is at most real of complex number. So, real part of a complex number is less than equal to absolute value of the complex number and the absolute value of this whenever as soon as you take that the minus sign goes away and then you take it inside. So, its sum of absolute values just take less than equal to, now let us take out the first term and inside this we have let us forget about this j sitting in the denominator.

So, this is only reducing the exponent so forget about the j so you just take this as, so this again becomes a geometric series and in this geometric series z absolute value of z over absolute value of z i is at most half. So, it converges and remember that z i, it is always converges to at most two, no matter what z ii am choosing here right.

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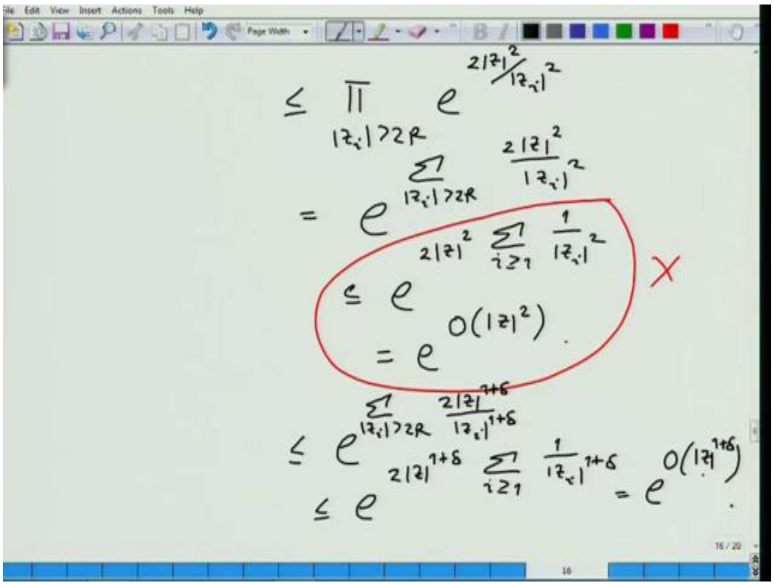

So, I can write this as e to the 2 z square by z i square and this of course, now I can write as and this sum we just showed the previous lemma is bounded sum over i, 1 over absolute z i square. So, in fact we can do slightly better here, in fact let us go, let me take these out and do the slightly better analysis which will use later on. So, see z over z i is always less than 1 in this sum that is by choice so z over z i squared is less than equal to z over z i to the power 1 plus epsilon or one plus delta. So, this is I can write this as less than equal to e to the sum over z i greater than 2 R to z to the 1 plus delta by z i to the 1 plus delta.

So, what we conclude is that the absolute value of the product is bounded by to the order, absolute value of z to the 1 plus delta. So, we are not only sure that the product is bounded we are here we are also showing that the product is of order 1. So, it is a function its analytic function of order 1 and that completes the proof of the theorem because all we needed to show was.

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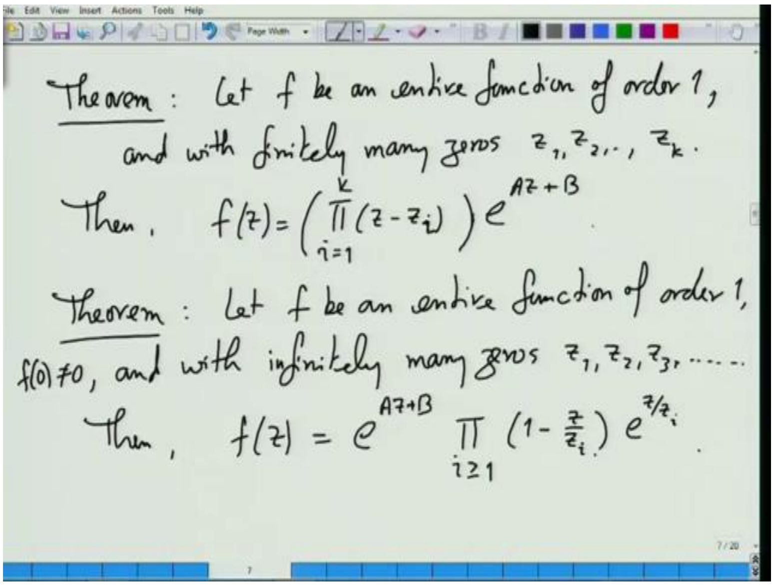

So, that this product is bounded, now that not does not complete the proof, I am sorry, I should take that back that is showing that this function is analytic function. But, an entire function of order 1 with precisely the same set of zeroes of that, so that we have concluded.

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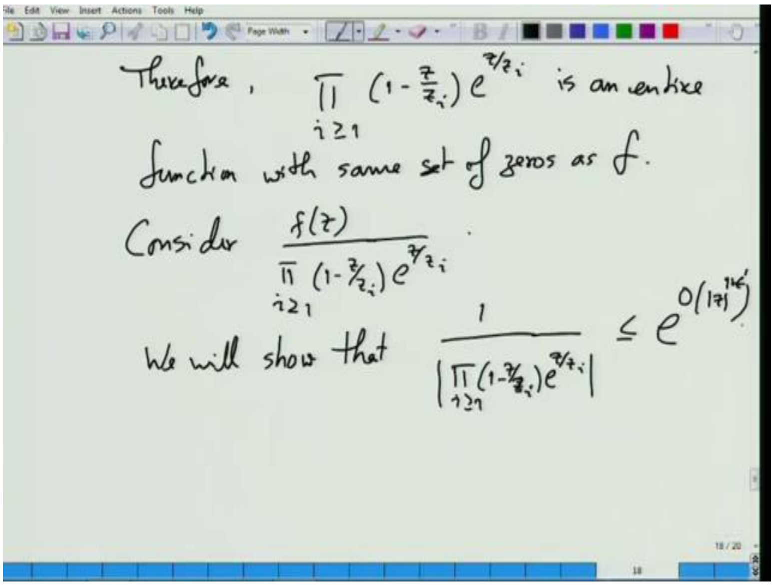

Now, consider f z divided by this this function is anentire function this function has no zeroes also because all the zeroes cancel out of f and this. So, it is an entire function without zeroes if we can get that the order of this function is 1 we have proved the theorem we know that the order of f is 1. So, if we can prove that the order of 1 over this product is 1 then we are done, now we will show that one over absolute value of this is at most e to the order z 1 plus some epsilon prime. Now, we do this using the same thing as just did actually is just a little bit of an extension of what we just did.

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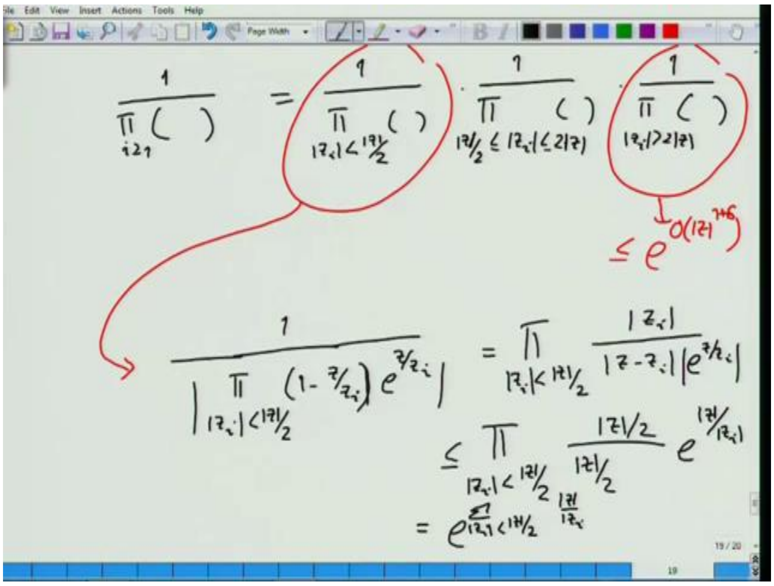

So, 1 over the product i greater than equal to 1 of this, we write it as 1 over product absolute value of z i is less than absolute value of z by 2 time 1 over 2 z. So, these 3 part yes given any z split the roots in 3 groups absolute value less than absolute value of z by 2 between z by 2 and 2 z and more than 2 z. So, this is the only infinite product, the other two are finite products, for this infinite product we have just shown that the product is bounded by this which is a function of order 1. So, this we have already done, this is already done, so what is left with these two finite products if we consider this.

So, this is equal to 1 over product z i less than z by 2, 1 minus z over z i times e to the z over z i in absolute value. So, let us say not consider these not equal to just consider this absolute value is equal to, let us take the product, now this I can ignore why because it gap between z and z i. So, it is at least absolute value of z by 2, I do not want to ignore this, all I am saying is this isthis is less than equal to product z i less than z by 2 absolute value of z i is bounded by.

So, of course z by 2 absolute value of z minus z i is at least absolute value of z by 2, so I can replace it by z by 2 and there I can I need to bring e up there when I take. Now, of course this cancels each other out, so all that you are left with is e to the sum z i less than z by 2 z over z i equal to this.

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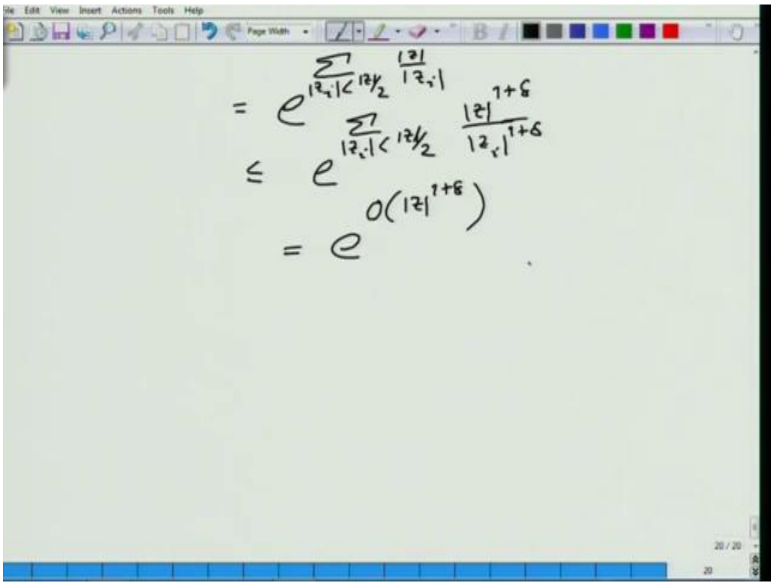

So, how do we handle this, we cannot handle the sum, can we handle the sum we need ayou cannot sum one over z i you do not know if that is bounded. But, remember this is a finite sum not only is this finite sum it is only up to z i is less than z by 2, so this is at most. So, that means z is more than z i always in this sum, so I do the reverse of what I did earlier and replace this by z to the 1 plus delta over z i to the 1 plus delta. Now, as always true, I am increasing the exponent, so this becomes finite and now we can bound this, so that is it for today.